A SPATIAL ANALOGUE OF MAY'S THEOREM

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Abstract. In a spatial model with Euclidean preferences, we establish that the *geometric median* satisfies Maskin monotonicity, anonymity, and neutrality. For three agents, it is the unique such rule.
Keywords: Euclidean preferences, Maskin monotonicity, Nash implementation, geometric median

1. Introduction

An early social-choice theoretic foundation for majority rule was provided by May (1952). In an environment with a group of agents who choose one of the alternatives based on strict preferences, he shows that majority rule is the unique rule satisfying three natural axioms. The first of these axioms, anonymity, requires that the names of the agents do not matter. The second, neutrality, requires that the names of the alternatives do not matter. The third, positive responsiveness, means that in any given profile, if the collection of agents who prefer the chosen alternative increases with respect to set inclusion, then that alternative should remain chosen.

We work in an environment of Euclidean preferences; that is, policy space is given by Euclidean space, and each agent has a "favorite" alternative. The further a policy is from this favored alternative (the ideal or bliss point), the worse it is for the agent. Our aim is to provide a notion of majority rule in such a context analogous to May's. That is, we investigate this environment axiomatically. Early attempts to provide a notion of majority rule in spatial environments (such as Plott (1967)) were guided by generalizing the classic results of Hotelling (1929) and Black (1948) to spatial environments. That is, they took the notion of majority rule core as *given* and tried to compute the corresponding equilibrium concept. In fact, this equilibrium, termed the *median in all directions*, very seldom exists.

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In an earlier paper (Brady and Chambers (2015)), we have shown that a concept called the *geometric median* emerges as the unique social choice rule satisfying a host of compelling properties. In this work, we show that it satisfies the natural counterparts of anonymity, neutrality, and positive responsiveness in a spatial framework. We also establish that, for three agents, it is the unique such rule.[1] A related paper is Duggan (2015), who also provides a spatial generalization of May's Theorem. His work can be distinguished from ours on several grounds: first, he works with more general single-peaked preferences; but he does so only in one-dimensional environments. Second, instead of working with a social choice rule, he works with a rule which aggregates preferences into a single social preference. Third, instead of a positive responsiveness condition, he utilizes different transitivity conditions.

Generalizing anonymity to spatial environments is straightforward. Describing the content of the remaining two axioms (neutrality and positive responsiveness) takes some work. With regard to neutrality, we cannot "rename" alternatives arbitrarily. If we were to do so, the resulting preference profile may not be Euclidean. To this end, we interpret neutrality as an equivariance of the social rule to isometries: it does not matter which coordinate system we use to describe ideal points. This axiom formalizes the idea that the names of alternatives should not matter, and is the counterpart of the classical neutrality axiom in our setting.

Positive responsiveness also has a natural generalization to this environment. Maskin Monotonicity (Maskin (1999)) states that, if the chosen alternative moves up in the ranking of all agents, it remains chosen. This is entirely analogous to May's criterion that if the chosen alternative moves up in everybody's ranking, it remains chosen. Thus, Maskin Monotonicity seems an appropriate generalization of May's positive responsiveness condition to spatial environments.

The geometric median (or medians in the case of an even number of agent) is any point which minimizes the aggregate Euclidean distance to the agents' ideal points. We show that, in general, the geometric median satisfies these three properties. Moreover, in the case of three agents, it is the *unique* such rule. More generally, we do not know whether the geometric median is the unique rule satisfying the properties, but it seems plausible that with a high enough dimension of the underlying Euclidean space, it will be.

[1]Other works using the geometric median in economics or political science research include Cervone, Dai, Gnoutcheff, Lanterman, Mackenzie, Morse, Srivastava, and Zwicker (2012), Baranchuk and Dybvig (2009), and Chung and Duggan (2014). In particular, the latter work describes an interesting generalization of the concept to general convex preferences.

The paper is organized as follows. Section 2 presents the model. Section 3 provides the results. Section 4 concludes.

2. The Model

Let $X = \mathbb{R}^d$ be the policy space. For any $x, y \in X$ let $\|x - y\|$ denote the Euclidean distance. Let $N = \{1, \ldots, n\}$ be a finite set of agents. Each agent $i \in N$ is equipped with a preference relation $\succsim_i$ (with associated strict preference $\succ_i$). Preferences are assumed to be Euclidean so that for each $i \in N$, $\succsim_i$ can be represented by an "ideal point", $z_i \in X$, with the property that for any $x, y \in X$, $x \succsim_i y$ if and only if $\|x - z_i\| \leq \|y - z_i\|$. We define an *isometry* as any distance preserving mapping $f : X \to X$ so that for all $x, y \in X$ we have $\|x - y\| = \|f(x) - f(y)\|$.[2]

An aggregation or *social choice rule* is a mapping $\varphi : X^N \to X$. Since there is a one-to-one relationship between preference profiles and a set of points in X, we will use the notation $Z \in X^N$ to indicate a preference profile of the agents that is represented by the points $Z = (z_i)_{i \in N}$ where $z_i \in X$ for each i.

For $i \in N$ and $z_i, x \in X$, let $UC_i(z_i, x) = \{y \in X \mid \|y - z_i\| \leq \|x - z_i\|\}$ be the upper contour set for the preference relation represented by the point z_i at the point x. This is simply the set of all outcomes agent i weakly prefers to x. We will say that the preference relation represented by a point z_i' is a monotonic transformation of the preference relation represented by z_i at a point x if $UC_i(z_i', x) \subseteq UC_i(z_i, x)$. Let $MT(z_i, x)$ be the set of all monotonic transformations of the preference relation represented by the point z_i at the point x and $MT(Z, x)$ be the set of all monotonic transformations of a preference profile represented by the set of points Z at a point x.

For $x, y \in X$ we will denote the line segment with x and y as endpoints by $\overline{xy} = \{t \in X \mid \|x - t\| + \|t - y\| = \|x - y\|\}$.

We say that an outcome $x \in X$ is weakly Pareto efficient (WPE) if there does not exist $y \in X$ such that $y \succ_i x$ for all $i \in N$. The outcome is Pareto efficient (PE) if there does not exist $y \in X$ such that $y \succsim_i x$ for all $i \in N$ and $y \succ_j x$ for at least one $j \in N$. We will say that a social choice rule φ satisfies *Pareto efficiency* if for any $Z \in X^N$, $\varphi(Z)$ is PE.

For an agent i with ideal point $z_i \in X$ we will often use the utility representation $u_i(y) = -(y - z_i) \cdot (y - z_i)$ for $y \in X$ to represent i's preference relation. For a profile of preferences $Z \in X^N$ and utility representations $(u_i(y))_{i \in N}$, consider the sets

$$comp(Z) = \{a = (a_1, \ldots, a_n) \mid \exists y \in X \text{ such that } a_i \leq u_i(y) \text{ for all } i \in N\}$$

[2]Since $X = \mathbb{R}^d$, isometries are bijections that correspond to reflections, rotations, and translations.

and

$$con(Z) = \left\{ x \in X \mid x = \sum_{i=1}^{n} \lambda_i z_i \text{ such that } \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

The first set is said to be the comprehensive hull of the utility possibility set, where by comprehensive we mean that for any $a \in comp(Z)$ and $b \leq a$ coordinate-wise, then $b \in comp(Z)$. The latter set is simply the convex hull of the set of ideal points. For a set $S \subset \mathbb{R}^d$, we let $bd(S)$ denote its boundary and $int(S)$ denote its interior.

For a set of points $(a_1, \ldots, a_n)$ with each $a_i \in X$, we define a geometric median as a solution to the following minimization problem:

$$\min_{x \in X} \sum_{i=1}^{n} \|x - a_i\|. \tag{1}$$

That is, the geometric median minimizes the sum of distances between itself and all of the points.

The geometric median for a finite set of points always exists; further, it is unique if n is odd or if the points $(a_1, \ldots, a_n)$ are not collinear (Haldane (1948)). In the case of collinear points, there could be multiple geometric medians, in particular when the number of distinct points is even. In this case, the geometric median is a set-valued concept; any selection from this set would result in a single-valued rule.

For simplicity, we assume throughout that n is odd (to ensure uniqueness). In the conclusion, we discuss how our results should be modified in the case of an even number of agents. For a preference profile $Z \in X^N$, we will let x_Z^* denote the geometric median of the ideal points.

It can be shown that (1) has a convenient, simplified dual characterization (Güler (2010)). To see this, first note that for any $x \in X$ we have $\|x\| = \max_{\|y\| \leq 1} \langle x, y \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product. This allows us to rewrite (1) as the minimax problem

$$\min_{x \in X} \max_{\|y_i\| \leq 1} \sum_{i=1}^{n} \langle a_i - x, y_i \rangle. \tag{2}$$

The dual to (1) can then be regarded as the maximin problem corresponding to (2). After some simplification, the dual problem becomes

$$\max \left\{ \sum_{i=1}^{n} \langle a_i, y_i \rangle \,\middle|\, \|y_i\| \leq 1, 1 \leq i \leq n, \sum_{i=1}^{n} y_i = 0 \right\}. \tag{3}$$

These duality results give us a simple way of checking whether or not a point is a geometric median. For a candidate point, x, first consider all i for which $x \neq a_i$. For these points, y_i is the unit vector in the direction $a_i - x$. For i with $x = a_i$, y_i can be any point inside the unit ball. If $\sum_{i=1}^{n} y_i = 0$, then x is a geometric median.

We now briefly describe the axioms we will impose on a social choice rule.

Axiom 2.1. A social choice rule φ satisfies *anonymity* if for every bijection $\sigma : N \to N$ and for every $Z \in X^N$ we have $\varphi(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) = \varphi(Z)$.

In words, anonymity states that the outcome from our social choice rule is invariant to changing the names of the agents.

Axiom 2.2. A social choice rule φ satisfies *neutrality* if for any isometry f and any $Z \in X^N$ we have $\varphi(f(z_1), \ldots, f(z_n)) = f(\varphi(z_1, \ldots, z_n))$.

Neutrality states that the social choice resulting from any reflection, rotation, or translation of agents' ideal points is the same as applying the reflection, rotation, or translation to the social choice from the untransformed ideal points.

The final axiom says that the social choice is preserved through monotonic transformations.

Axiom 2.3. A social choice rule φ satisfies *Maskin monotonicity* or *positive responsiveness* if for all $Z \in X^N$ and for any $Z' \in MT(Z, \varphi(Z))$ we have $\varphi(Z') = \varphi(Z)$.

3. Results

In this section, we propose a class of social choice rules that satisfy the axioms discussed in the previous section. We also show that Pareto Efficiency of a social choice rule is a consequence of imposing two of our axioms. Further, Nash-implementation is briefly discussed as we show that our proposed class meets conditions sufficient for such implementation. Finally, we show that this class of social choice rules is the only class satisfying our axioms for the case of $n = 3$.

Let us assume that n is odd. Consider the social choice rule φ such that for any $Z \in X^N$, $\varphi(Z) = x^*_Z$. That is, given any preference profile, the social choice is always the point in X that minimizes the total distance between itself and the agents' ideal points. This choice has a nice appeal in many settings. For example, the choice could be over the location of a supply distribution hub given the location of n factories. Then this rule selects the location that minimizes the total transportation cost between the hub and factories. Or, in a political science context, the choice for a socially optimal candidate would be the location that is minimally far from the policy relevant locations of the set of n voters.

Our first result shows that this rule satisfies our axioms.

Proposition 3.1. *Let φ be a social choice rule such that for any $Z \in X^N$, $\varphi(Z) = x^*_Z$. Then φ satisfies anonymity, neutrality, and Maskin monotonicity.*

Before presenting the proof, we first provide two lemmas, whose proofs can be found in Brady and Chambers (2015). The first characterizes monotonic transformations and the second establishes that the geometric median satisfies Maskin monotonicity.

Lemma 3.1. *Suppose $z_i, z'_i \in X$ represent two preference relations. For $x \in X$, $z'_i \in MT(z_i, x)$ if and only if $z'_i \in \overline{z_i x}$.*

The following result was originally established by Gini and Galvani (1929).

Lemma 3.2. *For $Z \in X^N$, if $Z' \in MT(Z, x^*_Z)$, then $x^*_{Z'} = x^*_Z$.*

We now prove Proposition 3.1.

Proof. It is trivial that φ satisfies anonymity.

Let f be any isometry. By uniqueness of the geometric median it follows that

$$\sum_{i=1}^{n} \|x^*_Z - z_i\| < \sum_{i=1}^{n} \|x - z_i\|$$

for any $x \in X \backslash \{x^*_Z\}$. Since f is an isometry we have

$$\sum_{i=1}^{n} \|x^*_Z - z_i\| = \sum_{i=1}^{n} \|f(x^*_Z) - f(z_i)\| < \sum_{i=1}^{n} \|f(x) - f(z_i)\|$$

for any $x \in X \backslash \{x^*_Z\}$. Since f is bijective it follows that $f(x^*_Z)$ is the geometric median for the set of points $(f(z_1), \ldots, f(z_n))$ and thus $f(\varphi(Z)) = f(x^*_Z) = \varphi(f(z_1), \ldots, f(z_n))$.

Finally, φ satisfying Maskin monotonicity immediately follows from Lemma 3.2. □

We did not impose any efficiency assumptions in our axiomatization. As we show next, PE of a social choice rule immediately follows from satisfying neutrality and Maskin monotonicity. Thus, the social choice rule such that $\varphi(Z) = x^*_Z$ satisfies PE by Proposition 3.1.

Proposition 3.2. *Let $Z \in X^N$ be a preference profile and $x \in X$ an outcome. Then, x is PE if and only if $x \in con(Z)$.*

Lemma 3.3. *For $Z \in X^N$, the set comp(Z) is convex.*

Proof. Take any $a, b \in comp(Z)$ so that there exists $y_a, y_b \in X$ such that $u_i(y_a) \geq a_i$ and $u_i(y_b) \geq b_i$ for all $i \in N$. It follows by convexity of Euclidean distance that for any $t \in [0,1]$ and for all $i \in N$ we have

$$u_i(ty_a + (1-t)y_b) \geq tu_i(y_a) + (1-t)u_i(y_b) \geq ta_i + (1-t)b_i.$$

Since X is convex and $y_a, y_b \in X$, it follows that $ty_a + (1-t)y_b \in X$ for any $t \in [0,1]$. Thus, $ta + (1-t)b \in comp(Z)$ by definition, and so $comp(Z)$ is a convex set. $\square$

Lemma 3.4. *For $Z \in X^N$ and an outcome $x \in X$, if x is PE then $u(x) = (u_1(x), \ldots, u_n(x)) \in bd(comp(Z))$.*

Proof. Fix $Z \in X^N$ and suppose $x \in X$ is PE but $u(x) \notin bd(comp(Z))$. Since $u(x)$ is not on the boundary, there exists $a = (a_1, \ldots, a_n) \in comp(Z)$ such that $a_i > u_i(x)$ for all i. By definition of $comp(Z)$ there exists $z \in X$ such that $u_i(z) \geq a_i > u_i(x)$ for all i, contradicting x is PE. $\square$

We now prove Proposition 3.2.

Proof. First, note that $x = \sum_{i=1}^n \lambda_i x_i$ for $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, and $x_i \in X$ if and only if x solves

$$\max_{y \in X} -\sum_{i=1}^n \lambda_i (y - x_i) \cdot (y - x_i). \tag{4}$$

Suppose $x \in X$ is PE. By Lemma 3.4, $u(x) = (u_1(x), \ldots, u_n(x)) \in bd(comp(Z))$ and by Lemma 3.3 $comp(Z)$ is convex. Thus, it follows by the supporting hyperplane theorem (see Theorem 6.8 in Güler (2010)) that there exists $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha \neq 0$ such that $\alpha \cdot u(x) \geq \alpha \cdot a$ for all $a \in comp(Z)$. Further, $\alpha_i \geq 0$ for all i. If not, e.g. if $\alpha_i < 0$, by comprehensivity we could find $\tilde{a} \in comp(Z)$ with $\tilde{a}_i < 0$ small enough such that $\alpha \cdot u(x) < \alpha \cdot \tilde{a}$. Define $\lambda_i = \frac{\alpha_i}{\sum_{j=1}^n \alpha_j}$ so that $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, and $\lambda \cdot u(x) \geq \lambda \cdot a$ for all $a \in comp(Z)$. It follows that x solves

$$\max_{y \in X} \sum_{i=1}^n \lambda_i u_i(y), \tag{5}$$

which is of the same form as (4). Thus, $x = \sum_{i=1}^n \lambda_i z_i$ and we can conclude that $x \in con(Z)$.

Now suppose $x \in con(Z)$ and so it follows that x solves (5). However, suppose that x is not PE so that there exists $x' \in X$ such that $x' \succsim_i x$ for all i with $x' \succ_j x$ for at least one j. Since x solves (5) and x' Pareto dominates

x, then it must be the case that $\lambda_j = 0$ for all j such that $x' \succ_j x$. Let $x'' = tx + (1-t)x'$ for some $t \in (0,1)$ so that $x'' \in X$ by convexity of the outcome space. By strict convexity of Euclidean preferences, $x'' \succ_i x' \succsim_i x$ for all i, contradicting x as a solution to (5). □

Proposition 3.3. *Let φ be a social choice rule that satisfies Maskin monotonicity and neutrality. Then φ satisfies Pareto efficiency.*

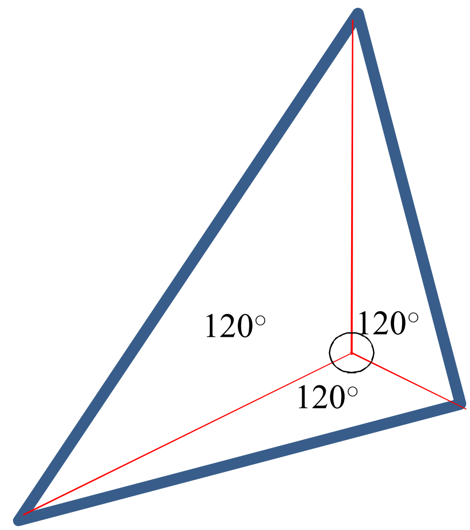
Proof. Suppose φ is a social choice rule that satisfies Maskin monotonicity and neutrality but not Pareto efficiency. Then, by Proposition 3.2, there exists $Z = (z_1, \dots, z_n) \in X^N$ such that $\varphi(Z) \notin con(Z)$. By the separating hyperplane theorem (see Theorem 6.9 in Güler (2010)), there exists a hyperplane $H_{(\alpha,c)} = \{x \in X \mid \alpha \cdot x = c\}$ for some $0 \neq \alpha \in X$ and $c \in \mathbb{R}$ that separates $con(Z)$ and $\{\varphi(Z)\}$. For each $i \in N$ let $z'_i \in X$ satisfy $z'_i \in \overline{z_i\varphi(Z)}$ and $z'_i \in H_{(\alpha,c)}$, so z'_i is the intercept with the hyperplane of the line segment connecting agent i's ideal point to the social choice. It follows by Lemma 3.1 that $Z' = (z'_1, \dots, z'_n) \in MT(Z, \varphi(Z))$ and thus $\varphi(Z') = \varphi(Z)$ by Maskin monotonicity. Let $f : X \to X$ be the reflection about the hyperplane $H_{(\alpha,c)}$ and define $z''_i = f(z'_i)$ for each $i \in N$. Since $z'_i \in H_{(\alpha,c)}$ it follows that $z''_i = f(z'_i) = z'_i$ and thus $\varphi(f(z'_1), \dots, f(z'_n)) = \varphi(Z')$. However, $\varphi(Z') \neq f(\varphi(Z'))$, contradicting neutrality □

3.1. **The three agent case.** We now show for $n = 3$ that the only social rule satisfying our axioms is the rule that selects the geometric median from the set of ideal points. Before presenting the result, we first state some facts about the geometric median for three points in $\mathbb{R}^2$ (see Deimling (2011) p. 325-326 and Coxeter (1989) p. 21-22). Let $A = (a_1, a_2, a_3)$ be the points. First, suppose that the three points are not collinear and that $bd(con(A))$ forms a triangle (denoted as $\triangle_{a_1a_2a_3}$). Suppose all interior angles of the triangle are less than 120°.[3] In this case the geometric median is the unique point in $int(con(A))$ such that the angle between any two line segments connecting the geometric median to the vertices of the triangle a_i and a_j with $i \neq j$ (denoted $\angle_{a_i x^*_A a_j}$) is 120°. See Figure 1. For the special case in which $\triangle_{a_1a_2a_3}$ is equilateral, the geometric median will coincide with the intersection of the three medians of the triangle.[4] If the triangle is isosceles, then the geometric median lies on the axis of symmetry. If $\triangle_{a_1a_2a_3}$ has an angle that is at least 120°, then the geometric median corresponds to the obtuse-angled vertex. If the points are

[3]An interior angle is simply the angle formed by two adjacent sides of the triangle

[4]A median of a triangle is any of the line segments connecting a vertex to the midpoint of the opposite side of the triangle

FIGURE 1. The geometric median



collinear then the geometric median is the point lying between the other two or where multiple points are located if the points are not distinct.

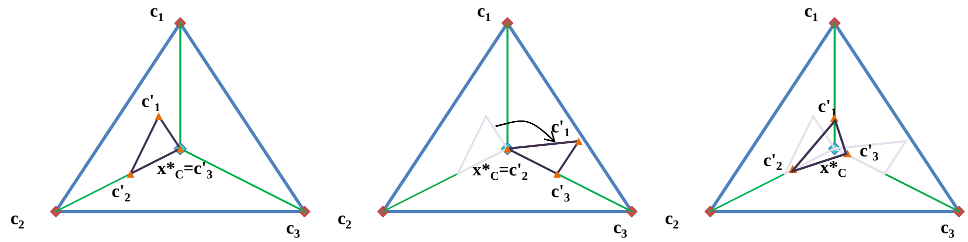
We make note of a few preliminary results that will be used in the proof. In what follows $B_{P,\varepsilon}$ will denote an ε-ball about a point $P \in X$ for some $\varepsilon > 0$.

Lemma 3.5. *Let $A' = (a'_1, a'_2, a'_3)$ be such that $\triangle_{a'_1 a'_2 a'_3}$ has all interior angles less than or equal to $120°$. Then $A' \in MT(A, x^*_A)$ for some A such that $\triangle_{a_1 a_2 a_3}$ is equilateral.*

Proof. Let $\triangle_{c_1 c_2 c_3}$ be some arbitrary equilateral triangle and x^*_C the associated geometric median. Since $x^*_C \in int(con(C))$, it follows that we can find an $\varepsilon > 0$ such that the associated $B_{x^*_C,\varepsilon} = \{c \in int(con(C)) \mid \|c - x^*_C\| < \varepsilon\} \subset int(con(C))$. Consider $\triangle_{c'_1 c'_2 c'_3} \cong \triangle_{a'_1 a'_2 a'_3}$ such that $con(C') \subset B_{x^*_C,\varepsilon}$, $c'_2 \in \overline{c_2 x^*_C}$, $c'_3 = x^*_C \in \overline{c_3 x^*_C}$, and $c'_1 \in con(\triangle_{c_1 x^*_C c_2})$.[5] Now consider a movement of $\triangle_{c'_1 c'_2 c'_3}$ such that $c'_2 = x^*_C \in \overline{c_2 x^*_C}$, $c'_3 \in \overline{c_3 x^*_C}$, and $c'_1 \in con(\triangle_{c_1 x^*_C c_3})$. Note that this movement is a continuous "shift" of c'_2 along $\overline{c_2 x^*_C}$ and c'_3 along $\overline{c_3 x^*_C}$ that sends c'_1 from $con(\triangle_{c_1 x^*_C c_2})$ to $con(\triangle_{c_1 x^*_C c_3})$. It thus follows that at some point along this movement we have $c'_i \in \overline{c_i x^*_C}$ for each i and thus $C' \in MT(C, x^*_C)$ by Lemma 3.1. See Figures 2(A)-2(C) for an illustration of this

[5] Note that $c'_1 \in con(\triangle_{c_1 x^*_C c_2})$ will follow since it is assumed $\triangle_{c'_1 c'_2 c'_3}$ has all angles less than or equal to $120°$ while $\angle_{c_1 x^*_C c_2} = 120°$

FIGURE 2. Finding $C' \in MT(C, x^*_C)$



(A) (B) (C)

procedure. Since $\triangle_{c'_1c'_2c'_3} \cong \triangle_{a'_1a'_2a'_3}$, the result follows by simply scaling both $\triangle_{c'_1c'_2c'_3}$ and $\triangle_{c_1c_2c_3}$ appropriately and then applying any needed isometries so that $\triangle_{c'_1c'_2c'_3} = \triangle_{a'_1a'_2a'_3}$ and then defining A to be the scaled and transformed C. $\square$

Lemma 3.6. *Let $A' = (a'_1, a'_2, a'_3)$ be such that $\triangle_{a'_1a'_2a'_3}$ is scalene*[6] *and has an angle greater than $120°$. Then $A' \in MT(A, x^*_A)$ for some A such that $\triangle_{a_1a_2a_3}$ is isosceles and the measure of the obtuse angles of the two triangles are the same.*

Proof. Without loss of generality let a'_1 be the obtuse-angled vertex of $\triangle_{a'_1a'_2a'_3}$. Let $\triangle_{c_1c_2c_3}$ be an arbitrary isosceles triangle such that $\angle_{c_2c_1c_3} = \angle_{a'_2a'_1a'_3}$. Note that this implies $x^*_C = c_1$. Since $\triangle_{a'_1a'_2a'_3}$ is scalene, it follows that either $\angle_{a'_1a'_2a'_3} < \angle_{a'_1a'_3a'_2}$ or vice versa. Without loss of generality, assume the former holds. Since $\triangle_{c_1c_2c_3}$ is isosceles and $\angle_{c_2c_1c_3} = \angle_{a'_2a'_1a'_3}$ it then follows that $\angle_{a'_1a'_2a'_3} < \angle_{c_1c_2c_3} = \angle_{c_1c_3c_2} < \angle_{a'_1a'_3a'_2}$. Consider $c'_3 \in \overline{c_3x^*_C} = \overline{c_3c_1}$ such that $\angle_{a'_1a'_2a'_3} = \angle_{c_1c_2c'_3}$ and $\angle_{c_1c'_3c_2} = \angle_{a'_1a'_3a'_2}$.[7] Setting $c'_2 = c_2$ and $c'_1 = c_1$ gives us $C' \in MT(C, x^*_C)$ by Lemma 3.1 and $\triangle_{c'_1c'_2c'_3} \cong \triangle_{a'_1a'_2a'_3}$. The result follows by simply scaling both $\triangle_{c'_1c'_2c'_3}$ and $\triangle_{c_1c_2c_3}$ appropriately and then applying any needed isometries so that $\triangle_{c'_1c'_2c'_3} = \triangle_{a'_1a'_2a'_3}$ and then defining A to be the scaled and transformed C. $\square$

Now suppose A consists of distinct collinear points, and without loss of generality, assume $a_3 \in \overline{a_1a_2}$. Further, assume $P \neq a_i$ for any i and that $P \in \overline{a_1a_2}$. We now show how to construct $A' \in MT(A, P)$ such that a'_3 is the midpoint of $\overline{a'_1a'_2}$, something we refer to as the collinear midpoint construction

[6] A scalene triangle has all three interior angles of different measure.

[7] Note that as $c_3 \to c_1$ we have $\angle_{c_1c_2c_3} \to 0$ so finding such a c'_3 is always possible by the Intermediate Value Theorem

in the proof. First, assume $a_3 \in \overline{Pa_2}$. If $\|P - a_3\| < \frac{1}{2}\|P - a_2\|$ then choose $a_3' = a_3$ and $a_2' \in \overline{Pa_2}$ such that $\|P - a_3'\| = \frac{1}{2}\|P - a_2'\|$. If $\|P - a_3\| \geq \frac{1}{2}\|P - a_2\|$ then choose $a_2' = a_2$ and $a_3' \in \overline{Pa_3}$ such that $\|P - a_3'\| = \frac{1}{2}\|P - a_2'\|$. Choosing $a_1' = P$ gives us $A' \in MT(A, P)$ such that a_3' is the midpoint of $\overline{a_1'a_2'}$. Now, if $a_3 \in \overline{Pa_1}$, we can repeat the same procedure but reverse the roles of a_1 and a_2 so that a_1' and a_3' satisfy $\|P - a_3'\| = \frac{1}{2}\|P - a_1'\|$ and $a_2' = P$.

We now present our main result.

Theorem 3.1. *Suppose $n = 3$ and let φ be a social choice rule. Then φ satisfies anonymity, neutrality, and Maskin monotonicity if and only if for any $Z \in X^N$, $\varphi(Z) = x_Z^*$.*

Proof. Proposition 3.1 showed that the geometric median satisfies our three axioms. We prove the converse by cases. First, note that by Proposition 3.2 and Proposition 3.3 it must be the case that $\varphi(Z) \in con(Z)$. To this end, it is without loss to assume that $d = 2$, as the smallest hyperplane containing any three points has at most two dimensions, and all monotonic transformations of those three points with respect to an element in the convex hull also lie in that hyperplane.

Case 1 Let $Z \in X^N$ be a preference profile such that $\triangle_{z_1 z_2 z_3}$ is equilateral. Since the geometric median lies at the point of intersection of the three medians for the triangle, if we can show that $\varphi(Z)$ must lie on one of these medians chosen arbitrarily, then the claim will be true for Case 1. Let y be the midpoint between z_2 and z_3 and $\overline{z_1 y}$ the corresponding median. Suppose however that $\varphi(Z) \notin \overline{z_1 y}$. Let $f : X \to X$ be a reflection in $\overline{z_1 y}$ and define $z_i' = f(z_i)$ for each $i \in N$. It follows that f simply switches the vertices z_2 and z_3 in $\triangle_{z_1 z_2 z_3}$ so that $z_2' = z_3$ and $z_3' = z_2$. Note that f is equivalent to a bijection that switches agent 2's ideal point with agent 3's and, thus, by anonymity it must be the case that $\varphi(Z') = \varphi(Z)$. However, by neutrality $\varphi(f(z_1), f(z_2), f(z_3)) = \varphi(Z') = f(\varphi(Z))$, a contradiction. Thus, $\varphi(Z) \in \overline{z_1 y}$ and we must have $\varphi(Z) = x_Z^*$ when $\triangle_{z_1 z_2 z_3}$ is equilateral.

Case 2 Suppose now $\triangle_{z_1 z_2 z_3}$ is has all interior angles less than or equal to 120°. By Lemma 3.5 there exists $\hat{Z}$ such that $Z \in MT(\hat{Z}, x_{\hat{Z}}^*)$ and $\triangle_{\hat{z}_1 \hat{z}_2 \hat{z}_3}$ is equilateral. By Case 1, it follows that $\varphi(\hat{Z}) = x_{\hat{Z}}^*$. By Lemma 3.2 it follows that $x_Z^* = x_{\hat{Z}}^*$. The result then follows by Maskin montonicity.

Case 3 Consider now $\triangle_{z_1 z_2 z_3}$ that is isosceles with an interior angle greater than 120°. Suppose, without loss of generality, that z_1 is the obtuse-angled vertex so that $x_Z^* = z_1$. By appealing to arguments similar to Case 1, it is easy to see that $\varphi(Z)$ must lie on the axis of symmetry. Suppose $\varphi(Z) \in int(con(Z))$ so it follows that $\angle_{z_2 \varphi(Z) z_3} > 120°$ and $\angle_{z_2 \varphi(Z) z_1} = \angle_{z_3 \varphi(Z) z_1}$. Note that we can find $z_2' \in \overline{z_2\varphi(Z)}$ and $z_3' \in \overline{z_3\varphi(Z)}$ such that $\angle_{z_2'z_1z_3'} \leq 120°$ and $\triangle_{z_1z_2'z_3'}$ is isosceles with an axis of symmetry through z_1. Choosing $z_1' = z_1$ gives us $Z' \in MT(Z, \varphi(Z))$ and thus $\varphi(Z') = \varphi(Z)$ by Maskin monotonicity. By Case 2, it follows that $\varphi(Z') = x^*_{Z'}$ and thus $\angle_{z_i'\varphi(Z')z_j'} = 120°$ for all $i \neq j$. Thus, it follows by Lemma 3.1 that $\angle_{z_i\varphi(Z)z_j} = 120°$ for all $i \neq j$. But no such $\varphi(Z) \in int(con(Z))$ that lies on the axis of symmetry exists since $\angle_{z_2\varphi(Z)z_3} > 120°$ by assumption. It then follows that the only choices for $\varphi(Z)$ are z_1 and the midpoint of $\overline{z_2z_3}$. Suppose then that $\varphi(Z)$ is the midpoint of $\overline{z_2z_3}$. Then, it is easy to see that we can find $Z' \in MT(Z, \varphi(Z))$ such that $\triangle_{z_1'z_2'z_3'}$ is equilateral[8] and $\varphi(Z')$ is the midpoint of $\overline{z_2'z_3'}$, which contradicts Case 1. Thus we must have $\varphi(Z) = x^*_Z = z_1$.

Case 4 Suppose now $\triangle_{z_1z_2z_3}$ is scalene with an interior angle greater than 120°. Without loss of generality, assume z_1 is the obtuse-angled vertex. By Lemma 3.6 there exists $\hat{Z}$ such that $Z \in MT(\hat{Z}, x^*_{\hat{Z}})$ and $\triangle_{\hat{z}_1\hat{z}_2\hat{z}_3}$ is isosceles with $\angle_{\hat{z}_2\hat{z}_1\hat{z}_3} = \angle_{z_2z_1z_3}$. By Case 3, it follows that $\varphi(\hat{Z}) = x^*_{\hat{Z}} = \hat{z}_1$. By Lemma 3.2 it follows that $x^*_Z = x^*_{\hat{Z}}$. The result then follows by Maskin montonicity.

Case 5 The last case to consider is when Z is a set of collinear points. If $z_1 = z_2 = z_3$ then trivially we must have $\varphi(Z) = z_1 = x^*_Z$ since that is the only choice in $con(Z)$. Suppose the points in Z are distinct and that $z_3 \in \overline{z_1z_2}$ so that $x^*_Z = z_3$. If $\varphi(Z) \neq z_3$ then we can find $Z' \in MT(Z, \varphi(Z))$ such that z_3' is the midpoint of $\overline{z_1'z_2'}$ and, without loss of generality, $z_1' = \varphi(Z') = \varphi(Z)$ by Maskin monotonicity[9]. Let $f : X \to X$ be a reflection in the line perpendicular to $\overline{z_1'z_2'}$ running through z_3' and let $z_i'' = f(z_i')$. By neutrality we must have $\varphi(Z'') = f(\varphi(Z)) = z_1''$ but by anonymity, we must have $\varphi(Z'') = z_2''$ a contradiction. Thus, we must have $\varphi(Z) = x^*_Z$ when the points are collinear and distinct. Note that if the points were not distinct e.g. $z_1 \neq z_2 = z_3$ so that $x^*_Z = z_3$ still, but $\varphi(Z) \neq z_3$, then the previous argument still goes through by choosing $z_2' = z_2$, $z_3' \in \overline{\varphi(Z)z_3}$ such that $\|\varphi(Z) - z_3'\| = \frac{1}{2}\|\varphi(Z) - z_2'\|$ and $z_1' = \varphi(Z)$. Thus, in all collinear cases we have $\varphi(Z) = x^*_Z$, which completes the proof. □

4. Conclusion

This paper has considered a natural generalization of the classical notions of May to spatial environments. It remains an open question as to whether these conditions uniquely characterize the geometric median for more than

[8]This is achieved by moving z_2 and z_3 in tandem towards $\varphi(Z)$ until each side of the triangle is equal in length

[9]This follows by the midpoint collinear construction outlined previously

three agents. In particular, multi-valued concepts would need to be admitted. In Brady and Chambers (2015), we handled this issue by studying the *smallest* rule satisfying a host of properties.

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