An axiomatic theory of political representation

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June 2005

Abstract

We discuss the theory of gerrymandering-proof voting rules. Our approach is axiomatic. We show that, for votes over a binary set of alternatives, any rule that is unanimous, anonymous, and gerrymandering-proof must decide a social outcome as a function of the proportions of agents voting for each alternative, and must either be independent of this proportion, or be in one-to-one correspondence with the proportions. In an extended model in which the outcome of a vote at the district level can be a composition of a governing body (with two possible parties), we discuss the quasi-proportional rules (characterized by unanimity, anonymity, gerrymandering-proofness, strict monotonicity, and continuity). We show that we can always (pointwise) approximate a single-member district quota rule with a quasi-proportional rule. We also discuss a more general environment, where there may be more than two parties. JEL classification: D63, D70. Keywords: gerrymandering, representative systems, proportional representation, social choice, quasi-arithmetic means.

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1 Introduction

In representative democracies, there are two standard methods of assigning representatives to districts. One, the single-member, or winner-take-all method, assigns a unique representative to each district of voters as some function of their votes. The other method is the method of proportional representation. In this method, representatives of each of the parties are assigned to each district in proportion equal to the number of votes each of the parties received in the districts.

The two methods each have their strengths. The winner-take-all method gives each district a unique representative respecting its "collective interest." The proportional method accurately reflects the composition of votes received for different alternatives. In particular, under proportional representation systems, the benefit to strategically constructing districts is significantly reduced.

We study an abstract theory of representation. Our particular interest is in voting rules for which there is no benefit to strategically constructing districts. At least one rule and system of representation enjoys this property: majority rule with proportional representation. Our goal is to obtain a broader understanding of these systems. We ask the following question: "For which voting rules is it without loss of generality to group agents into voting districts?"

Our formal model is implicitly a two-stage model. We focus on the study of "rules." Each voter selects (votes for) an alternative, resulting in a vector of alternatives indexed by the names of the voters (sometimes referred to as a vote-profile or a list of votes). We do not model the procedure by which voters select alternatives; one might naturally imagine that they announce their favorite alternative according to some preference relation.

Voters are grouped into districts. Each district selects a district outcome according to the rule. Outcomes can be alternatives over which votes were cast in the first stage, but they may include other objects as well. For example, the outcome may be a "tie," while we might not let voters vote for such an object in the first stage. Outcomes of a vote can be compromises between alternatives over which voters vote. A tie, for example, is a kind of compromise outcome. Likewise, another compromise outcome might involve a coin-flip over the alternatives.

We do not explicitly model a voter who acts as a representative for the district. Instead, we imagine that the role of such a representative would be to vote in a second stage for her district's outcome. The fact that she acts as a representative for her district is interpreted as meaning that if all members of the district had voted for the district outcome, the social outcome would not change. We formalize this by imagining that, in each district, the voters' votes are replaced by the district outcome obtained by the first stage election (we also sometimes refer to this vector of outcomes as a vote profile). To accommodate the implicit second stage of voting, a voting rule must also select a unique outcome for every possible list of outcomes that may be voted for in the implicit second stage. The same rule is then applied to these modified votes, resulting in a social outcome. Similar implicit two-stage aggregation procedures have been studied by Murakami [16, 17], Fishburn [9, 10], and Fine [7]. The following example is intended to clarify how the model works.

Example: We here imagine two *alternatives* $A = \{1, -1\}$, over which voters may vote. The possible *outcomes* are elements of the set $X = \{-1, 0, 1\}$. Voters in the initial stage may not vote for the "outcome" 0, which is interpreted as a tie between the two alternatives. Imagine a set of agents $N = \{1, 2, 3, 4, 5, 6\}$ and two possible lists of districts $N_1 = \{1, 2\}$, $N_2 = \{3, 4\}$, and $N_3 = \{5, 6\}$; and $M_1 = \{1, 2, 3, 4\}$, $M_2 = \{5\}$, and $M_3 = \{6\}$. The voting rule under consideration is plurality rule. Among districts, votes are tallied for each alternative; the one receiving the most votes wins, otherwise it is a tie. In the second stage, representative votes (which are the *district outcomes*) are tallied. A representative vote may therefore be for a tie. Consider the following list of votes: $(1, 1, 1, -1, -1, -1) \in A^N$. Then $(1, 1) \in A^{N_1}$, $(1, -1) \in A^{N_2}$, and $(-1, -1) \in A^{N_3}$. The outcome in N_1 is 1, in N_2, it is 0; and in N_3, it is -1. The profile of votes in the second stage is therefore $(1, 1, 0, 0, -1, -1) \in X^N$. Note that 0 is not a legitimate vote for individual voters in the first stage, but as a district outcome, it is a legitimate vote in the second stage. Applying plurality rule to this profile results in the *social outcome* of 0, a tie. Consider now the districts M_1, M_2, M_3, $(1, 1, 1, -1) \in A^{M_1}$, $(-1) \in A^{M_2}$, and $(-1) \in A^{M_3}$. The district outcome for M_1 is 1; for M_2 is -1, and for M_3 is -1. Thus, the list of second stage votes is given by $(1, 1, 1, 1, -1, -1) \in X^N$. Applying plurality rule to this profile results in the social outcome of 1. The choice districts influences the *social outcome*.

As the example illustrates, the social outcome generally depends on the choice of districts. When there exists a society of voters and a list of first stage votes over alternatives for which this can happen, we say that "gerrymandering" is possible. Otherwise, we say the rule is "gerrymandering-proof".

In another work [6], we consider rules with a finite set of outcomes, each of which is an alternative for which voters may vote. Any rule of this type which is democratic (in the sense of being anonymous and reflecting the will of the people when a unanimity of voters vote for a certain alternative) and gerrymandering-proof exhibits pathologies. It must be a type of "unanimity rule," whereby outcomes are partially ordered. Any voter may veto an outcome with an outcome that is ranked higher.

Our first main result is a related statement dealing with more general environments. We allow the set of alternatives and outcomes to differ (alternatives are always possible outcomes, but other outcomes may be admitted). We further allow these sets to be infinite. Consider two possible alternatives over which voters may vote in the first stage. We study the restriction of the rule to profiles of votes consisting of exactly those alternatives (effectively the same as imagining there are only two alternatives over which voters can vote). Our result states that there are exactly two possibilities for such environments. One possibility is that the *social outcome* selected for such environments is independent of the vote profile (thus, it is a constant function of the vote profile). The other possibility is that the rule depends only of the proportion of voters voting for each alternative, and that there exists a one-to-one correspondence between this proportion and the set of possible outcomes.

One immediate implication of this result is that the set of possible outcomes which can be realized when voters vote for these two alternatives must either be a singleton or must be countably infinite. Hence, to rule out pathologies, a rule must admit a countably infinite set of *outcomes*.

Proportional representation can be understood as a voting rule whose possible district and social outcomes are elements of a countably infinite set. In this case the set of possible outcomes are the *compositions of a governing body*. Systems of proportional representation, at least when perfectly implemented, are independent of the way that districts are drawn (for an interesting axiomatic study of the integer problem in proportional representation, see Balinski and Young [3]).

To explore this idea, we study an environment with an underlying binary social decision. In the initial stage, voters vote for one of two underlying alternatives. District outcomes, however, consist of the proportion of voters voting for each alternative. In the second stage, we imagine the representative votes to be for objects such as: "a governing body which is composed 57% of agents in favor of alternative 0, and 43% of agents in favor of 1." It is important to emphasize that in the first stage, voters do not vote for such constructs. These outcomes are important in the implicit second stage, when *votes corresponding to district outcomes* are aggregated.

The interpretation of the set of outcomes as compositions of a governing body is not important. Another natural interpretation is that the outcomes are lotteries over the two degenerate alternatives. Moreover, one may even choose to think of the outcomes as forming a classical unidimensional Euclidean policy space.

Are there rules that are both gerrymandering-proof and democratic when we admit a large set of outcomes? A broad class of such rules is known from the mathematics literature. Imagine voters vote over the alternatives $\{0, 1\}$ and that district and social outcomes are elements of $[0, 1]$. Define the "proportional rule" as that rule which simply takes the arithmetic mean of the votes that voters submit. This rule is gerrymandering-proof and corresponds to a system of proportional representation. However, there is no reason to think the arithmetic mean is special. In fact, any "quasi-arithmetic mean" will work just as well. Define a *quasi-proportional rule* as a rule that takes a quasi-arithmetic mean over all votes received.

The abstraction away from standard proportional representation is more than just a mathematical exercise. There are important practical implications of the analysis. To this end, we study "how close" we can come to a single-member district system with a gerrymandering-proof rule. We establish that there exists a *sequence* of voting rules which are gerrymandering-proof, but which "converge" to majority rule with single-member districts, in a formal sense. The benefit of such a result from an institutional design standpoint is that there exists a system which is gerrymandering-proof, but for which each district "almost" gets its own representative. In fact, we show that this approximation result actually holds for any "quota rule."[1]

These results do not hold as we move away from binary decisions to decisions involving more than two alternatives. In fact, we show that essentially the only natural method of representation in this more general environment is the proportional rule.

Section 2 introduces the formal model. Section 3 provides a fundamental result motivating the remaining part of the study. Section 4 discusses our model of proportional representation. Section 5 concludes.

2 The model

Let X be an arbitrary set of outcomes. These correspond to outcomes which are obtained as the result of votes both in districts and socially. There is an infinite set of potential agents, which we without loss of generality index by the natural numbers $\mathbb{N}$. At any given time, we will only consider finite subsets of $\mathbb{N}$. The set of finite subsets of $\mathbb{N}$ is denoted by $\mathcal{N}$. A rule is a function $f : \bigcup_{N \in \mathcal{N}} X^N \to X$, recommending for each society N of voters, and each $x \in X^N$ some outcome.

Alternatives, $A \subset X$, are a subset of the possible outcomes. These are a subset of elements over which individual voters vote in the first stage. We will introduce an axiom of gerrymandering-proofness which depends on A. The larger is A, the stronger the axiom. In the example in the introduction, $A = \{-1, 1\}$ whereas $X = \{-1, 0, 1\}$. In this example, X allows for ties at the district and social level, while voters in the first stage cannot themselves vote for a "tie." The rule f can be applied both within districts to individual votes for alternatives and socially to profiles which consist of the district outcomes.

Variants of the following conditions were studied in Chambers [6]. Kolmogorov [14] and Nagumo [18] also study these axioms in the context of the real numbers (see Theorem 2 below).

2.1 Democratic principles

The first property we discuss in this section states that a rule should respect the "will of the people" when this "will" is unambiguous. It applies to profiles of votes for *outcomes*, as opposed to alternatives. This reflects the fact that unanimity should apply in the implicit second stage of voting.

For all $N \in \mathcal{N}$ and all $x \in X$, let x^N be a vector in X^N such that for all $i \in N$, $x_i^N = x$. For all $N \in \mathcal{N}$, all $x \in X^N$, and all $M \subset N$, let x_M be the restriction of x to X^M.

Unanimity: For all $N \in \mathcal{N}$ and all $x \in X$, $f\left(x^N\right) = x$.

[1] A quota rule is a rule for which there exists some status quo alternative 0, and a **quota** $q \in [0, 1]$, such that alternative 1 wins if the proportion of agents voting for 1 is greater than or equal to q.

The next axiom states that a rule should be ignorant of the names of agents. We also state it with respect to outcomes, as opposed to alternatives.

Anonymity: For all $N, N' \in \mathcal{N}$ such that $|N| = |N'|$, all bijections $\sigma : N \to N'$, and all $x \in X^N$ and $x' \in X^{N'}$ such that for all $i \in N$, $x_i = x'_{\sigma(i)}$, $f(x) = f(x')$.

2.2 Gerrymandering-proofness

Informally, gerrymandering-proofness states that for any population of voters and any collection of first stage votes for alternatives, it is without loss of generality to partition the set of agents into districts, find the outcome for each district, and then treat each district as if each agent in the district had voted for the outcome selected for the district. We state the axiom parametrically, so that it may depend on $A \subset X$, the alternatives over which individuals vote in the first stage.

A-Gerrymandering-proofness: For all $N \in \mathcal{N}$, all partitions $\{N_1, ..., N_K\}$ of N, and all $x \in A^N$, $f(x) = f\left(f(x_{N_1})^{N_1}, ..., f(x_{N_K})^{N_K}\right)$.

Under the unanimity principle, A-gerrymandering-proofness is equivalent to the stricter statement that for all $N \in \mathcal{N}$, all $M \subset N$, and all $x \in A^N$, $f(x) = f\left(f(x_M)^M, x_{N \setminus M}\right)$. This latter version of A-gerrymandering-proofness is more useful in the proofs of theorems.

Note that if $A \subset B \subset X$, then B-gerrymandering-proofness is stronger than A-gerrymandering-proofness. Therefore, the smaller is the set A under consideration, the weaker the axiom. To this extent, our main results will rely on the case in which A is a binary set. If A is a singleton, then under the unanimity principle, A-gerrymandering-proofness is vacuous.

The axiom of X-gerrymandering-proofness was first stated by Kolmogorov [14] and Nagumo [18]. These authors were concerned with the axiomatization of the *quasi-arithmetic means*, in which the axiom plays a central role. We later state their theorem and show how it relates to ours.

Any anonymous rule can be specified without reference to the specific names of agents. In the proofs of results in which anonymity plays a role, we often exploit this fact without mention, disregarding the variable N.

The following axiom states that only the *proportions* of votes received for each alternative are used in determining the social outcome. It is not very compelling by itself, but it will be shown to be an implication of other properties below in Lemma 1.

Let m be an integer, let $N \in \mathcal{N}$, and let $x \in X^N$. Let $N' \in \mathcal{N}$ be such that $|N'| = m|N|$. A vector $x' \in X^{N'}$ is an **m-replica of x** if there exists a partition of N' into m sets of size $|N|$, say $\{N_1, ..., N_m\}$ such that for all N_i, there exists a bijection $\sigma_i : N \to N_i$ so that for all $j \in N$, $x_j = x'_{\sigma(j)}$. For all $N \in \mathcal{N}$, $x \in X^N$, $m \in \mathbb{N}$, x^m denotes an m-replica of x.

A-Replication invariance: Let m be an integer. Let $N \in \mathcal{N}$ and let $x \in A^N$. Let x' be an m-replica of x. Then $f(x') = f(x)$.

3 A theorem of the alternative for representation

The following observation is useful:

Lemma 1: If a rule satisfies unanimity, anonymity, and A-gerrymandering-proofness, then it satisfies A-replication invariance.

Proof: Let $N \in \mathcal{N}$ and let $x \in A^N$. Let x' be an m-replica of x. Then by definition of x', $f(x') = f\left(\underbrace{x, ..., x}_{m}\right)$. By A-gerrymandering-proofness, $f(x') = f\left(\underbrace{f(x), ..., f(x)}_{m|N|}\right)$. By unanimity, $f\left(\underbrace{f(x), ..., f(x)}_{m|N|}\right) = f(x)$. Thus $f(x') = f(x)$.$\blacksquare$

The following theorem gives a general result on the structure of democratic and gerrymandering-proof rules. Let $x, y \in X$, and let $A = \{x, y\}$. For $N \in \mathcal{N}$, say that $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}^N$ **is an $\{\mathbf{x}, \mathbf{y}\}$-profile** if there exists $i \in N$ such that $z_i = x$ and $j \in N$ such that $z_j = y$. Thus, z is an $\{x, y\}$-profile if all voters vote for either x or y, and at least one voter votes for x and one votes for y.

Theorem 1 is a "theorem of the alternative." It states that in an environment in which voters vote over two alternatives x and y, if a rule satisfies our primary axioms, then there are two (mutually exclusive) possibilities. The first, pathological, possibility is that the rule is constant on the set of all $\{x, y\}$-profiles. Such a rule does not recognize the proportions of agents voting for each alternative x and y. Such rules are investigated in Chambers [6]. The second possibility is that the rule is generated by an injection between the set of proportions of voters voting for x, and the set of outcomes. This second possibility requires that the set of outcomes is at least countable, but it also requires that a rule can only be based on the set of proportions of voters voting for each alternative. Moreover, it requires that a rule completely discriminate among proportions. We first present an informal description of the proof, then a rigorous argument.

In the proof, we construct a function h which takes as arguments pairs of positive rational numbers (α, β). The output of h for any such pair is an outcome, it is the unique outcome recommended for a society for which a fraction $\frac{\alpha}{\alpha+\beta}$ of the agents vote for x and a fraction $\frac{\beta}{\alpha+\beta}$ vote for y. It is easily verified that the equivalence classes for this function are "convex" cones. The claim in Theorem 1 is that either there is only one equivalence class of h, or that there exists a countable number of equivalence classes, each of which is a ray. When we assume that the statement of Theorem 1 is false, we are assuming that there exists an equivalence class of h which is neither a ray and which is also not the entire domain of h. A key step in the proof of Theorem 1 is the verification that if $(\alpha, 1-\alpha)$ and $(\beta, 1-\beta)$ lie in such a cone, so that $h(\alpha, 1-\alpha) = h(\beta, 1-\beta)$, then adding the same vector to both $(\alpha, 1-\alpha)$ and $(\beta, 1-\beta)$ does not change this equality. We then demonstrate that for an appropriate choice of $(\alpha, 1-\alpha)$ and $(\beta, 1-\beta)$, this property is violated, resulting in a contradiction.

Theorem 1: Suppose that a rule f satisfies unanimity, anonymity, and $\{x,y\}$-gerrymandering-proofness. Then one and only one of the following is true. i) f is constant on the set of $\{x,y\}$-profiles, ii) there exists an injective function $g^{(x,y)} : (0,1) \cap \mathbb{Q} \to X$ such that for all $N \in \mathcal{N}$, if $z \in \{x,y\}^N$ is an $\{x,y\}$-profile, $f(z) = g^{(x,y)}\left(\frac{|\{i \in N : z_i = x\}|}{|N|}\right)$.[2]

Proof: Step 1: Construction of an auxiliary function

First, we construct an auxiliary function $h : \mathbb{Q}^2_{++} \to X$ in the following manner. For all pairs $(\alpha, \beta) \in \mathbb{Q}^2_{++}$, we may write $\alpha = \frac{m(\alpha)}{n}$, and $\beta = \frac{m(\beta)}{n}$, where $m(\alpha), m(\beta)$, and n are natural numbers greater than zero. We define $h(\alpha, \beta) = f\left(x^{m(\alpha)}, y^{m(\beta)}\right)$. Although the representation of α and β in terms of ratios of natural numbers is not unique, it is unique up to scalar multiplication. Therefore, by $\{x,y\}$-replication invariance, h is well-defined.

Step 2: Establishing "additivity" of the auxiliary function

We establish that for all $(\alpha, \beta), (\alpha', \beta') \in \mathbb{Q}^2_{++}$, if $h(\alpha, \beta) = h(\alpha', \beta')$, then $h(\alpha + \alpha', \beta + \beta') = h(\alpha, \beta)$. To this end, suppose that (α, β) and (α', β') are such that $h(\alpha, \beta) = h(\alpha', \beta')$. Label $z \equiv h(\alpha, \beta)$. There exists some n large enough so that $\alpha = \frac{m(\alpha)}{n}, \beta = \frac{m(\beta)}{n}, \alpha' = \frac{m(\alpha')}{n}, \beta' = \frac{m(\beta')}{n}$. Thus, $\alpha + \alpha' = \frac{m(\alpha) + m(\alpha')}{n}$ and $\beta + \beta' = \frac{m(\beta) + m(\beta')}{n}$. By definition of h,

$$h(\alpha + \alpha', \beta + \beta') = f\left(x^{m(\alpha) + m(\alpha')}, y^{m(\beta) + m(\beta')}\right).$$

Rewriting,

$$f\left(x^{m(\alpha) + m(\alpha')}, y^{m(\beta) + m(\beta')}\right) = f\left(\left(x^{m(\alpha)}, y^{m(\beta)}\right), \left(x^{m(\alpha')}, y^{m(\beta')}\right)\right).$$

By $\{x,y\}$-gerrymandering-proofness,

$$\begin{aligned} & f\left(\left(x^{m(\alpha)}, y^{m(\beta)}\right), \left(x^{m(\alpha')}, y^{m(\beta')}\right)\right) \\ = \; & f\left(f\left(x^{m(\alpha)}, y^{m(\beta)}\right)^{m(\alpha)+m(\beta)}, f\left(x^{m(\alpha')}, y^{m(\beta')}\right)^{m(\alpha')+m(\beta')}\right). \end{aligned}$$

[2] For any $A \subset X$ and $\{x,y\} \subset A$, A-gerrymandering proofness is clearly stronger than $\{x,y\}$-gerrymandering proofness. Such stronger assumptions will have the same implications for $\{x,y\}$-profiles.

But the preceding is $f\left(z^{m(\alpha)+m(\beta)}, z^{m\left(\alpha'\right)+m\left(\beta'\right)}\right)$, so that by unanimity, $f\left(z^{m(\alpha)+m(\beta)}, z^{m\left(\alpha'\right)+m\left(\beta'\right)}\right) = z = h(\alpha, \beta)$.

The preceding paragraph tells us that the equivalence classes for the function h are $\mathbb{Q}$-convex cones, closed under addition and rational scalar multiplication. This last property will be called **rational homogeneity** of h.

Step 3: Establishing "translation invariance" of the auxiliary function

Next, we claim that for all $(\alpha, \beta), \left(\alpha', \beta'\right) \in \mathbb{Q}^2_{++}$, if $\alpha + \beta = \alpha' + \beta'$, and $h(\alpha, \beta) = h\left(\alpha', \beta'\right)$, then for all $\left(\alpha'', \beta''\right) \in \mathbb{Q}^2_{+}$ (here, pairs of nonnegative rational numbers), $h\left(\alpha + \alpha'', \beta + \beta''\right) = h\left(\alpha' + \alpha'', \beta' + \beta''\right)$. We will call this property **translation invariance of equivalence classes**. To see this, again note that we may choose n large so that $\alpha = \frac{m(\alpha)}{n}$, $\beta = \frac{m(\beta)}{n}$, $\alpha' = \frac{m\left(\alpha'\right)}{n}$, $\beta' = \frac{m\left(\beta'\right)}{n}$, $\alpha'' = \frac{m\left(\alpha''\right)}{n}$, and $\beta'' = \frac{m\left(\beta''\right)}{n}$. Clearly, $m(\alpha) + m(\beta) = m(\alpha') + m\left(\beta'\right)$. Write

$$h\left(\alpha + \alpha'', \beta + \beta''\right) = f\left(x^{m(\alpha)+m\left(\alpha''\right)}, y^{m(\beta)+m\left(\beta''\right)}\right).$$

Rewriting, we obtain

$$f\left(x^{m(\alpha)+m\left(\alpha''\right)}, y^{m(\beta)+m\left(\beta''\right)}\right) = f\left(\left(x^{m(\alpha)}, y^{m(\beta)}\right), \left(x^{m\left(\alpha''\right)}, y^{m\left(\beta''\right)}\right)\right).$$

By $\{x, y\}$-gerrymandering-proofness, we obtain

$$\begin{aligned} & f\left(\left(x^{m(\alpha)}, y^{m(\beta)}\right), \left(x^{m\left(\alpha''\right)}, y^{m\left(\beta''\right)}\right)\right) \\ = \quad & f\left(f\left(x^{m(\alpha)}, y^{m(\beta)}\right)^{m(\alpha)+m(\beta)}, f\left(x^{m\left(\alpha''\right)}, y^{m\left(\beta''\right)}\right)^{m\left(\alpha''\right)+m\left(\beta''\right)}\right). \end{aligned}$$

As

$$f\left(x^{m(\alpha)}, y^{m(\beta)}\right) = h(\alpha, \beta) = h\left(\alpha', \beta'\right) = f\left(x^{m\left(\alpha'\right)}, y^{m\left(\beta'\right)}\right),$$

and as $m(\alpha) + m(\beta) = m(\alpha') + m\left(\beta'\right)$, conclude

$$\begin{aligned} & f\left(f\left(x^{m(\alpha)}, y^{m(\beta)}\right)^{m(\alpha)+m(\beta)}, f\left(x^{m\left(\alpha''\right)}, y^{m\left(\beta''\right)}\right)^{m\left(\alpha''\right)+m\left(\beta''\right)}\right) \\ = \quad & f\left(f\left(x^{m\left(\alpha'\right)}, y^{m\left(\beta'\right)}\right)^{m\left(\alpha'\right)+m\left(\beta'\right)}, f\left(x^{m\left(\alpha''\right)}, y^{m\left(\beta''\right)}\right)^{m\left(\alpha''\right)+m\left(\beta''\right)}\right). \end{aligned}$$

By $\{x, y\}$-gerrymandering-proofness,

$$\begin{aligned} & f\left(f\left(x^{m\left(\alpha'\right)}, y^{m\left(\beta'\right)}\right)^{m\left(\alpha'\right)+m\left(\beta'\right)}, f\left(x^{m\left(\alpha''\right)}, y^{m\left(\beta''\right)}\right)^{m\left(\alpha''\right)+m\left(\beta''\right)}\right) \\ = \quad & f\left(\left(x^{m\left(\alpha'\right)}, y^{m\left(\beta'\right)}\right), \left(x^{m\left(\alpha''\right)}, y^{m\left(\beta''\right)}\right)\right). \end{aligned}$$

Rewriting obtains

$$f\left(\left(x^{m(\alpha')},y^{m(\beta')}\right),\left(x^{m(\alpha'')},y^{m(\beta'')}\right)\right)=f\left(x^{m(\alpha')+m(\alpha'')},y^{m(\beta')+m(\beta'')}\right).$$

This is in turn equal to $h\left(\alpha'+\alpha'',\beta'+\beta''\right)$. Hence $h\left(\alpha+\alpha'',\beta+\beta''\right) = h\left(\alpha'+\alpha'',\beta'+\beta''\right)$.

Step 4: Using the auxiliary function to establish the theorem[3]

Next, to show that either i) or ii) in the statement of the theorem must be true, suppose that they are both false. They are clearly mutually exclusive statements. As ii) is false, there exist $\alpha,\beta\in\mathbb{Q}\cap(0,1)$ and $x\in X$ such that $\alpha<\beta$, and $h(\alpha,1-\alpha)=h(\beta,1-\beta)$. By the preceding arguments, for all $\gamma\in\mathbb{Q}\cap(\alpha,\beta)$, $h(\gamma,1-\gamma)=h(\alpha,1-\alpha)$. To see this, note that as $\alpha<\gamma<\beta$, there exists $\theta\in\mathbb{Q}\cap(0,1)$ such that $(\gamma,1-\gamma)=\theta(\alpha,1-\alpha)+(1-\theta)(\beta,1-\beta)$. We know that $h(\theta(\alpha,1-\alpha))=h(\alpha,1-\alpha)$ and $h((1-\theta)(\beta,1-\beta))=h(\beta,1-\beta)=h(\alpha,1-\alpha)$ by rational homogeneity of h, consequently,

$$\begin{aligned} h(\gamma,1-\gamma) &= h(\theta(\alpha,1-\alpha)+(1-\theta)(\beta,1-\beta)) \\ &= h(\alpha,1-\alpha), \end{aligned}$$

by additivity.

This establishes that the set of $\alpha\in\mathbb{Q}\cap(0,1)$ for which $h(\alpha,1-\alpha)=x$ is a non-degenerate interval. As i) is false, h is nonconstant, so there exists some $\gamma<1$ such that $h(\gamma,1-\gamma)\neq x$. Without loss of generality, let us suppose that for all α for which $h(\alpha,1-\alpha)=x$, $\gamma>\alpha$. Define $\gamma^*\equiv\sup\{(\alpha,1-\alpha):h(\alpha,1-\alpha)=x\}$, then $0<\gamma^*<1$. Define $\alpha^*=\inf\{(\alpha,1-\alpha):h(\alpha,1-\alpha)\}$. Clearly $\alpha^*<\gamma^*$.

Let $\beta^*\in(\alpha^*,\gamma^*)$. Let $\varepsilon\in\mathbb{Q}$ be small and positive. By rational homogeneity, $h(\beta^*-\varepsilon,1-\beta^*)=h\left(\frac{\beta^*-\varepsilon}{1-\varepsilon},\frac{1-\beta^*}{1-\varepsilon}\right)$. Let $\eta\in\mathbb{Q}$ be small and positive. By definition of γ^*, $h(\gamma^*+\eta,1-(\gamma^*+\eta))\neq x$. By rational homogeneity, $h(\gamma^*+\eta-\varepsilon,1-(\gamma^*+\eta))=h\left(\frac{\gamma^*+\eta-\varepsilon}{1-\varepsilon},\frac{1-(\gamma^*+\eta)}{1-\varepsilon}\right)$. By choosing ε and η appropriately, we guarantee that $\frac{\beta^*-\varepsilon}{1-\varepsilon},\frac{\gamma^*+\eta-\varepsilon}{1-\varepsilon}\in(\alpha^*,\gamma^*)$.[4] Consequently

$$\begin{aligned} h(\beta^*-\varepsilon,1-\beta^*) &= x \\ h(\gamma^*+\eta-\varepsilon,1-(\eta^*+\eta)) &= x. \end{aligned}$$

By translation invariance of equivalence classes, it follows that

$$h((\beta^*-\varepsilon,1-\beta^*)+(\varepsilon,0))=h((\gamma^*+\eta-\varepsilon,1-(\gamma^*+\eta))+(\varepsilon,0)).$$

[3] Our original proof of Step 4 was significantly more complicated. We thank the associate editor for suggesting the following proof.

[4] One can ensure that $\frac{\beta^*-\varepsilon}{1-\varepsilon}\in(\alpha^*,\gamma^*)$ by choosing $\varepsilon\in\left(\frac{\beta^*-\gamma^*}{1-\gamma^*},\frac{\beta^*-\alpha^*}{1-\alpha^*}\right)$ (a non-empty interval when $\beta^*<1$). To ensure $\frac{\gamma^*+\eta-\varepsilon}{1-\varepsilon}$, one then needs to choose $\eta\in((\alpha^*-\gamma^*)+(1-\alpha^*)\varepsilon,(1-\alpha^*)\varepsilon)$; this can also be done as $\alpha^*<\gamma^*$.

However, $h(\beta^* - \varepsilon, 1 - \beta^*) = x$ and $h(\gamma^* + \eta, 1 - (\gamma^* + \eta)) \neq x$, a contradiction. ■

Theorem 1 states that if there are only a finite number of outcomes for binary environments, then f must be constant on all $\{x, y\}$-profiles. The immediate implication of this result is that to eliminate the possibility of gerrymandering, either a pathological rule must be used, or *a countably infinite set of outcomes must be admitted*.

In systems of proportional representation, the social outcome of a vote profile is *not* an alternative for which agents may vote, but a *composition of a governing body*. There are a countable number of possible compositions of governing bodies.

There are other ways of introducing outcomes into the model. Allowing district and social outcomes of a vote profile to be lotteries over the two alternatives, which are resolved after the second stage of voting, results in a continuum. Other types of mixing are possible. For example, if voters vote over how much out of two levels to spend on a public project, social and district outcomes may be any amount of money in between.

4 Infinite sets of outcomes

4.1 The quasi-arithmetic means and quasi-proportional representation

Building on the preceding section, we here explore the implications of allowing the set of outcomes to be infinite. Our aim is to discuss a notion of proportional representation.

In arbitrary infinite sets, many bizarre rules can be constructed satisfying the axioms of Theorem 1; even with the restrictive X-gerrymandering-proofness. A general characterization of the family seems out of reach at this time. However, we will be content in this section to study the special case of the unit interval (and later on any finite-dimensional simplex).

We consider a binary social choice model. There are two alternatives over which individual voters in society may vote in the first stage, say $A = \{0, 1\}$. The outcomes are given by $X = [0, 1]$, representing the proportion of a governing body representing each of these two alternatives.

Our goal is to understand democratic and gerrymandering-proof rules. In our model, proportional representation is defined as the following rule: for all $N \in \mathcal{N}$ and all $x \in [0,1]^N$, $f(x) = \sum_{i \in N} \frac{x_i}{|N|}$. Thus, f is the arithmetic mean of those alternatives which receive votes.

Other notions of mean have been defined in the mathematics literature. They are called the quasi-arithmetic means. Thus, let $g : [0, 1] \to \mathbb{R}$ be a continuous, strictly increasing function. The quasi-arithmetic mean (with respect to g) is defined as follows: for all $N \in \mathcal{N}$, for all $x \in [0,1]^N$, $f(x) = g^{-1}\left(\frac{\sum_{i \in N} g(x_i)}{|N|}\right)$. The quasi-arithmetic means satisfy all of the axioms of Theorem 1 (they not only

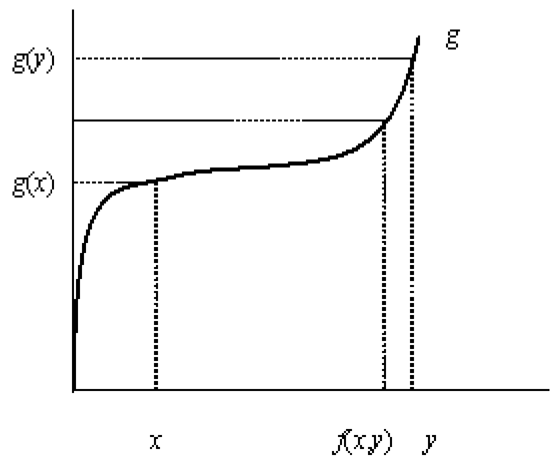

Figure 1: A quasi-proportional rule

satisfy $\{0,1\}$-gerrymandering proofness, they also satisfy $[0,1]$-gerrymandering-proofness). The function g continuously transforms outcome space into $\mathbb{R}$. Votes for outcomes are transformed by g. The average of these transformed votes is taken; then this average is mapped back to the original outcome space via the inverse of g. Such means define a society-specific notion of average. In this context, we will call a rule which is generated by a quasi-arithmetic mean a **quasi-proportional rule**.

Figure 1 depicts a typical quasi-proportional rule. There is a continuous and strictly increasing function g. Fix any two points $x, y \in [0,1]$. We compute $f(x,y)$ as follows. First, find each of $g(x)$ and $g(y)$. Then, take the average of these two points. Finally, $f(x,y)$ is found as the inverse (under g) of this average.

All quasi-proportional rules satisfy the axioms of Theorem 1. Other rules exist which satisfy the axioms of Theorem 1 as well. For example, the "positional dictatorship," defined as $f(x) = \min_{i \in N} \{x_i\}$ for all $N \in \mathcal{N}$ and all $x \in [0,1]^N$ or the "product rule," defined as $f(x) = \prod_{i \in N} x_i$ for all $N \in \mathcal{N}$ and all $x \in [0,1]^N$ both satisfy them. In fact, the positional dictatorship is a natural generalization of a unanimity-type rule with 0 as a status quo.

The preceding rules all have several characteristics in common. Firstly, they are continuous in all parameters. Importantly, they are also "monotonic," in the sense that if all votes (over outcomes) move weakly to the right, then so does the recommended social outcome.

Continuity: For all $N \in \mathcal{N}$, the rule f is continuous over X^N.

Monotonicity: For all $N \in \mathcal{N}$, for all $x, y \in X^N$, if for all $i \in N$, $x_i \leq y_i$, $f(x) \leq f(y)$.

A stronger version of monotonicity is also useful.

Strict monotonicity: For all $N \in \mathcal{N}$, for all $x, y \in X^N$, if for all $i \in N$, $x_i \leq y_i$ and $x \neq y$, $f(x) < f(y)$.

The following well-known theorem characterizes all strictly monotonic and continuous rules satisfying the democratic and gerrymandering-proofness properties. Versions of Theorem 2 were first proved by Kolmogorov [14], Nagumo [18], and de Finetti [8]. We will not give a proof of this well-known result. A careful inspection of the proof shows that only $\{0,1\}$-gerrymandering-proofness is required to establish the characterization. The monotonicity axiom has been weakened in a pair of papers by Fodor and Marichal (see [11], Theorem 6-8 and the erratum [12]).

Theorem 2: Suppose that $X = [0,1]$. A rule satisfies unanimity, anonymity, $\{0,1\}$-gerrymandering-proofness, continuity, and strict monotonicity if and only if it is a quasi-proportional rule.

While this theorem posits axioms similar to those of Theorem 1, it relies critically on monotonicity and continuity. These are meaningful only because X is linearly ordered. Theorem 1 discards all order and topological structure on X. In this sense, Theorem 1 results in a less tractable yet more general result. In particular quasi-proportional rules fall into case *ii*) in Theorem 1. In fact, strict monotonicity is easily seen to imply case *ii*) of Theorem 1 under the remaining axioms of Theorem 2. Thus, let $N \in \mathcal{N}$ and let $x, y \in \{0,1\}^N$ where $\frac{|\{i \in N : x_i = 1\}|}{|N|} < \frac{|\{i \in N : y_i = 1\}|}{|N|}$. By anonymity, we may without loss of generality assume that $x_i \leq y_i$ for all $i \in N$, and that there exists $i \in N$ for which $x_i < y_i$, so that $f(x) < f(y)$. Part *ii*) of Theorem 1 then follows from $\{0,1\}$-replication invariance.

4.2 Using lotteries to approximate single-member district systems

If $A = X = \{0,1\}$, majority rule with single-member districts is not A-gerrymandering-proof. We introduce a gerrymandering-proof method by which a society can come as close as they like to a majority rule system with single-member districts. Let us suppose that $A = \{0,1\}$ and that $X = [0,1]$. The set X is now interpreted as the set of lotteries over the alternatives. Again, the interpretation is that all uncertainty is resolved after the outcome is realized (the outcome here is a lottery).

Let $N \in \mathcal{N}$ and let $x \in A^N$. For such a profile, majority rule is the rule for which $f^{maj}(x) = 1_{\left\{x: \frac{\sum_{i \in N} x_i}{|N|} \geq .5\right\}}$.[5] We claim that there exists a sequence $\{f^m\}_{m=1}^{\infty}$ of quasi-proportional rules such that for all $N \in \mathcal{N}$ and all $x \in A^N$, $f^m(x) \to f^{maj}(x)$.

The result is about pointwise approximation. How closely the rule approximates majority rule depends on the specific vote profile in question. If we assume an ex-ante upper bound on the cardinality of the set of agents, we obtain a uniform approximation result. Given is a natural number n and $\varepsilon > 0$. There exists a quasi-proportional rule for which for all societies of size less than n, for all vote profiles of alternatives, the quasi-proportional rule recommends an outcome within ε of what majority rule with single-member districts would recommend. Formally, given $n > 0$ and $\varepsilon > 0$, there exists $M > 0$ so that for all $m \geq M$, all $N \in \mathcal{N}$ for which $|N| \leq n$, for all $x \in A^N$, $\left\| f^m(x) - f^{maj}(x) \right\| < \varepsilon$.

The result is not specific to majority rule. We show how to prove it for the **quota rules**. A quota rule is a rule for which a fixed proportion of agents q is required in order for society to select the outcome 1, otherwise the outcome 0 is selected.

The quota rules are parametrized by a value in $(0,1)$.[6] Thus, let $q \in (0,1)$. Define the quota rule f^q as follows. For all $N \in \mathcal{N}$, and for all $x \in [0,1]^N$,

$$f^q(x) \equiv \left\{ \begin{array}{c} 1 \text{ if } \frac{\sum_{i \in N} x_i}{|N|} \geq q \\ 0 \text{ if } \frac{\sum_{i \in N} x_i}{|N|} < q \end{array} \right\}.$$

Naturally, we could replace the weak inequality with strict and vice-versa.

Theorem 3 (Quota Rule Approximation): Suppose that $A = \{0,1\}$, and let $X = [0,1]$. Let f^q be a quota rule. Then there exists a sequence $\{f^m\}_{m=1}^{\infty}$ of quasi-proportional rules such that for all $N \in \mathcal{N}$ and all $x \in A^N$, $f^m(x) \to f^q(x)$. Moreover, for all $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists some quasi-proportional rule f such that for all $N \in \mathcal{N}$ for which $|N| \leq n$ and for all $x \in A^N$, $\|f(x) - f^q(x)\| < \varepsilon$.

Proof: Step 1: Establishing the pointwise convergence result

Let f^q be a quota rule. For all $m \in \mathbb{N}$, such that $1/m < q$ and $m > 2$, define the piecewise linear (in three pieces) function $h^m : [0,1] \to [0,1]$ by

$$h^m(x) \equiv \left\{ \begin{array}{c} (mq-1)\,x \text{ for } 0 \leq x < \frac{1}{m} \\ \frac{x}{m-2} + q - \left(\frac{m-1}{m}\right)\left(\frac{1}{m-2}\right) \text{ for } \frac{1}{m} \leq x \leq 1 - \frac{1}{m} \\ m(1-q)\,x + 1 - m(1-q) \text{ for } 1 - \frac{1}{m} < x \leq 1 \end{array} \right\}.$$

[5] The function 1 is the "indicator function," taking a value of 1 on the set and 0 otherwise. Note that this specification breaks ties in favor of alternative 1; this has no effect on the results.

[6] We could actually allow $q = 0, 1$, but these correspond to unanimity rules. We know unanimity rules are already $\{0,1\}$-gerrymandering-proof. Moreover, the min and max rules for the extended model coincide with the unanimity rules. These rules are also $\{0,1\}$-gerrymandering-proof. Therefore, there is no need to discuss approximation in this case.

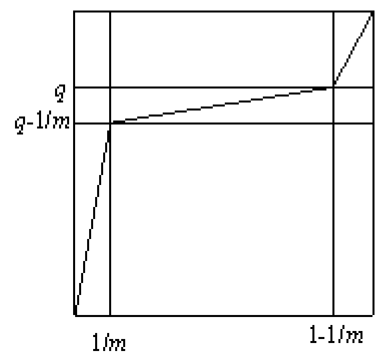

Figure 2: The function h^m in the proof of Theorem 3

Each h^m is continuous and strictly monotonic. Let f^m be the quasi-proportional rule defined with the function h^m. We claim that the first statement in the claim of Theorem 3 holds with respect to the sequence $\{f^m\}_{m=1}^{\infty}$. To see this, let $N \in \mathcal{N}$ and $x \in A^N$, and suppose that $\frac{\sum_{i \in N} x_i}{|N|} \geq q$. In particular, then, as x consists solely of zeroes and ones, and as $h^m(0) = 0$ and $h^m(1) = 1$, we conclude that $\frac{\sum_{i \in N} h^m(x_i)}{|N|} \geq q$. Moreover, $(h^m)^{-1}(q) = 1 - \frac{1}{m}$, and as h^m is monotonic, so is its inverse, hence $(h^m)^{-1}\left(\frac{\sum_{i \in N} h^m(x_i)}{|N|}\right) \geq 1 - \frac{1}{m}$, so that $(h^m)^{-1}\left(\frac{\sum_{i \in N} h^m(x_i)}{|N|}\right) \rightarrow 1$.

Suppose next that $\frac{\sum_{i \in N} x_i}{|N|} < q$. In particular, then, as x consists solely of zeroes and ones, and as $h^m(0) = 0$ and $h^m(1) = 1$, we conclude that $\frac{\sum_{i \in N} h^m(x_i)}{|N|} < q - \eta$ for some $\eta > 0$ and all m. Thus, there exists an M large enough so that for all $m > M$, $\frac{\sum_{i \in N} h^m(x_i)}{|N|} < q - \frac{1}{m}$. But $(h^m)^{-1}\left(q - \frac{1}{m}\right) = \frac{1}{m}$. Hence, by monotonicity of $(h^m)^{-1}$, $(h^m)^{-1}\left(\frac{\sum_{i \in N} h^m(x_i)}{|N|}\right) \leq (h^m)^{-1}\left(q - \frac{1}{m}\right)$. Thus, $(h^m)^{-1}\left(\frac{\sum_{i \in N} h^m(x_i)}{|N|}\right) \rightarrow 0$.

Step 2: Establishing the uniform bound result

To verify the second statement, let $n \in \mathbb{N} \cup \{0\}$ and let $\mathbb{Q}_n \equiv \left\{\frac{m}{k} : m, k \in \mathbb{N}, k \leq n\right\}$. The set $\mathbb{Q}_n$ is finite. The result therefore follows from the pointwise convergence result. ■

Figure 2 depicts the function h^m as described in the proof of Theorem 3. The pointwise approximation result holds only on $\bigcup_{N \in \mathcal{N}} A^N$ and not $\bigcup_{N \in \mathcal{N}} X^N$. For example, consider a society consisting of one agent, $N = \{1\}$, and consider

$(3/4) \in X^N$. Then $f^{1/2}((3/4)) = 1$, yet for every quasi-proportional rule f, $f((3/4)) = 3/4$.

The quota rule approximation theorem should *not* be interpreted as the statement that quota rules are "approximately" gerrymandering-proof. Rather, the way to read it is that for any quota rule, there exists a gerrymandering-proof rule which pointwise approximates it. The implications of Theorem 3 are that one can design institutions which are not susceptible to gerrymandering, and for which each district "almost" gets its own representative.

4.3 On quasi-proportional representation for non-binary environments

In this section, we consider some finite set Y. The set of outcomes $X = \Delta(Y) \equiv \left\{p \in \mathbb{R}^Y_+ : \sum_{x \in Y} p(x) = 1\right\}$ is the set of lotteries over Y. As this section postulates no notion of gerrymandering-proofness, we do not need to introduce a set of alternatives A to state any results. However, A may be understood as being identified with the set $\{\delta_x : x \in Y\}$, where δ_x is the degenerate probability measure placing probability one on x.

All of the axioms of Theorem 2 are well-defined in this environment, with the exception of monotonicity, which needs to be reformulated. In the two-alternative case, monotonicity meant that if every agent increased their vote, a corresponding increase in the outcome would obtain. In the two-alternative case; however, lotteries are naturally completely ordered. Here, they are not. However, there is still a natural definition of monotonicity that we can discuss.

Generalized monotonicity For all $N \in \mathcal{N}$, all $x \in Y$ and all $p, p' \in \Delta(Y)^N$ such that for all $i \in N$, $p_i(x) \geq p'_i(x)$, $f(p)(x) \geq f(p')(x)$.

We know from results in the functional equations literature that generalized monotonicity is enough to force us to use proportional representation when coupled with unanimity and anonymity. A good reference is Ju, Miyagawa, and Sakai (Corollary 9) [13].

Theorem 4 (Ju, Miyagawa, Sakai): Let Y be a finite set, where $|Y| \geq 3$. Suppose $X = \Delta(Y)$. A rule satisfies unanimity, anonymity, and generalized monotonicity if and only if it is the proportional rule, where for all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) = \frac{\sum_{i \in N} x_i}{|N|}$.

Theorem 4 does not rely on any gerrymandering-proofness axiom or continuity. In environments with three or more alternatives, generalized monotonicity implies a type of additive separability which already forces us to use the proportional rule. Needless to say, the proportional rule is X-gerrymandering proof. This suggests that generalized monotonicity is quite strong, and that a search for other, weaker, notions of monotonicity would be fruitful.

5 Conclusion

In this work, we show that admitting a countably infinite number of outcomes may allow us to construct gerrymandering-proof rules. Such constructs allow us to bypass the impossibility result of Chambers [6]. In the case of binary voting environments, one can construct rules which approximate any quota rule (to an arbitrarily high degree) and are gerrymandering-proof.

We also study this issue in non-binary environments. We show that in such environments, the proportional rule is essentially the only natural (*i.e.* monotonic) rule available. Thus, the possibilities in this environment are much more restricted.

The literature on stochastic social choice, initiated by Arrow's [2] Possibility Theorem, contains ideas bearing resemblance to those presented here.[7] Arrow's theorem states that a preference aggregation rule satisfying several intuitive properties does not exist. One traditional way of "resolving" this non-existence result is to consider a larger domain of outcomes (in most cases, the set of lotteries over preferences) while maintaining that individuals only have preferences over degenerate alternatives.[8] The "random dictator rule," whereby social preference is a uniform lottery over individual preferences, is usually understood to satisfy the stochastic counterparts of the Arrow axioms. While lotteries are considered in both papers, the results in this paper (see for example Theorem 4) are concerned with the expectations of these lotteries, whereas stochastic choice is concerned with the lotteries themselves. Moreover the sets of axioms investigated in these works apply to completely different domains. There is also no logical intersection between them (even the Pareto property of these papers is conceptually stronger than the unanimity property discussed here). More importantly, from a conceptual standpoint, in Arrow's framework, *even if the set of degenerate alternatives is infinite*, an impossibility result obtains. This is not the case in our work.

Thus, while there is no formal relation between the work of these papers and the results presented here, both sets of results underscore the importance of domain and range restrictions in formal social choice models. More generally, from a mathematical perspective, properties of the domain and range are critical in the theory of functional equations (see Aczél [1]).

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