

# Haves and Have-Nots: A Theory of Economic Sufficientarianism

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## Abstract

In this paper, we introduce a generalization of the concept of sufficientarianism, extending it to assess allocations across multiple consumption goods. For a fixed society of agents, sufficientarianism posits that allocations are compared according to the number of individuals whose consumption is considered to be sufficient. Central to our discussion is a newly introduced ethical principle of ‘sufficientarian judgment.’ This principle asserts that if starting from an allocation in which all agents have identical consumption and a change in one agent’s consumption hurts society, then there would be no change in the consumption of any other agent that could subsequently benefit society. Sufficientarianism is shown to be equivalent to sufficientarian judgment, symmetry, and separability. We investigate our axioms in both specific economic environments and abstract environments. Finally, we argue formally that sufficientarian judgment is closely related to the leximin principle.

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# 1 Introduction

Consider a society in which every individual is assured of fundamental necessities like clean water, food, and education. This scenario, regardless of individual wealth or the means through which individuals attain their status, arguably dominates another societal structure where some members are deprived of essential resources such as potable water or adequate food. In this paper, we examine a novel interpretation of economic *sufficientarianism*, a compelling distributive justice criterion recently introduced into the economics literature by [Alcantud et al. \(2022\)](#) and [Bossert et al. \(2022, 2023\)](#). The goal of sufficientarianism is to ensure each individual has a sufficient level of well-being. This work departs from the concept of well-being as primitive, directly working instead with economic observables.

The basic idea of sufficientarianism is that potential allocations of resources for a society are compared according to the number of individuals meeting an exogenously specified threshold of well-being. The more agents who reach the level of well-being, the better.

Our work’s contribution is mainly threefold. First, we investigate a framework in which sufficientarianism is based on directly observable consumption, rather than well-being.<sup>1</sup> We envision ranking allocations of consumption goods for a fixed and finite collection of individuals. We have in mind ranking allocations, for example, consisting of food or clean water. Second, we provide a novel axiomatization of sufficientarianism within a completely abstract framework. When describing the set of goods that are deemed “sufficient,” we allow total generality, and the set can take any form. This axiomatization endogenizes the sufficient set, rather than taking an exogenous sufficientarian threshold as a primitive in any of the axioms. This axiomatization could also be understood as endogenizing the concept of welfare. The set of “sufficient” consumption bundles consists of those whose welfare is deemed high enough, under

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<sup>1</sup>At the same time, there is nothing precluding the interpretation of the objects we study as levels of well-being rather than consumption bundles.

the assumption that all agents have the same welfare for the same bundles. Third, we also investigate the concept of sufficientarianism in concrete economic environments, so that if a consumption bundle  $x$  is deemed sufficient, so is any bundle larger than  $x$ .

Our work also describes an ethical condition that we believe to be novel and highly related to the concept of sufficientarianism: for lack of a better term, we call this principle *sufficientarian judgment*. Roughly speaking, sufficientarian judgment asks that there should be an absolute priority on helping the worst off. More formally, we do not take individual preferences as primitives. Instead, our primitive is a ranking over allocations. To formalize the concept of one agent being worse off than another, we adopt a minimally restrictive approach. Suppose we begin with an allocation where all agents have identical consumption, we imagine a social planner as identifying all agents as equally well-off within such a situation. <sup>2</sup>

Now imagine a scenario where only the consumption of a specific agent, Alice, changes, while all remaining agents maintain their previously identical consumption. If the allocation where Alice has the new consumption has rendered the society *strictly worse off* than the original allocation, we can infer that Alice has been made worse off than the remaining agents.<sup>3</sup> Sufficientarian judgment asks that there is no way to improve upon this new allocation by altering any other agent's consumption. We cannot compensate society either fully or partially for Alice's loss by rewarding any other agent, say Bob. Neither enhancing nor decreasing Bob's consumption ameliorates the social welfare impacted by the change in Alice's consumption. The only way to improve society's welfare is exclusively through an adjustment to Alice's consumption.

To this end, sufficientarian judgment is a very strong requirement. It asks society to prioritize the least fortunate in an absolute way. It also has some more implications: rewarding more fortunate agents

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<sup>2</sup>This seems to be the idea behind egalitarian-equivalence for example, see [Pazner and Schmeidler \(1978\)](#).

<sup>3</sup>Our framework envisions that this "worse off" or implicit welfare ranking is the same for all agents.

cannot benefit society in any sense. Although this is a strong property, a particularly rich class of rankings satisfies it. We will describe some of these in the discussion of the independence of the axioms.

Our main result establishes that a ranking over allocations satisfies sufficientarian judgment, along with two well-established axioms of separability and symmetry, if and only if it is sufficientarian.

Previous axiomatic works related to sufficientarianism base their results on a principle called "prioritarian threshold," along with other ethical axioms (Mariotti and Veneziani, 2009, 2012, 2013; Alcantud, 2013; Lombardi et al., 2016). As the name suggests, a crucial concept in the prioritarian threshold property involves constructing a sufficient set via a threshold. The context for this principle is that the observable is a collection of real-valued welfare levels, rather than allocations of goods. The axiom is parametric in a sense, taking a threshold level as a parameter that defines the sufficient set.

In contrast, our work endogenizes the notion of a sufficient set. We can compare our work to the previous works by considering more structured economic environments. Though our main result takes place in an abstract environment, we develop several corollaries that apply to concrete economic environments.

For example, if consumption lies in a set with an order structure, then a natural monotonicity property implies that the sufficient set is closed upwards (upper-comprehensive). Our work extends naturally to classical commodity space, or more generally to any meet-semilattice. In this framework, we describe a natural generalization of the version of sufficientarianism presented in Alcantud et al. (2022). This generalization arises when we imagine that the sufficient set is determined by a threshold: There is a fixed threshold bundle  $\beta$ , and the sufficient set consists of all bundles at least as large as  $\beta$ . We can capture this version of sufficientarianism (called *threshold sufficientarianism*) by a simple axiom, asking that different goods in commodity space cannot substitute for one another. That is, if a decrease in food for Alice renders society worse off, increasing her other aspects of consumption like clean water will not bring society back to the starting point. This axiom, which we call  $\geq$ -complements, often suffices to axiomatize

threshold sufficientarianism, but sometimes an additional continuity condition is needed.

Finally, we also show that sufficientarianism is related to the leximin property of ranking via a weakening of sufficientarian judgment. Imagine that we can assign a utility to each bundle and that the utility of a bundle can take a value of either 0 or 1. Then maximizing any anonymous Bergson-Samuelson social welfare function will maximize the number of agents getting a utility of 1. This goes in particular for the leximin ranking and for the sufficientarian ranking.

However, utilizing a leximin ranking after assigning a utility to all bundles more generally results in a ranking that satisfies a weakening of sufficientarian judgment, which we term *weak sufficientarian judgment*. This property also begins with the idea that, starting from an allocation in which all agents consume identically, a change in Alice's consumption hurts society. The distinction for weak sufficientarian judgment is that it allows a subsequent change in Bob's consumption to cause a social gain, but can never bring society back to the initial social welfare level before Alice suffers from the consumption shock.

Associating a utility with each bundle, and then ranking allocations according to the imputed leximin ranking between the allocations gives a ranking of allocations that satisfies this property. We also show that for two agents, such binary relations are the only ones that satisfy weak sufficientarian judgment, symmetry, and separability. We call these binary relations *endogenous leximin*.

The paper proceeds as follows. We conclude the introduction with some related literature; then we proceed directly to the model and abstract result (Section 2). Section 3 investigates our concepts in concrete economic models, in particular proposing characterizations of threshold sufficientarianism. Section 4 describes the weakening of the sufficientarian judgment and the connection to the leximin ranking. Finally, Section 5 concludes.

## 1.1 Related Literature

[Alcantud et al. \(2022\)](#) introduced two substantive axioms, termed *absolute individual improvement* and *prioritarian threshold*. The first of these asks that if one allocation  $b$  strictly socially dominates another allocation  $a$ , and in  $b$  some agent's welfare is not maximal, then replacing that agent's welfare by the maximum possible welfare in  $a$ , and by anything else in  $b$  does not lead to a reversal in the ranking. The second of these, prioritarian threshold, references an exogenous sufficientarian threshold. It asks for some level of welfare  $\beta$  such that if any agent consumes strictly less than  $\beta$ , then replacing the welfare of all agents by  $\beta$  makes society strictly better off. Our axiom of sufficientarian judgment is logically independent of each of these two, and in a strong sense: in the setting of their paper, we can construct binary relations satisfying all of the axioms used in our main representation except separability, but neither of theirs, and we can construct binary relations satisfying all of the axioms used in their main representation except separability, but not sufficientarian judgment. Other papers also directly address the topic of sufficientarianism: [Bossert et al. \(2022, 2023\)](#).

The sufficientarian judgment axiom, as well as the notion of sufficientarianism, is closely related to a collection of lattice-theoretic axioms employed in characterizing similar representations. The connection here is best understood in a two-agent environment. Suppose that  $a, b, c \in \mathbb{R}$ , where these numbers represent welfare levels. Sufficientarian judgment is the requirement that  $(b, c)$  is indifferent to the socially worse of  $(b, b)$  or  $(a, c)$ . The axiom requires that the binary relation commute with respect to “meets” in a lattice-theoretic sense, when one of the allocations is constant. See for example [Miller \(2008\)](#); [Chambers and Miller \(2014a,b, 2018\)](#), where the constancy hypothesis is not imposed. In these works, it is explained that the main idea is originally due to [Kreps \(1979\)](#) in his study of preference for flexibility over menus. Related is the work of [Hougaard and Keiding \(1998\)](#); [Christensen et al. \(1999\)](#), where the lattice-theoretic version of the property appears in the cardinal form.

The properties of sufficientarian judgment and weak sufficientarian judgment are similar in spirit to basic concepts such as the Pigou-Dalton transfer principle or other generalizations in economic environments. References include [Fleurbaey \(2007\)](#), [Fleurbaey and Tadenuma \(2014\)](#), or [Sprumont \(2012\)](#). These works present forms of absolute priority for the worst off. To understand this claim, consider a two-agent economy. Weak sufficientarian judgment asks that, starting from an allocation in which both agents consume the same, if we change 1's consumption, and get a worse social bundle, then any change in agent 2's consumption cannot bring society back to the welfare level of the starting point. What this means is that the original bundle strictly dominates any bundle in which agent 1's imputed individual welfare has lowered from it.

What we call “endogenous leximin” is a concept that has existed in different forms in the literature for some time, although we are not aware of any work on this topic in a totally abstract environment. Most (but not all) of this work supposes that individuals have different preferences, and represents these preferences via some type of canonical utility representation. [Sprumont \(2012\)](#) describes a special case of endogenous leximin in an economic environment where individuals have preferences, and the common ranking underlying endogenous leximin is naturally in accordance with these preferences. The monograph by [Fleurbaey and Maniquet \(2011\)](#) contains many references to works studying rankings of allocations where individuals' preferences differ, and where a generalization of endogenous leximin allows each individual to possess their own utility function. For example, axiomatizations based on the classical money metric representation or on the representation first described by [Wold \(1944\)](#) have been studied. In particular, they describe axiomatizations of this type of endogenous leximin ranking in several classical economic environments.

In extending sufficientarianism to a multidimensional framework, this paper navigates previously unexplored territory, diverging from the predominantly welfaristic focus in the literature. We recognize a parallel between our approach to multidimensional sufficientarianism and the broader economic themes

of multidimensional poverty measures (Atkinson, 2003; Alkire and Foster, 2011) and Nussbaum’s capability approach (Nussbaum, 2011), which also emphasize the complexity of well-being across multiple dimensions. Our approach builds on these foundations by considering sufficientarian orderings in a multidimensional setting, capturing the nuanced relationships between dimensions.

## 2 The Model and Results

In this section, we explore the concept of economic sufficientarianism in an abstract environment.

Let  $A$  be a nonempty set, denoting the set of possible consumption bundles. For example,  $A$  may consist of a classical commodity space, it may consist of unordered bundles of goods, or it may consist of a set of life outcomes. Alternatively,  $A$  may index characteristics that may be associated with a given individual: health status, life satisfaction, and so forth. Finally, in accordance with the classical study of sufficientarianism,  $A$  may consist of welfare levels. Elements of  $A$  are the objects which are assigned to agents. Typical elements of  $A$  will be denoted by  $a, b \in A$ .

Let  $N = \{1, \dots, n\}$  be a finite set, where  $n \geq 2$  denotes the number of individuals in the society.  $N$  represents the set of agents in society.

We can think of elements of  $A^N$  as *allocations*. Every  $x \in A^N$  associates a unique member of  $A$  with each member of society. We typically view a member  $x \in A^N$  as a function  $x : N \rightarrow A$ : so, for example, for a map  $\sigma : N \rightarrow N$ , the notation  $x \circ \sigma$  means  $(x \circ \sigma)_i = x_{\sigma(i)}$ . Further, we often identify a constant function with the value that function takes, so that *e.g.*  $b$  may refer either to  $b \in A$  or  $b \in A^N$ .

Consider an element  $x$  in  $A^N$ , where the  $i$ th component of  $x$  is denoted by  $x_i \in A$ . Our objective is to study rankings over the elements of  $A^N$ . Define a binary relation  $\succsim$  on  $A^N$ . Examination of this binary relation is our main focus. Next, we specify the properties that this relation may possess.

The following two are standard properties, though the second of them is not explicitly stated in



Alcantud et al. (2022). It follows as a consequence of some of their other axioms.

**Weak order:**  $\succeq$  is a weak order.<sup>4</sup>

**Symmetry:** Let  $\sigma : N \rightarrow N$  be a bijection and let  $a \in A^N$ . Then  $a \sim a \circ \sigma$ .

As in Alcantud et al. (2022), we postulate a simple separability condition on  $\succeq$ . The interpretation of this axiom is much the same as it is in that paper. For any  $x, y \in A^N$  and  $M \subseteq N$ ,  $(x_M, y_{-M})$  denotes the member of  $A^N$  for which  $(x_M, y_{-M})_i = x_i$  when  $i \in M$  and  $(x_M, y_{-M})_i = y_i$  when  $i \notin M$ .

**Separability:** For any  $M \subseteq N$ , and any  $x, y, x', y' \in A^N$ ,  $(x_M, y_{-M}) \succeq (x'_M, y_{-M})$  iff  $(x_M, y'_{-M}) \succeq (x'_M, y'_{-M})$ .

This form of separability is written in a fashion that is apparently stronger than the one in Alcantud et al. (2022). However, a standard induction argument establishes that it is equivalent to theirs under the weak order hypothesis.

We now introduce our main innovation. First, some notation. Let  $a, b, c \in A$  and  $i, j \in N$  for which  $i \neq j$ . Then  $b$ , with an abuse of notation, also represents the member of  $A^N$  for which  $b_k = b$  for all  $k \in N$ . We use the notation  $a_i b$  to represent that member of  $A^N$  for which  $(a_i b)_i = a$  and for all  $k \neq i$ ,  $(a_i b)_k = b$ , and  $c_j(a_i b)$  to be the member of  $A^N$  for which  $(c_j(a_i b))_i = a$ ,  $(c_j(a_i b))_j = c$  and for all  $k \notin N \setminus \{i, j\}$ ,  $(c_j(a_i b))_k = b$ .

**Sufficientarian judgment:** Let  $a, b, c \in A$  and suppose that  $i \neq j$ . Then  $b > a_i b$  implies  $a_i b \succeq c_j(a_i b)$ .

In words, sufficientarian judgment says the following. Starting from an allocation in which all agents are treated identically in terms of consumption, receiving  $b$ , suppose a change in  $i$ 's alternative to  $a$  forces social welfare to drop strictly. Given that  $\succeq$  is intended to reflect social welfare, under the separability and transitivity hypotheses, this happens because  $i$  becomes worse off than the other agents in terms of

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<sup>4</sup>That is,  $\succeq$  is complete and transitive.

the judgment of the social ranking.<sup>5</sup> Here, what sufficientarian judgment asks is that this drop in social welfare can never be compensated, either fully or partially, by giving some other agent  $j$  consumption of  $c$ . This is true no matter how much of a gain  $c$  would cause to agent  $j$ . Society should not treat  $i$  and  $j$ 's welfare as substitutes, at least in the case in which  $i$ 's welfare level has become less than  $j$ 's.

Sufficientarian judgment applies only when the starting position of all agents is identical (the alternative  $b \in A$  is consumed by all agents), though the intuition would also be valid in case the starting position of  $i$  is worse than that of agent  $j$ . That said, we can only meaningfully speak of a starting position of  $i$  being worse than that of  $j$  under the hypotheses of symmetry and separability.

In the context of sufficientarian judgment, we can infer that agent  $i$  is worse off than the remaining agents precisely because in moving from  $b$  to  $a_i b$ , society has become worse off. This involves an implicit assumption that all members of society were equally well off when consuming  $b$ ; an implicit interpersonal comparison. The idea behind the axiom then is that society should prioritize raising the welfare of poor agents as much as possible, to the extent that any gain in welfare for a richer agent would have no effect on the social ranking.

On the other hand, if a loss in social welfare results from a loss of welfare of a rich agent, we want to allow society to benefit from a gain in welfare to a poor agent. To understand, suppose  $A = \mathbb{R}$ , where  $N = \{1, 2\}$ , and that we are considering  $x \in A^N$  given by  $(5, 0)$ . It is reasonable to assume that  $(5, 0) \succ (0, 0)$ ; so that a loss of 5 units for the richer agent (agent 1) hurts society. At the same time, it is reasonable to suppose that  $(0, 5) \succ (0, 0)$  as well: that loss can be compensated for by a gain in utility by agent 2.

What sufficientarian judgment rules out is a situation whereby, for example,  $(5, 5) \succ (0, 5)$  and

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<sup>5</sup>Observe that the conjunction of transitivity and separability imply a form of the Pareto property: for any  $i \in N$  and  $a, b \in A$ , we may define  $a \succeq_i b$  if  $a_i c \geq b_i c$  for any  $c \in A$ . It is obvious that separability and transitivity jointly imply that for any  $x, y \in A^N$ , if  $x_i \succeq_i y_i$  for all  $i \in N$ , then  $x \succeq y$ . Thus,  $\succeq$  is "Paretian" with respect to the individual preferences  $\succeq_i$ .

$(0, 7) \succ (0, 5)$ . In this example, agent 1's move from 5 to 0 has decreased social welfare. Starting from an implicit assumption that the two agents are equally well off under  $(5, 5)$ , this means that agent 1 has become worse off in the move to  $(0, 5)$ . She is now the “poorer” of the two agents. We want to preclude that this drop in wealth for agent 1 can be compensated for by enriching agent 2, which rules out  $(0, 7) \succ (0, 5)$ . Thus, it asserts that the priority should be on bringing the agent whose welfare dropped back up to the welfare of the remaining agents; in this sense it is a strongly egalitarian property.

We say that a binary relation  $\succeq$  is *sufficientarian* if there exists a set  $S \subseteq A$  such that for all  $x, y \in A^N$ ,  $x \succeq y$  iff  $|\{i \in N : x_i \in S\}| \geq |\{i \in N : y_i \in S\}|$ . We refer to the set  $S$  as the *sufficient set*. This formalizes the notion of economic sufficientarianism. The set  $S$  becomes a parameter of the binary relation, the goal being to ensure as many agents as possible have consumption in  $S$ . The sufficient set  $S$  may be empty (or the entire set), in which case the binary relation  $\succeq$  coincides with total indifference. In case this is a concern, the following (standard) property rules this out.

**Non-degeneracy:** There exists  $x, y \in A^N$  for which  $x \succ y$ .

**Theorem 1.** *A binary relation  $\succeq$  is sufficientarian iff it satisfies weak order, symmetry, separability, and sufficientarian judgment. These axioms are independent.*

*Proof.* Let  $\succeq$  satisfy the axioms. Define the ranking  $\geq^*$  on  $A$  by  $a \geq^* b$  iff  $a_i c \succeq b_i c$  for some  $c \in A$ .

Observe that by separability,  $a_i c \succeq b_i c$  iff  $a_i d \succeq b_i d$  for any  $c, d \in A$ . Obviously  $\geq^*$  is a weak order.

We claim that  $\geq^*$  has at most two indifference classes. To this end, and by contradiction, suppose that there are  $a^*, b^*, c^* \in X$  for which  $a^* \succ^* b^* \succ^* c^*$ .

We invoke separability. First we simplify notation. Fix agents  $1, 2 \in N$ . We define the binary relation  $\succeq^2$  on  $A^{\{1,2\}}$  by  $(a, b) \succeq^2 (c, d)$  if for any  $x, y \in A^N$  for which  $x_{\{1,2\}} = (a, b)$  and  $y_{\{1,2\}} = (c, d)$ , where  $x_i = y_i$  for all  $i \in N \setminus \{1, 2\}$ , we have  $x \succeq y$ . By separability and weak order,  $\succeq^2$  is itself a weak order.

Now observe that  $(b^*, b^*) \succ^2 (b^*, c^*)$  by separability (since  $b^* \succ^* c^*$ ). It follows that  $b^* \succ c_2^* b^*$ , also by separability. By sufficientarian judgment,  $c_2^* b^* \succeq a_1^*(c_2^* b^*)$ , and so by separability it follows that  $(b^*, c^*) \succeq^2 (a^*, c^*)$ . But this contradicts separability (as  $a^* \succ^* b^*$ ).

So there can be at most two indifference classes for  $\succeq^*$ . If there is only one indifference class, then it is obvious that  $\succeq$  is total indifference and is hence consistent with sufficientarianism for  $S = N$ . So let us suppose that there are two indifference classes. Denote by  $S \subseteq A$  the higher of these two indifference classes.

Let  $x, y \in A^N$ . We claim that if  $|\{i \in N : x_i \in S\}| \geq |\{i \in N : y_i \in S\}|$ , then  $x \succeq y$ . Similarly if  $|\{i \in N : x_i \in S\}| > |\{i \in N : y_i \in S\}|$ , then  $x \succ y$ . By symmetry and transitivity, we may assume without loss that for each of  $x$  and  $y$ , we have  $x_1 \geq^* x_2 \dots \geq^* x_n$  and  $y_1 \geq^* y_2 \dots \geq^* y_n$ .

Suppose that  $|\{i \in N : x_i \in S\}| \geq |\{i \in N : y_i \in S\}|$ . This implies that for all  $i \in N$ , we have  $x_i \geq^* y_i$ . Consequently, by repeatedly invoking separability,  $(x_1, x_2, \dots, x_n) \succeq (y_1, x_2, \dots, x_n) \succeq (y_1, y_2, x_3, \dots, x_n) \dots \succeq (y_1, \dots, y_n)$  so that  $x \succeq y$  by transitivity. Similarly, if  $|\{i \in N : x_i \in S\}| > |\{i \in N : y_i \in S\}|$ , then there is in addition some  $i \in N$  for which  $x_i \succ^* y_i$ . Consequently, in the preceding chain, one of the instances of  $\succeq$  is actually  $\succ$  and so  $(x_1, \dots, x_n) \succ (y_1, \dots, y_n)$  by weak order.

For the converse, weak order, symmetry, and separability are easily verified.

For sufficientarian judgment, let  $a, b, c \in A$ . Suppose that  $b \succ a_i b$ . By definition, this means that  $b \in S$  and  $a \notin S$ . Since  $b \in S$ , in either case, whether  $c \in S$  or  $c \notin S$ , we have  $a_i b \succeq c_j(a_i b)$ .

On the independence of the axioms: We illustrate by examples that the four axioms are independent. These examples will illustrate the general structure of rules satisfying these axioms.

### Weak order

Let  $\succeq$  be the relation defined by  $x \succeq y$  iff  $y = x \circ \sigma$  for some permutation  $\sigma : N \rightarrow N$ . Symmetry and sufficientarian judgment are automatically satisfied (sufficientarian judgment because the antecedent condition is never satisfied, since there are no strict rankings according to this relation). Separability is

similarly easy to verify. This ranking does not satisfy non-degeneracy; however, to modify it so that it does, suppose for example that  $N = \{1, 2\}$  and let  $A = \{a, b, c, d\}$ . Add to  $\succeq$  the rankings  $(a, b) > (c, d)$ ,  $(b, a) > (c, d)$ ,  $(a, b) > (d, c)$ ,  $(b, a) > (d, c)$ . Observe that anonymity, separability, and symmetry remain satisfied. Sufficientarian judgment is satisfied as the antecedent condition still never holds. Weak order remains violated.

### Symmetry

Say that a binary relation  $\succeq$  is *weighted sufficientarian* if there is a function  $\lambda : N \rightarrow \mathbb{R}_{++}$  and a set  $S \subseteq A$  such that for all  $x, y \in A^N$ ,  $x \succeq y$  if and only if  $\sum_{i \in N} \lambda_i \mathbf{1}_S(x_i) \geq \sum_{i \in N} \lambda_i \mathbf{1}_S(y_i)$ , where  $\mathbf{1}_S(x)$  is the usual indicator function, returning 1 when  $x \in S$  and 0 otherwise.<sup>6</sup> These objects generally violate symmetry. For example, when  $\lambda(i) \neq \lambda(j)$  for some  $i, j \in N$ , symmetry will be violated. A related generalization, which would feature each individual possessing their own sufficient set  $S_i$ , generally fails sufficientarian judgment. Another standard class of solutions are *dictatorships*, where there is a weak order  $\succeq$  on  $A$  such that for all  $x, y \in A^N$ ,  $x \succeq y$  iff  $x_i \succeq y_i$ .

### Separability

There are many binary relations violating separability but satisfying our remaining properties. A canonical example is obtained in the case of  $A = [0, 1]$ , whereby for all  $x, y \in A^N$ ,  $x \succeq y$  if and only if  $\min_i x_i \succeq \min_i y_i$ .

### Sufficientarian judgment

There are many binary relations that violate sufficientarian judgment but satisfying our remaining properties. Let  $u : A \rightarrow \mathbb{R}$  be a function and define  $x \succeq y$  iff  $\sum_i u(x_i) \geq \sum_i u(y_i)$ . Then as long as the image of  $u$  contains at least three values, this binary relation violates sufficientarian judgment but satisfies the remaining properties.

□

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<sup>6</sup>Or just an ordinal “qualitative measure” ranking the sets.

**Remark 1.** *Sufficientarian judgment could be replaced by the following dual condition in Theorem 1 or any of the results to follow. The proof is essentially identical to that of Theorem 1 and is thus omitted.*

**Dual Sufficientarian Judgment:** *Let  $a, b, c \in A$  and suppose that  $i \neq j$ . Then  $a_i b > b$  implies  $c_j(a_i b) \succeq a_i b$ .*

*The interpretation of this dual condition has less ethical content than does sufficientarian judgment but asks that if social welfare goes up in replacing agent  $i$ 's consumption with  $a$  from  $b$ , then one cannot bring it back down by changing some other agent  $j$ 's consumption.*

### 3 Structured economic environments

In this section, we suppose that  $A$  represents a classical economic environment, for example, a commodity space. We endow  $A$  with a binary relation  $\geq$ , which we will assume to be a preorder.<sup>7</sup>

The framework of [Alcantud et al. \(2022\)](#) may be viewed as a special case of this general structure. Their assumption is that  $A = [0, 1]$ . Their axioms specifically reference  $\geq$ , interpreted as the usual ordering on  $[0, 1]$ .

The added structure suggests the following property:

**Monotonicity:** Let  $x, y \in A^N$  for which for all  $i \in N$ ,  $x_i \geq y_i$ . Then  $x \succeq y$ .

Additionally, say that a subset  $S \subseteq A$  is *monotone* (or *upper comprehensive*) if  $a \in S$  and  $b \geq a$  implies  $b \in S$ .

We say that a ranking is *monotone-sufficientarian* if there is a monotone set  $S \subseteq A$  such that for all  $a, b \in A^N$ ,  $a \succeq b$  iff  $|\{i \in N : a_i \in S\}| \geq |\{i \in N : b_i \in S\}|$ .<sup>8</sup>

The following corollary of Theorem 1 asserts that the addition of monotonicity to the preceding

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<sup>7</sup>A preorder is a reflexive and transitive binary relation. This assumption is without loss for our purposes. If  $\geq$  is not a preorder the following monotonicity assumption would be equally valid for the smallest preorder containing it.

<sup>8</sup>This family of rankings was suggested to us by an anonymous associate editor.

axioms ensures that the sufficient set  $S$  is monotone. Its proof is obvious.

**Corollary 1.** *A binary relation  $\succeq$  is monotone sufficientarian iff it satisfies weak order, symmetry, separability, sufficientarian judgment, and monotonicity.*

We are now in a position to compare our results with those of [Alcantud et al. \(2022\)](#). Say a binary relation  $\succeq$  is *threshold sufficientarian* if there exists  $\beta \in A$  for which for all  $x, y \in X^N$ ,  $x \succeq y$  iff  $|\{i \in N : x_i \geq \beta\}| \geq |\{i \in N : y_i \geq \beta\}|$ .

Let us now consider the specific environment in which  $A = [0, 1]$  with its usual order  $\geq$ . This is the environment studied by [Alcantud et al. \(2022\)](#). Every threshold sufficientarian binary relation in this case is monotone sufficientarian. The converse is not true, but the generalization over threshold sufficientarian rules is largely technical.<sup>9</sup>

To this end, we will now discuss the axioms of [Alcantud et al. \(2022\)](#). Our goal is to show that there is no logical relation between their axioms and sufficientarian judgment. We will do so by establishing that there are binary relations satisfying all of the axioms of Corollary 1 except for separability, but which violate the two following axioms. We will also show that there are binary relations satisfying all of the axioms from their Theorem 1 except for separability, but which violate sufficientarian judgment.

The novel axioms of theirs which have not so far appeared are the following. The motivation of these properties can be found in [Alcantud et al. \(2022\)](#).

**Absolute individual improvement:** Let  $x, y \in A^N$  and let  $i \in N$ . Suppose that  $x > y$  and  $x_i < 1$ . Then for all  $b_i \in A$ ,  $(1, x_{-i}) \succeq (b_i, y_{-i})$ .

**Prioritarian threshold:** There exists  $\beta \in (0, 1)$  for which for all  $x \in A^N$  for which there exists  $i \in N$

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<sup>9</sup>For an example of such a ranking, fix some  $\beta \in A$ . Define a ranking  $\succeq$  for which for all  $x, y \in A^N$ ,  $x \succeq y$  iff  $|\{i \in N : x_i > \beta\}| \geq |\{i \in N : y_i > \beta\}|$ . A rule like this might be called “strictly sufficientarian,” but the main distinction is that only the individuals getting strictly more than  $\beta$  have a sufficient amount. Those getting exactly  $\beta$  do not consume a sufficient amount.

such that  $x_i < \beta$  and for all  $j, k \in N \setminus \{i\}$ ,  $x_j = x_k$ , we have  $\beta > x$ .

Our first example is a class of binary relations satisfying all of the axioms from Corollary 1 except for separability, but which violates each of Absolute individual improvement and Prioritarian threshold. This class also satisfies homotheticity whenever homothetic expansions are meaningful. Roughly, this class is a kind of generalized maxmin class, whereby any two allocations which are comonotonic (that is, if agent  $i$  is strictly richer than  $j$  in one allocation,  $i$  is weakly richer than  $j$  in the other) are judged according to a weighted maxmin criterion. The binary relation is then completed by symmetry.

**Example 1.** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$  be a fixed increasing sequence of real numbers. For any  $x \in [0, 1]^N$ , let  $d(x)$  denote the non-increasing rearrangement of  $x$ . Define a utility function  $U : A^N \rightarrow \mathbb{R}$  via  $U(x) = \min_{i \in N} \{\alpha_i d(x)_i\}$ . Finally let  $\succeq$  be the binary relation on  $A^N$  represented by  $U$ . This construction is depicted in Figure 1, where  $A = [0, 1]$  and  $N = \{1, 2\}$ . The two-dimensional picture on  $[0, 1]^2$  therefore represents the level sets of  $\succeq$ . It is easy to verify by inspection that sufficientarian judgment, monotonicity, and symmetry are satisfied here. Absolute individual improvement is generally violated; let us suppose that  $N = \{1, 2\}$ : observe that if  $0 < a < 1$ , then  $(a, a) \succ (a - \epsilon, a - \epsilon)$  for  $\epsilon > 0$ , but if  $\epsilon > 0$  is small enough then  $(1, a - \epsilon) \succ (a, a)$ . Prioritarian threshold is similarly violated: for any  $\beta \in (0, 1)$ , by choosing  $\epsilon > 0$  small,  $(\beta - \epsilon, 1) \succ (\beta, \beta)$ .

Our second example is a binary relation satisfying all the axioms of Theorem 1 of Alcantud et al. (2022) except for separability, but which does not satisfy sufficientarian judgment. This example also satisfies symmetry. The example consists of a ranking which coincides with threshold sufficientarianism except when for both allocations, all agents reach the sufficientarian threshold. In this case, it ranks allocations monotonically but in a way in which all allocations for which at least one agent consumes 1 are indifferent.

**Example 2.** First, fix  $\tau \in (0, 1)$ . The binary relation  $\succeq$  behaves as the threshold sufficientarian rule with



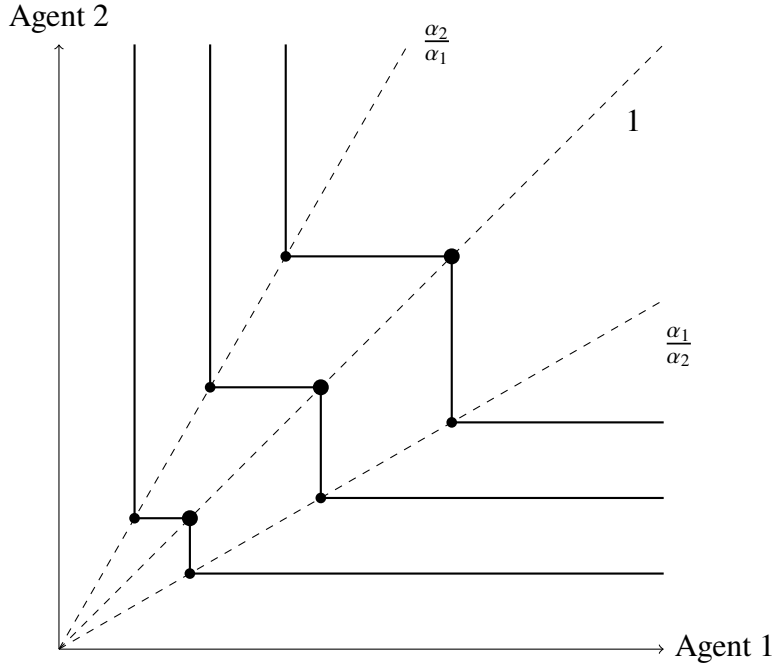


Figure 1: Level sets for Example 1, N=2 Case

respect to  $\beta = \tau$  for any  $x, y \in A^N$  where  $\min_{i \in N} \min\{x_i, y_i\} < \tau$ .

If for all  $i \in N$ ,  $\min\{x_i, y_i\} \geq \tau$ , then we define  $x \succeq y$  if and only if  $\prod_{i \in N} (1 - x_i) \leq \prod_{i \in N} (1 - y_i)$ .

It is again simple to see that symmetry and monotonicity are satisfied here. Absolute individual improvement is also satisfied: suppose that  $x > y$ , where  $x_j < 1$ . If  $\min_{i \in N} \min\{x_i, y_i\} < \tau$ , then this implies that  $\min_i \{y_i\} < \tau$ ; otherwise, according to the sufficientarian ranking,  $y \succeq x$ . Therefore, the constraint remains valid when replacing agent  $j$ 's consumption by 1, and so the result follows from the fact that sufficientarianism satisfies absolute individual improvement. On the other hand, if  $\min_{i \in N} \min\{x_i, y_i\} \geq \tau$ , then in replacing agent  $j$ 's consumption by 1 leads to an allocation which dominates all allocations.

Prioritarian threshold follows simply by taking  $\tau$  to be the  $\beta$  in the axiom; it is straightforward to verify the claim as sufficientarianism satisfied prioritarian threshold.

Finally, sufficientarian judgment is violated here: take any  $\tau < a' < a < 1$  and observe that  $(a, a') \succ (a', a')$ . However,  $(1, a') \succ (a, a')$ .

### 3.1 On threshold sufficientarianism

In this section, we propose some simple axioms that allow us to meaningfully talk about threshold sufficientarianism in a multidimensional environment. To this end, we assume that  $A$  is endowed with a partial order  $\geq$ .<sup>10</sup> Further, we assume that for each pair  $a, b \in A$ , there is a unique greatest lower bound (or *meet*) according to  $\geq$ ; we denote this greatest lower bound by  $a \wedge b$ . For a set of objects  $B$ ,  $\bigwedge B$  denotes the greatest lower bound of  $B$  (when it exists) and similarly, for an indexed set of objects  $\{x^\lambda\}_{\lambda \in \Lambda}$ ,  $\bigwedge_{\lambda} x^\lambda$  denotes  $\bigwedge \{x^\lambda : \lambda \in \Lambda\}$ . Formally, we assert that the pair  $(A, \geq)$  is a meet-semilattice. Definitions here are standard, see *e.g.* [Davey and Priestley \(2002\)](#).

We formulate a simple axiom that requires that different goods cannot substitute for each other in the sufficient set. Obviously, whether this axiom makes any sense depends on what we imagine consumption goods to be. Let us consider the standard meet-semilattice of  $\mathbb{R}_+^2$  with its usual pointwise order. Elements of  $\mathbb{R}_+^2$  are interpreted as bundles of two different consumption goods. Our goal is to require the following property, stated informally: If the loss of one good for an agent makes society strictly worse off, then society cannot revert to the original level of welfare by compensating the same agent with any other good.

To see how this works, consider  $(a_1, a_2)$ ,  $(b_1, a_2)$  and  $(b_1, b_2)$ . Suppose now that society determines that changing  $i$ 's consumption from  $(a_1, a_2)$  to  $(b_1, a_2)$  leads to a strict decrease in social welfare. That is, the loss of good 1 for agent  $i$  renders the society worse off. We want to ask that there is no amount of good 2 with which we could compensate agent  $i$  that allows the social welfare to be fully compensated for this loss:  $(a_1, a_2) \succ (b_1, b_2)$ . Arguably, the axiom is sensible in cases where the goods are food and clean water. It would make less sense if the goods were broccoli and cauliflower.

Moving away from  $\mathbb{R}_+^2$  to the more general lattice-theoretic environment, the requirement is that if changing agent  $i$ 's consumption from  $a$  to  $a \wedge b$  leads to a strict decrease in the social ranking, then

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<sup>10</sup>A partial order is reflexive, transitive, and antisymmetric.

moving from  $a$  to  $b$  must similarly lead to a strict social decrease. This motivates the following general concept.

**$\geq$ -Complements:** Let  $i \in N$ , and let  $a, b, c \in A$ . Suppose that  $a_i c > (a \wedge b)_i c$ . Then  $a_i c > b_i c$ .

With this in mind, let us now discuss the concepts of interest here. One was already mentioned before: say that a binary relation  $\succeq$  is *threshold sufficientarian* if there exists  $\beta \in A$  for which  $x \succeq y$  iff  $|\{i : x_i \geq \beta\}| \geq |\{i : y_i \geq \beta\}|$ . Our next definition allows us to capture a modest generalization of threshold sufficientarianism. The generalization permits technical concepts such as “strict sufficientarianism” as described in the preceding section, as well as more economically relevant examples, such as commodity spaces where some commodities are deemed irrelevant.

Say that  $\mathcal{F} \subseteq A$  is a  $\geq$ -*filter* if the following two properties are satisfied.

1. If  $a \in \mathcal{F}$  and  $b \geq a$ , then  $b \in \mathcal{F}$ .
2. If  $a, b \in \mathcal{F}$ , then  $(a \wedge b) \in \mathcal{F}$ .

We call a sufficientarian binary relation whose sufficient set is a filter a *generalized threshold sufficientarian* binary relation. For a threshold sufficientarian binary relation, the set  $S$  takes the form  $S = \{a \in A : a \geq \beta\}$ .<sup>11</sup> It is easily verified that every such set is a filter. Nevertheless, there are filters which are not threshold sufficientarian. Here are two examples we have in mind.

1. Let  $A = \mathbb{Z}^2$ . Then the set  $S = \{(a_1, a_2) : a_1 \geq 0\}$  is a filter.
2. Let  $A = \mathbb{R}$ . Then the set  $S = \{a : a > 0\}$  is a filter.

The first of these models a multi-commodity environment, where one of the commodities (the second) does not matter. It is redundant in terms of sufficiency. We would like to allow for this generalization in case some of the commodities, whose consumption can come in negative units, are not deemed necessary

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<sup>11</sup>A filter corresponding to a threshold sufficientarian rule is called a *principal filter* in the mathematics literature.

for the agents. This type of generalized threshold sufficientarian rule is an economically meaningful generalization of the concept of threshold sufficientarianism. The second of these is close to being sufficientarian, but the sufficient set does not contain its lower bound. We referred to such an object as a “strict” sufficientarian binary relation earlier. This example constitutes a technical generalization of threshold sufficientarianism.

Our goal is to characterize generalized threshold sufficientarian binary relations through the use of  $\geq$ -complements. As a corollary, we will also provide conditions under which we can infer that a binary relation is threshold sufficientarian. The following technical discussion is useful for this second goal.

We say that  $(A, \geq)$  satisfies the *descending chain condition (DCC)* if for every sequence  $a_1 \geq a_2 \geq a_3 \dots$ , there exists  $k$  such that for all  $l, m \geq k$ ,  $a_l = a_m$ . The DCC is a weaker condition than finiteness. For example,  $A = \mathbb{Z}_+^n$  with its usual pointwise order satisfies the DCC. The relevance of the condition is that it is a necessary and sufficient condition that a semilattice must satisfy for all filters to be principal filters.

We may choose to work with semilattices that do not satisfy the DCC. We can recover threshold sufficientarianism by working with a slightly stronger notion of a semilattice. We say that  $(A, \geq)$  is *meet-complete* if every nonempty subset  $B \subseteq A$  has a greatest lower bound, which we denote by  $\bigwedge B$ . For example,  $\mathbb{R}_+^n$  is meet-complete, whereas  $\mathbb{R}^n$  is not.

The following axiom is meaningful for meet-complete semilattices only.

**Descending continuity:** Let  $(\Lambda, \geq')$  be a nonempty linearly ordered set, and let  $\{x^\lambda\}_{\lambda \in \Lambda} \in A^N$  be a set of allocations for which for all  $\lambda, \lambda' \in \Lambda$  and all  $i \in N$ , if  $\lambda' \geq' \lambda$  then  $x_i^\lambda \geq x_i^{\lambda'}$  and  $x^\lambda \sim x^{\lambda'}$ . Then  $(\bigwedge_\Lambda x_1^\lambda, \dots, \bigwedge_\Lambda x_n^\lambda) \sim x^{\lambda'}$  for any  $\lambda' \in \Lambda$ .

The following is our main characterization of generalized threshold sufficientarianism.

**Theorem 2.** *A binary relation  $\succsim$  is generalized threshold sufficientarian iff it satisfies weak order,*

*symmetry, separability, sufficientarian judgment, monotonicity, and  $\geq$ -complements.*

*Proof.* We know by Corollary 1 that  $\succeq$  is monotone-sufficientarian. Let us call its corresponding sufficient set  $S$ . Our goal is to establish that  $S$  is a filter. That  $S$  is closed upwards follows as  $S$  is a monotone set. Now, suppose that  $a, b \in S$ . We want to claim that  $(a \wedge b) \in S$ , so suppose by means of contradiction that it is not.

Then for any  $c \in A$ , we have  $a_i c > (a \wedge b)_i c$ , by the definition of sufficientarianism. Therefore, by  $\geq$ -complements,  $a_i c > b_i c$ , which means that  $b \notin S$ , a contradiction. So  $S$  is a filter.

Finally, it is obvious that if  $\succeq$  is generalized threshold sufficientarian, then it satisfies  $\geq$ -complements. □

The following proposition qualifies when we additionally can assert that the binary relation is threshold sufficientarian.

**Proposition 1.** *The following statements are true.*

1. *If  $(A, \geq)$  satisfies the DCC, then a binary relation is threshold sufficientarian iff it is generalized threshold sufficientarian.*
2. *If  $(A, \geq)$  is a meet-complete semilattice, then a binary relation is threshold sufficientarian iff it is generalized threshold sufficientarian and it satisfies descending continuity.*

*Proof.* Both of these arguments are standard, for the first, see for example [Davey and Priestley \(2002\)](#).

Let us prove the first claim. Suppose that  $(A, \geq)$  satisfies the DCC, and suppose that  $\succeq$  is generalized threshold sufficientarian, with sufficient set  $S$  that is a filter. We claim that  $S$  is a principal filter. Suppose by means of contradiction that it is not. Then there is no  $\beta \in S$  for which  $S = \{a \in A : a \geq \beta\}$ . Therefore, for any  $a \in S$ , there exists  $b \in S$  for which  $b \geq a$  is false. As a consequence, we obtain that  $a > (a \wedge b)$ , and since  $S$  is a filter,  $(a \wedge b) \in S$ . Consequently, for every  $a \in S$ , there is  $a' \in S$  for which  $a > a'$ . Now

let  $a_1 \in S$  be arbitrary, and inductively construct  $a_n \in S$  from  $a_{n-1} \in S$  so that  $a_{n-1} > a_n$ . This sequence establishes that  $(A, \geq)$  violates the DCC.

Conversely, suppose that the DCC is violated for  $(A, \geq)$ , and let  $a_1 > a_2 > \dots$  be a strictly decreasing sequence. Define  $S = \bigcup_n \{a \in A : a \geq a_n\}$ . Observe  $S$  is a filter but cannot be principal: if  $S$  were principal, there would exist  $\beta \in S$  for which for all  $a \in A$ ,  $a \geq \beta$ . But since  $\beta \in S$ , by definition of  $S$ , there exists  $n$  for which  $\beta \in \{a \in A : a \geq a_n\}$ . Hence  $\beta \geq a_n > a_{n+1}$ , a contradiction.

For the second part, it is clear that any threshold sufficientarian ranking obeys descending continuity. Next, we may without loss assume that  $S \neq \emptyset$  (the case of  $S = \emptyset$  delivers the same representation as the case of  $S = A$ ). Observe that descending continuity implies that any  $\geq$ -chain  $Y \subseteq S$  is bounded below by  $\bigwedge Y \in S$ . Consequently, Zorn's Lemma implies that there is a  $\geq$ -minimal element  $\beta \in S$ . We claim that for all  $a \in S$ ,  $a \geq \beta$ , for if not, then there is  $a \in S$  for which  $\beta > a \wedge \beta$ , contradicting minimality of  $\beta$  as  $a \wedge \beta \in S$  because  $S$  is a filter.

□

**Remark 2.** *The axiom of descending continuity can often be formulated in a less restrictive version whose antecedent only considers sequences. This represents weakening the axiom by asking that it holds only when  $\Lambda = \mathbb{N}$  with its usual order. For instance, let us consider a decreasing sequence of allocations  $\{x^m\}_{m \in \mathbb{N}}$  with the property that  $x^m \sim x^{m'}$  for all  $m, m' \in \mathbb{N}$ . A weakening of the axiom would then require only that  $\bigwedge_{\mathbb{N}} x^m \sim x^{m'}$  for any chosen  $m'$ . This simpler form of the axiom is adequate for sets like  $\mathbb{R}_+^n$  with its conventional order. However, the scenario becomes more complex when we consider abstract but largely technical environments. Sequences no longer suffice and our indexing set must be taken to be some arbitrary linearly ordered set  $\Lambda$ . For example, if  $A$  represents the collection of all subsets of an uncountable set, and we define a binary relation by a sufficient set  $S$  that contains all subsets with complements of at most countable size, this generalized threshold sufficientarian relation would satisfy the*

sequential criterion but not the more comprehensive axiom as delineated in Proposition 1. This relation is clearly not threshold sufficientarian: by means of contradiction, if such a threshold  $\beta$  were to exist, it would necessarily be infinite with an at most countable complement. Otherwise  $\beta$  itself would be in the sufficient set yet not have a countable complement. So  $\beta$  is infinite with an at most countable complement. But then it follows that for any  $a \in \beta$ ,  $\beta \setminus \{a\}$  is also an infinite set with an at most countable complement. Yet,  $\beta \setminus \{a\} \geq \beta$  is false, contradicting the fact that  $\beta$  is a threshold.

## 4 On endogenous leximin and a weakening of sufficientarian judgment

In this section, we return to the abstract setup of Section 2, without any exogenous ordering on the set  $A$ . Our aim is to establish that a weakening of sufficientarian judgment leads to a familiar ethical principle. We study the relation between sufficientarianism and the leximin ranking. The leximin ranking  $\geq_l$  is a ranking on  $\mathbb{R}^n$ , which asserts that for any  $x, y \in \mathbb{R}^n$ ,  $x \geq_l y$  if, when each of  $x$  and  $y$  are rearranged in nondecreasing order,  $x$  dominates  $y$  according to the classical lexicographic relation. Thus, it ranks nondecreasing rearrangements lexicographically.

Our setup is abstract and thus the leximin ranking is not defined on  $A^N$  absent some ordering on  $A$ . Nevertheless, there are obvious parallels between sufficientarianism and the leximin ranking. We detail these below.

First, let us say that a weak order  $\succsim$  on  $A^N$  is an *endogenous leximin* ranking if the following is true. There exists a weak order  $\geq^*$  over  $A$  for which, for any  $x, y \in A^N$  the following two assertions hold.

1. If  $x_n \geq^* x_{n-1} \dots \geq^* x_1$  and  $y_n \geq^* y_{n-1} \dots \geq^* y_1$ , then  $x \succsim y$  if either for all  $i \in N$ ,  $x_i =^* y_i$ , or there exists an  $i \in N$  such that for all  $j < i$ ,  $x_j =^* y_j$  and  $x_i >^* y_i$ .

2. For any permutation  $\sigma : N \rightarrow N$ ,  $x \sim x \circ \sigma$ .

For a given weak order  $\geq^*$  on  $A$ , the two properties uniquely pin down a weak order over  $\succeq$ : for any pair  $x, y \in A^N$ , it is enough to assume by point (2) that  $x$  and  $y$  are “nondecreasing,” and then rank them according to (1). Transitivity of  $\succeq$  ensures that the ranking of the nondecreasing arrangements coincides with the ranking of  $x$  and  $y$ .

If, in the definition of endogenous leximin, the ranking  $\geq^*$  has a utility representation, the connection with the classical leximin ranking becomes clear. Suppose that  $\succeq$  is an endogenous leximin ranking, and  $\geq^*$  has a utility representation  $u$ , so that  $a \geq^* b$  iff  $u(a) \geq u(b)$ . Then  $x \succeq y$  iff  $(u(x_1), \dots, u(x_n)) \geq_l (u(y_1), \dots, u(y_n))$ . The phrase “endogenous” refers to the fact that the weak order  $\geq^*$  over  $A$  could be anything whatsoever, and need not be representable via a numerical scale.

We are unaware of any abstract studies of the endogenous leximin concept, though the idea has obviously existed in the literature for some time. [Sprumont \(2012\)](#) studies a special case of it, where the ranking  $\geq^*$  is tied to individual preferences via a kind of “agreement” property. More generally, the monograph by [Fleurbaey and Maniquet \(2011\)](#) describes many applications of the idea, where instead of a ranking  $\geq^*$  on  $A$ , a ranking  $\geq^*$  on  $A \times N$  is implicitly considered, and usually based on classical utility constructions. By contrast, individual preferences do not appear in our work.

The following proposition is immediate, so we state it without proof.

**Proposition 2.** *A ranking  $\succeq$  is sufficientarian if and only if it is endogenous leximin with weak order  $\geq^*$ , where  $\geq^*$  has at most two equivalence classes.*

Proposition 2 by itself does not say much about sufficientarian rankings. Any anonymous Bergson-Samuelson social welfare function applied to a ranking with two equivalence classes would give the same result. This is because there is only one monotone and symmetric method of ranking vectors of two-values.



Nevertheless, here we show that an obvious weakening of sufficientarian judgment is closely related to endogenous leximin. Instead of requiring that a loss in wealth for one agent cannot be compensated for partially or fully by a change in some other agent's consumption, we require instead only that it cannot be fully compensated.

**Weak sufficientarian judgment:** Let  $a, b, c \in A$  and suppose that  $i \neq j$ . Then  $b \succ a_i b$  implies  $b \succ c_j(a_i b)$ .

The following result establishes that endogenous leximin emerges upon weakening sufficientarian judgment to weak sufficientarian judgment. It is not a full characterization, nevertheless, it demonstrates a connection of leximin to sufficientarianism.

**Proposition 3.** *Any endogenous leximin binary relation  $\succeq$  satisfies weak order, symmetry, separability, and weak sufficientarian judgment. If  $|N| = 2$ , there are no other binary relations satisfying the four properties.*

*Proof.* Verification of the four properties for any endogenous leximin binary relation is routine. Separability is the only nontrivial property to verify, but this can easily be established using the fact that the leximin ranking is itself separable; see e.g. [d'Aspremont and Gevers \(1977\)](#); [Deschamps and Gevers \(1978\)](#); [Barbera and Jackson \(1988\)](#).

For the converse when  $|N| = 2$ , define  $\geq^*$  in the obvious way, namely  $a \geq^* b$  if for any  $c$  and any  $i \in N$ ,  $a_i c \succeq b_i c$ . Separability ensures  $\geq^*$  is well-defined while weak order ensures that  $\geq^*$  is also a weak order.

Now, let  $x, y \in A^N$ . By symmetry and transitivity, suppose without loss that  $x_1 \geq^* x_2$  and  $y_1 \geq^* y_2$ .

We need to ensure that

1. if  $x_2 >^* y_2$ , then  $x > y$
2. if  $x_2 \sim^* y_2$  and  $x_1 >^* y_1$ , then  $x > y$ .

3. if  $x_2 \sim^* y_2$  and  $x_1 \sim^* y_1$ , then  $x \sim y$ .

The third case is obvious by separability and transitivity. In terms of the second case, we have that  $x_1 >^* y_1$  and  $x_2 \geq^* y_2$ , so again by separability and weak order we must have  $x > y$ .

For the first case, if  $x_1 \geq^* y_1$ , then the argument is identical to the second case and we are done. So finally suppose that  $x_2 >^* y_2$  and  $y_1 >^* x_1$ . Observe that  $x = (x_1, x_2) \succeq (x_2, x_2)$ , by separability and  $x_1 \geq^* x_2$ . Again by separability,  $(x_2, x_2) > (x_2, y_2)$ . Now by weak sufficientarian judgment,  $(x_2, x_2) > (y_1, y_2) = y$ . Consequently by transitivity, we establish that  $x > y$ .  $\square$

Finally, we show that the four axioms are not enough to characterize endogenous leximin when  $|A| \geq 3$ . We conjecture that straightforward strengthenings of weak sufficientarian judgment or of the other axioms should suffice for such a characterization, but we leave this to future research.

**Example 3.** Let  $N = \{1, 2, 3\}$ , and  $A = \{a, b, c\}$ . Define a binary relation  $\succeq$  on  $A^N$  by  $(c, c, c) > (b, c, c) > (b, b, c) > (a, c, c) > (b, b, b) > (a, b, c) > (a, b, b) > (a, a, c) > (a, a, b) > (a, a, a)$ , where symmetry and transitivity characterize the remaining parts of the binary relation. It is straightforward to verify that all of the axioms are satisfied here.

It is also easy to see that, were  $\succeq$  endogenous leximin, then  $c >^* b >^* a$  would be necessary, as  $(c, c, c) > (b, c, c)$  and  $(b, b, b) > (a, b, b)$ . Yet endogenous leximin would then require that  $(b, b, b) > (a, c, c)$ , which is false.

## 5 Conclusion

In this work, we have studied the concept of sufficientarianism applied to an abstract framework and also to more structured economic environments. Sufficientarianism was shown to be the unique binary relation satisfying a collection of axioms, namely, symmetry, separability, and sufficientarian judgment.

There are several possible extensions for our work. As a first point, our work is fixed population, and might amount to ‘head-counting.’ A work that allows one to consider multidimensional allocations across different populations would be similarly interesting; see, for example, [Bossert et al. \(2022, 2023\)](#).

Second, sufficientarianism lends itself to a natural cardinal ranking, in which we only count the number of people who reach a given threshold. One could consider imposing a random threshold and studying the expected number of people to reach the threshold. In the case of  $X = \mathbb{R}_+$ , we could clearly end up with the set of all (linear) rankings satisfying a weak form of monotonicity in doing so. But in higher dimensional environments, we simply have not studied what this approach would generate. It is conceivable that the sufficientarian rankings generate as their expectation some type of supermodular and increasing rankings. We leave this to future research.

Threshold sufficientarianism is meaningful for any meet-semilattice, though we have only provided a characterization in semilattices that are meet-complete. Characterizing it more generally may be interesting. Of course, weakening axioms, especially that of completeness of the social ranking, might prove interesting as well.

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