A Test for Risk-Averse Expected Utility

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Abstract

We provide a GARP-like test for the risk-averse expected utility hypothesis with multiple commodities. Our test can be viewed as a natural counterpart of a classical test of expected utility, due to Fishburn (1975), in a demand-based framework.

1 Introduction

The recent contribution of Kubler et al. (2014) provides a GARP-like test for expected utility maximization in a contingent-consumption environment. In an environment with a single consumption good and finite states of the world, they establish an acyclicity condition on observed data which is both necessary and sufficient for a finite list of observed price and consumption pairs to be consistent with the hypothesis of expected utility maximization. Thus, their paper provides a counterpart of the classical work of Afriat (1967) with the added restriction that rationalizations be expected utility.

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As Kubler et al. (2014) note, their test is universal in nature, removing all existential quantification. Their test amounts to verifying that the product of certain cycles of risk-neutral prices be bounded above by one. Our aim in this note is to provide a *different* universal test. Our test should be distinguished from the Kubler et al. (2014) test in three ways. First, it applies to any finite number of consumption goods, whereas the test of Kubler et al. (2014) only applies for a single consumption good. Secondly, our test is intimately tied to the classical von Neumann-Morgenstern axioms of expected utility theory, and thus has a simple economic intuition. On the other hand, our test involves universal quantification over a potentially infinite set of objects, while the test in Kubler et al. (2014) can be reduced to universal quantification over a finite set.

Our test is perhaps most closely related to an early revealed preference test of expected utility due to Fishburn (1975). Fishburn constructs a test for an abstract environment of choice over lotteries with a finite set of outcomes. One observes a finite set of binary comparisons; some are weak, and some are strict. Fishburn provides necessary and sufficient conditions for there to exist an expected utility ranking which extends the observed finite set. Imagine that we observe x^k weakly preferred to z^k for $k = 1, \ldots, l$, and x^k strictly preferred to z^k for $k = l + 1, \ldots, K$. Fishburn establishes that these observations are consistent with expected utility maximization if there is no probability distribution over $\{1, \ldots, K\}$ which puts positive probability on $\{l + 1, \ldots, K\}$, and for which the expected x^k under this probability distribution is equal to the expected z^k.

Fishburn's test can be viewed as claiming that the smallest possible extension of the observed relations satisfying both independence and transitivity leads to no contradiction.

Our test works similarly. We have a finite list of commodities, and a finite set of states Ω. We observe a finite list of prices and consumption bundles chosen at those prices. Consumption in state ω at observation k is of the form $x^k_\omega \in \mathbb{R}^n_+$. Probabilities over Ω are known and are given by the full support distribution π.

We first ask: what could reveal a violation of the expected utility hypothesis in this context? There are only a finite set of states of the world, with known probabilities, but if the choices *were* rationalizable by an expected utility preference, there would be a natural extension to a preference over the set of all simple lotteries. One such violation would look like the following: suppose that for each x^k, there is some y^k which is feasible at prices p^k. In other words, x^k is *revealed preferred* to y^k. And suppose that there is some l for which y^l is strictly cheaper than x^l at prices p^l. Then x^l is *revealed strictly preferred* to y^l. Now, each y^k induces a lottery, and as such, it is meaningful to talk about mean-preserving spreads. Suppose we can find, for each k, a z^k which is a mean-preserving spread of y^k. Then if the data were expected utility rationalizable by a risk-averse preference, the lottery induced by x^k would be preferred to z^k for all k, and in fact, x^l would be strictly preferred to z^l.

We now have a finite collection of pairs of lotteries (x^k, z^k). These data can be tested with Fishburn's condition. If, in fact, they violate Fishburn's condition, then we know that the original data cannot be expected utility rationalizable.

So far this is very simple. In fact, what we show is that the converse is also true: if data are not risk-averse expected utility rationalizable, then we can find y^k and z^k as above, to violate Fishburn's condition. In fact, they can be chosen to violate Fishburn's condition in a very stark way: one must only test the uniform lottery over $\{1, \ldots, K\}$.

Moreover, the support of each z^k can be chosen to consist only of consumption that was actually observed demanded at some state; *i.e.* the support can be chosen amongst elements of the form x^k_ω. This resonates with the idea from Polisson and Quah (2013), who observe that in order to rationalize data, it is both necessary and *sufficient* to maintain consistency on the set of minimally extended "imaginary" data, constructed from those actually observed. However, while Polisson and Quah (2013) is concerned with developing Afriat-style algorithms (see Afriat (1967)) for testing decision models with money lotteries, our focus is developing universal statements about data from lotteries of general consumption bundles, which provides direct falsification of the expected utility model under risk aversion.

Importantly, we can write our conditions in a way which can be interpreted as a universal (even UNCAF) axiom; thus the condition can be explained as characterizing exactly which types of data are ruled out by the hypothesis of expected utility maximization (see Chambers et al. (2014)).

The idea of the proof is remarkably simple, and is a simple restatement of the dual set of linear inequalities stemming from the Afriat-style inequalities of Green and Srivastava (1986) or Varian (1983).

A host of other interesting papers have recently studied choice data in the context of expected utility maximization. In particular, Echenique and Saito (2013) investigates the subjective expected utility version of the model, which forms a kind of analogue of the Kubler et al. (2014) test. It would be interesting to propose a test of our structure in the subjective expected utility framework. Epstein (2000) investigates the empirical content of the notion of probabilistic sophistication (due to Machina and Schmeidler (1992)), providing a test which can refute the hypothesis.

2 The Model

We assume that there is a finite state space $\Omega = \{\omega | 1, 2, \ldots, S\}$ and a finite collection of consumption goods, labeled $1, 2, \ldots, N$. The agent is given an objective probability distribution over states $\pi \in \Delta(\Omega)$, for which for all $\omega \in \Omega$, $\pi_\omega > 0$. A *data set* consists of a K tuple for some finite K, $\mathcal{D} = \{(x^k, p^k)_{k=1}^K\}$, where $x^k \in \mathbb{R}_+^{SN}$ and $p^k \in \mathbb{R}_{++}^{SN}$.[1] The vector x^k represents an observed contingent consumption plan chosen by the agent under price p^k. In particular,

$$x^k = \begin{bmatrix} x_1^k \\ \vdots \\ x_\omega^k \\ \vdots \\ x_S^k \end{bmatrix} \quad p^k = \begin{bmatrix} p_1^k \\ \vdots \\ p_\omega^k \\ \vdots \\ p_S^k \end{bmatrix}$$

[1] As usual, $\mathbb{R}_+$ denotes the nonnegative reals and $\mathbb{R}_{++}$ denotes the positive reals. Hence, prices are strictly positive, but consumption can be zero.

and

$$x^k_\omega = \begin{bmatrix} x^k_{\omega,1} \\ \vdots \\ x^k_{\omega,N} \end{bmatrix} \quad p^k_\omega = \begin{bmatrix} p^k_{\omega,1} \\ \vdots \\ p^k_{\omega,N} \end{bmatrix}$$

where for all ω, k, n, $x^k_{\omega,n}$, $p^k_{\omega,n} \in \mathbb{R}_+$.

We say that $\mathcal{D}$ is *risk-averse expected utility rationalizable* if there exists a concave and nondecreasing $u : \mathbb{R}^N_+ \to \mathbb{R}$ for which for all k, x^k solves

$$\max \sum_\omega \pi_\omega u(x_\omega)$$

subject to $p^k \cdot x \leq p^k \cdot x^k$.

Some definitions are in order. Let $C = \mathbb{R}^{NS}_+$ denote the set of contingent consumption plans. A pair of revealed preference relations $\succeq^C$ and $\succ^C$ can be defined on C.

For $x, y \in C$, $x \succeq^C y$ if $x = x^k$ for some k and $p^k \cdot y \leq p^k \cdot x^k$. For $x, y \in C$, $x \succ^C y$ if $x = x^k$ for some k and $p^k \cdot y < p^k \cdot x^k$. $\succeq^C$ is intended to represented a revealed weak preference and $\succ^C$ a revealed strict preference.

Given a data set $\mathcal{D}$, we collect all the consumption bundles x^k_ω that are observed in the data:

$$\mathcal{X} = \{x \in \mathbb{R}^N | x = x^k_\omega \text{ for some } k \text{ and } \omega\}.$$

Denote the set of all simple lotteries on $\mathbb{R}^N_+$ with finite support (simple lotteries) by $\Delta_s(\mathbb{R}^N_+)$. Denote the set of all lotteries on $\mathcal{X}$ by $\Delta(\mathcal{X})$. Note that any state contingent consumption plan $x^k \in C$ induces an element in $\Delta(\mathcal{X})$ that places probability π_ω on x^k_ω. In what follows, we will abuse notation and let x^k denote both the contingent consumption plan and the lottery it induces in $\Delta(\mathcal{X})$.

The pair of relations $\succeq^C$, $\succ^C$ on C naturally induce a pair of relations on $\Delta_s(\mathbb{R}^N_+)$. Moreover, to test the hypothesis of risk aversion, it is natural to extend the previously defined revealed preference relations. For example, suppose that $x \succeq^C y$, and $z \in \Delta_s(\mathbb{R}^N_+)$ is a mean-preserving spread of y viewed as a lottery. If our decision maker's behavior is consistent with risk-averse expected utility maximization, it follows that x should be preferred to z. These ideas motivate the following definitions.

For $y, z \in \Delta_s(\mathbb{R}^N_+)$, $y \succeq^{m.p.s.} z$ if z is a mean-preserving spread of y. Define the pair of binary relations $\succeq^R$ and $\succ^R$ on $\Delta_s(\mathbb{R}^N_+)$ by

$$x \succeq^R z \text{ if there exists } y \text{ such that } x \succeq^C y \succeq^{m.p.s.} z$$

with

$$x \succ^R z \text{ if there exists } y \text{ such that } x \succ^C y \succeq^{m.p.s.} z$$

If the agent's behavior is consistent with risk-averse expected utility maximization, the pair of relations $\succeq^R, \succ^R$ will necessarily satisfy Fishburn's condition, *i.e.* if $x^k \succeq^R z^k$ for $k = 1, \ldots, l$, and $x^k \succ^R z^k$ for $k = l+1, \ldots, K$, then there are no $\{\mu_i\}_{i=1}^K$, with $\mu_k \geq 0$ for all k, $\sum_{k=l+1}^K \mu_k > 0$, and $\sum_1^K \mu_k x^k = \sum_1^K \mu_k z^k$. As we show in our main result, it turns out that a *sufficient* condition for the data $\mathcal{D}$ to conform with risk aversion and expected utility maximization is that the restriction of $\succeq^R, \succ^R$ to $\Delta(\mathcal{X})$ satisfies Fishburn's condition.

A remark is in order. Fishburn (1975) also considers the issue of testing the consistency of revealed preference relations with functional restrictions on the von Neumann-Morgenstern utility index. Specifically, he wants to test when observed data are consistent with the utility index u belonging to some convex cone $\mathcal{U}$. Again, he assumes a finite number of relations (which does not hold in our context). A natural guess is that if x^k is revealed weakly preferred to z^k for $k = 1, \ldots, l$ and revealed strictly preferred to z^k for $k = l+1, \ldots, K$, then if there is $\mu \in \Delta(K)$ for which $\mu(\{l+1, \ldots, K\}) > 0$ and $u \cdot \left(\sum_k \mu_k z^k\right) \geq u \cdot \left(\sum_k \mu_k x^k\right)$ for all $u \in \mathcal{U}$, then the observed data are inconsistent with expected utility maximization with utility index $u \in \mathcal{U}$ [2]. In our case, for example, we would consider the cone of nondecreasing and concave functions; the claim would then be that $\sum_k \mu_k z^k$ second order stochastically dominates $\sum_k \mu_k x^k$. Of course, the existence of such a μ refutes the hypothesis of expected utility rationalization with $u \in \mathcal{U}$, but for technical reasons, a converse

[2] Here we continue to use x and z for lotteries, and dot product for integration w.r.t. measures.

statement need not hold in general. However, we are able to show that owing to the special structure of linear pricing, a converse statement along the lines of this idea does in fact hold in the demand-based environment. In fact, it holds *even though observed revealed preference relations are infinite.*

Theorem 1. *The following are equivalent:*

I *For any* $\{z^k\}_{k=1}^K \subseteq \Delta(\mathcal{X})$ *for which* $x^k \succeq^R z^k$ *for all* k*, there is no* $\{\mu_k\}_{k=1}^K$ *such that* $\mu_k \geq 0$ *for all* k*,* $\sum_{\{k:x^k \succ^R z^k\}} \mu_k > 0$ *and* $\sum_1^K \mu_k x^k = \sum_1^K \mu_k z^k$.

II *For all* $S^k_\omega : K \times \Omega \to \mathbb{R}_+$ *for which for all* k, ω*,* $\sum_{l,\tau} S^k_\omega(l,\tau) = \pi_\omega = \sum_{l,\tau} S^l_\tau(k,\omega)$*,*

if for all k*,*

$$x^k \succeq^C \left(\frac{\sum_l \sum_\tau S^k_\omega(l,\tau)}{\pi_\omega} x^l_\tau \right)_{\omega \in \Omega}$$

then there is no k *for which* $x^k \succ^C \left(\frac{\sum_l \sum_\tau S^k_\omega(l,\tau)}{\pi_\omega} x^l_\tau \right)_{\omega \in \Omega}$.

III *For all* $\omega, \tau \in \Omega$ *and* $k, l \in \{1, ..., K\}$ *there exists* $u^k_w, u^l_\tau \geq 0$ *and* $\lambda_k, \lambda_l > 0$ *s.t.* $u^k_\omega \leq u^l_\tau + \lambda_l \frac{p^l_\tau}{\pi_\tau} \cdot (x^k_\omega - x^l_\tau)$.

IV *Data set* $\mathcal{D}$ *is risk-averse expected utility rationalizable.*

Before proceeding, we comment on cases I and II, which are our contribution. Case I considers the smallest possible preference extension "consistent" with the data, risk-aversion, and the expected utility hypothesis. It claims that if this extension is meaningfully defined; in that we cannot derive that a lottery x is strictly preferred to itself, then the data are expected utility rationalizable. Importantly, we only need to consider lotteries whose support are actual observed consumption bundles. This can be seen as a natural analogue of Fishburn's condition as applied to x^k and z^k.

Case II demonstrates a dual system of linear inequalities to the inequalities of case III. Case III was derived previously by Green and Srivastava (1986). The interpretation of the function S^k_ω is as a "spread" operator. For each k, the bundle $y^k = \left(\frac{\sum_l \sum_\tau S^k_\omega(l,\tau)}{\pi_\omega} x^l_\tau \right)_{\omega \in \Omega}$ is revealed weakly worse than x^k by demand behavior. Then, this particular contingent plan can be decomposed. In particular, since

$\sum_{l,\tau} S^k_\omega(l,\tau) = \pi_\omega$, the lottery z^k which places probability $S^k_\omega(l,\tau)$ on x^l_τ is a mean-preserving spread of y^k, so if the data are consistent with the hypothesis of expected utility maximization, z^k should be worse than x^k. The condition $\sum_{l,\tau} S^l_\tau(k,\omega) = \pi_\omega$ allows us to find a violation of Fishburn's condition with the lotteries x^k and z^k across all k if there is some x^k which is revealed strictly preferred to z^k.

Proof. The equivalence of III and IV is due to Green and Srivastava (1986).

We proceed to show that II and III are equivalent. To this end, suppose that III does not hold. Then there is no solution to the following linear system.[3] $Ab \geq 0$ and $\lambda \gg 0$, where

$$b = \begin{bmatrix} u^1_1 \\ \vdots \\ u^K_S \\ \lambda \end{bmatrix} \quad \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_K \end{bmatrix}$$

and A is equal to the top two quadrants of the matrix below:

$$T = \begin{array}{c} \\ \eta_{1,1,1,1} \\ \vdots \\ \eta_{k,\omega,l,\tau} \\ \vdots \\ \vdots \\ \eta'_k \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc|cccccc} u^1_1 & \ldots & u^k_\omega & \ldots & u^l_\tau & \ldots & u^K_S & & \lambda_1 & \ldots & \lambda_k & \ldots & \lambda_K \end{array} \\ \left[\begin{array}{ccccccc|ccccc} 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \ldots & 1 & \ldots & -1 & \ldots & 0 & 0 & \ldots & \frac{p^k_\omega}{\pi_k} \cdot (x^l_\tau - x^k_\omega) & \ldots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \hline \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 1 & \ldots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \end{array}\right] \end{array}$$

By construction of T and a standard theorem of the alternative (see for example Mangasarian (1994) p. 30), the nonexistence of b, λ such that $Ab \geq 0$ and $\lambda \gg 0$, is equivalent to the existence of $\eta \geq 0$ such that $T'\eta \leq 0$, where

[3] Vector inequalities are $x \geq y$ if $x_i \geq y_i$ for all i and $x \gg y$ if $x_i > y_i$ for all i.

$$\eta = \begin{bmatrix} \eta_{1,1,1,1} \\ \vdots \\ \eta^K_{K,S,K,S} \\ \eta' \end{bmatrix} \quad \eta' = \begin{bmatrix} \eta'_1 \\ \vdots \\ \eta'_K \end{bmatrix}$$

such that at least one $\eta'_k > 0$.

This implies that

$$\sum_{\omega} \sum_{(l,\tau)\neq(k,\omega)} \eta_{k,\omega,l,\tau} \frac{p^k_\omega}{\pi_\omega} \cdot (x^l_\tau - x^k_\omega) \leq 0 \; \forall \, k \tag{1}$$

with strict inequality for at least one k, and

$$\sum_{(l,\tau)\neq(k,\omega)} \eta_{k,\omega,l,\tau} = \sum_{(l,\tau)\neq(k,\omega)} \eta_{l,\tau,k,\omega} \; \forall \, k, \omega \tag{2}$$

Since $\eta_{k,\omega,k,\omega}$ does not show up in the two sets of relations above, and since the system is homogeneous and $p^k_\omega \cdot (x^k_\omega - x^k_\omega) = 0$, without loss we can normalize η such that for all ω, k, $\sum_l \sum_\tau \eta_{k,\omega,l,\tau} = \pi_\omega$ by picking $\eta_{k,\omega,k,\omega}$ appropriately and rescaling. Rearranging inequalities (1) gives

$$\begin{aligned} \sum_{\omega} \sum_{l} \sum_{\tau} \eta_{k,\omega,l,\tau} \frac{p^k_\omega}{\pi_\omega} \cdot (x^l_\tau - x^k_\omega) &= \sum_{\omega} p^k_\omega \cdot \Big(\sum_{l} \sum_{\tau} \frac{\eta_{k,\omega,l,\tau}}{\pi_\omega} x^l_\tau - \sum_{l} \sum_{\tau} \frac{\eta_{k,\omega,l,\tau}}{\pi_\omega} x^k_\omega \Big) \\ &= \sum_{\omega} p^k_\omega \cdot \Big(\sum_{l} \sum_{\tau} \frac{\eta_{k,\omega,l,\tau}}{\pi_\omega} x^l_\tau - x^k_\omega \Big) \leq 0 \end{aligned}$$

with at least one strict inequality. This together with (2) establishes the equivalence of II and III, by taking $S^k_\omega(l,\tau) = \eta_{k,\omega,l,\tau}$.

That IV implies I is straightforward; namely, observe that if $\mathcal{D}$ is risk-averse expected utility rationalizable, there is a concave nondecreasing $u : \mathbb{R}^N_+ \to \mathbb{R}$ for which for all $p, q \in \Delta_s(\mathbb{R}^N_+)$, $x^k \succeq^R z^k$ implies $u \cdot x^k \geq u \cdot z^k$, and $x^k \succ^R z^k$ implies $u \cdot x^k > u \cdot z^k$. Then the result follows by linearity.

We now show that I implies II. Suppose by means of contradiction that there is a solution to the system listed in II but that I is true. Let

$$y^k = \left(\frac{\sum_l \sum_\tau S^k_\omega(l,\tau)}{\pi_\omega} x^l_\tau \right)_{\omega \in \Omega}$$

By II, we have $x^k \succeq^C y^k \; \forall \, k$ with $\succ^C$ for at least one k.

Let z^k be the lottery that puts probability $S^k_\omega(l,\tau)$ on x^l_τ. Note that z^k is a mean preserving spread of y^k, so by definition, $x^k \succeq^R z^k \; \forall \, k$, with $\succ^R$ for at least one k.

For each (l,τ), the probability that $\sum_{k=1}^K \frac{1}{K} z^k$ attributes to x is given by $\sum_{\{(\tau,l):x^l_\tau = x\}} \frac{1}{K} \sum_k \sum_\omega S^k_\omega(l,\tau)$, which is $\sum_{\{(\tau,l):x^l_\tau = x\}} \frac{\pi_\tau}{K}$ by assumption. And by definition, the probability that $\sum_{k=1}^K \frac{1}{K} x^k$ attributes to x^l_τ is given by $\sum_{\{(\tau,l):x^l_\tau = x\}} \frac{\pi_\tau}{K}$.

This constitutes a contradiction to I (in particular, the contradiction comes in the form of a uniform distribution over the observations $1, \ldots, K$).

$\square$

An couple observations are in order. It can be shown that both (I) and (II) of our properties imply GARP. To see that (I) implies GARP, suppose that there is a revealed preference cycle $y^{k_1} \succeq^C y^{k_2} \succeq^C \ldots y^{k_m} \succ^C y^{k_1}$, where without loss we may assume there are no repetitions in the cycle. Since $\succeq^C$ implies $\succeq^R$ and $\succ^C$ implies $\succ^R$, we have $y^{k_1} \succeq^R y^{k_2} \succeq^R \ldots y^{k_m} \succ^R y^{k_1}$. Let $x^i = y^{k_i}$ and $z^i = y^{k_{i+1}}$ as in property (I), then a uniform distribution μ over the indices $i = 1, 2, ...m$ constitutes a violation of (I).

For (II), consider $S^k_\omega(l,\tau)$ in property II such that for $k = k_i$ for some i (k in the cycle)

$$S^{k_i}_\omega(l,\tau) = \begin{cases} \pi_\omega & \text{if } l = k_{i+1} \text{ and } \tau = \omega \\ 0 & \text{otherwise} \end{cases}$$

and for $k \neq k_i$ for any i (k not in the cycle)

$$S^k_\omega(l,\tau) = \begin{cases} \pi_\omega & \text{if } l = k \text{ and } \tau = \omega \\ 0 & \text{otherwise} \end{cases}$$

Then the cycle condition gives a violation of property (II), a contradiction.

3 Conclusion

We have developed a GARP-like test for the risk-averse expected utility environment with many commodities. Of interest for future research would be an analogous test in the subjective expected utility context, following the work of Echenique and Saito (2013).

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