The Axiomatic Structure of Empirical Content

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Decision/Choice Theory

- Normative vs. Positive
- Standard axiomatic approach “not enough” for many positive exercises

Example: Abstract choice over $\{x, y, z\}$.

Observe preference $x \succ y$, $y \succ z$, but not $x \succ z$.

Observed (revealed) preference not transitive. But maybe we simply didn't observe $x \succ z$.

The nature of falsifiable theories

Popper's theories:

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All swans are white:
$\forall s W(s)$

There exists a black swan:
$\exists s B(s)$

The nature of falsifiable theories

Popper's theories:

All swans are white:
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Falsifiable

There exists a black swan:
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Not falsifiable

The nature of falsifiable theories

Popper's theories:

All swans are white: $\forall s W(s)$ — Falsifiable

There exists a black swan: $\exists s B(s)$ — Not falsifiable — Verifiable

Basic goals

Introduce formal notion of falsifiability

All theories making no non-falsifiable claims have special type of universal axiomatization

Establish that broad class of revealed pref. exercises have nice falsifiability properties

Why should we care?

- A unified formal method for empirical content of economic models (Afriat, WARP, SARP, are all special cases)
- Can be brought to new theories as well: individual and group choice, game theory, production, decision theory, etc
- Exists simple algorithm for axiomatizing the empirical content of recursively axiomatized theories

Key aspects of our approach

Partial and finite observability

Absence of observation of black swan $\neq$ observation of absence of black swan

The model through extended example

We use the example of choice theory to describe our results

We imagine that we observe direct comparisons between pairs of objects

We may or may not be able to decide on which pairs are observed (field vs. experiments)

We want to formalize a hypothesis, or theory, about how these direct choices are governed.

What is the world here? The world is a set X, together with a binary relation $\succ^X$. We call this a *structure*.

We do not see all of X, nor do we see all of $\succ^X$. The world (and its cardinality) are unknown.

Instead, we see a finite subset of *data*: $D \subseteq X$, $\succ^D \subseteq \succ^X$

Structure and data

Structure:
$X = \{x, y, z, w\}$
$x \succ^X y$, $y \succ^X z$, $x \succ^X z$, $x \succ^X w$

Structure and data

Structure:
$X = \{x, y, z, w\}$
$x \succ^X y,\ y \succ^X z,\ x \succ^X z,\ x \succ^X w$

Data:
$D = \{x, y, z\}$
$y \succ^D z,\ x \succ^D z$

- Data "contained" in the structure

Structure:

$X = \{1, 2, 3, 4, 5, \ldots\}$

$1 \succ^X 2$, $2 \succ^X 3$, $3 \succ^X 4$,

$n \succ^X n+1$

Structure:
$X = \{1, 2, 3, 4, 5, \ldots\}$
$1 \succ^X 2$, $2 \succ^X 3$, $3 \succ^X 4$,
$n \succ^X n + 1$

Data:
$D = \{1, 2, 3, 4\}$
$2 \succ^D 3$, $3 \succ^D 4$

We do not see the structure, but we have a hypothesis, or **theory** about it

A **theory** is a class of structures closed under isomorphism. We seek to rationalize data with a structure from a theory.

Theories

Preference maximization

The theory of (strict) preference maximization is the class of structures $(X, \succ^X)$ for which $\succ^X$ is a strict ranking over X

Utility maximization

The theory of utility maximization is the class of structures $(X, \succ^X)$ for which $\succ^X$ is a strict ranking with a utility representation $u^X : X \to \mathbb{R}$, where $u^X(x) > u^X(y) \leftrightarrow x \succ^X y$

Preference max and utility max are *different* theories:

Lexicographic world: $X = \mathbb{R}^2$, $\succ^X$ lexicographic preference

Can data falsify a theory?

Data set $\mathcal{D}$:
$D = \{x, y, z\}$
$x \succ^D y$, $y \succ^D z$, $z \succ^D x$

Data exhibits a strict cycle–inconsistent with both preference maximization and utility maximization theories.

It falsifies those theories.
If data does not falsify a theory, we say it is rationalizable.

Note: No dataset taken from the lexicographic structure can falsify the theory of preference maximization.

Can we empirically distinguish between preference and utility maximization?

no

Data set $\mathcal{D}$ rationalizable by preference maximization iff it is rationalizable by utility maximization.

Empirical content ec of a theory: All structures which cannot be empirically distinguished from the theory. (*i.e.* structures that do not rationalize a falsifying data set)

The most permissive theory with the same predictions.

Example: Structure $(X, \succ^X)$, defined by $X = (x, y, z)$, with $x \succ^X y$ cannot rationalize anything falsifying preference maximization; it is part of the empirical content of preference maximization.

Example: Structure $(X, \succ^X)$, defined by $X = (x, y, z)$ with $x \succ^X y$, $y \succ^X z$, $z \succ^X x$ rationalizes cycles. So not part of ec.

Language and symbols

We introduce a symbol $\succ$ to describe the world

$\succ$ is not the same as $\succ^X$, but is used for writing axioms. It is a symbolic representation of $\succ^X$.

Axioms

- Examples of axioms using our symbol $\succ$:
- Irreflexivity: $\forall x, \neg(x \succ x)$
- Transitivity: $\forall x \forall y \forall z, ((x \succ y) \wedge (y \succ z)) \rightarrow x \succ z$
- Asymmetry: $\forall x \forall y, \neg\left((x \succ y) \wedge (y \succ x)\right)$
- Nonsatiation: $\forall x \exists y, y \succ x$

Empirical content of transitivity alone is the class of all structures: theory of transitivity makes no predictions.

Data set $D = \{x, y, z\}$ with $x \succ^D y$, $y \succ^D z$

$\succ^D$ can be extended to $x \succ^X y$, $y \succ^X z$, $x \succ^X z$

What about a cycle: $D = \{x, y, z\}$, $x \succ^D y$, $y \succ^D z$, $z \succ^D x$

Can extend to $x \succ^D y$, $y \succ^D z$, $z \succ^D x$, $y \succ^D x$, $z \succ^D y$, $x \succ^D z$

Violates asymmetry, but not transitivity

UNCAF

An axiom is an UNCAF axiom if it can be written as

$$\forall x_1 \forall x_2 ... \forall x_n, \neg \bigwedge_{i=1}^{K} \varphi_i$$

where each φ_i is $x_j \succ x_k$, $x_j = x_k$, or $x_j \neq x_k$ for some j, k.

- implies universal (Popper)
- rules out collection of observables holding simultaneously

Universal
Negation of
Conjunction of
Atomic
Formulas

Theorem

The empirical content of a given theory T is the theory axiomatized by all the UNCAF axioms satisfied by T.

Empirical content of theories is specified by UNCAF axioms: a special type of universal axiom

Examples of UNCAF axioms

Some UNCAF axioms satisfied by preference maximization:

$$\forall x, \neg(x \succ x)$$

$$\forall x \forall y, \neg\left((x \succ y) \wedge (y \succ x)\right)$$

$$\forall x \forall y \forall z, \neg\left((x \succ y) \wedge (y \succ z) \wedge (z \succ x)\right)$$

$$\forall x \forall y \forall z \forall w, \neg\left((x \succ y) \wedge (y \succ z) \wedge (z \succ w) \wedge (w \succ x)\right)$$

The empirical content of preference maximization

- Class of structures satisfying, for all $n \geq 1$,

$$\forall x_1 ... \forall x_n, \neg \left(\bigwedge_{i=1}^{n-1} (x_i \succ x_{i+1}) \wedge (x_n \succ x_1) \right)$$

- Absence of cycles.
- SARP

Result is flexible: allows us to take some properties as "given." *e.g.* suppose we know that the following asymmetry axiom is satisfied by design:

$$\forall x \forall y, \neg[(x \succ y) \wedge (y \succ x)]$$

Transitivity then implies

$$\forall x \forall y \forall z, \neg[(x \succ y) \wedge (y \succ z) \wedge (z \succ x)]$$

an UNCAF axiom: so transitivity has empirical content *given* previous asymmetry axiom.

A general approach

Result applies broadly, to any first-order language–allowing relation, function, and constant symbols (with appropriate definitions).

Many theories as we have defined them have no first order axiomatization at all (first order: "without quantifying over sets, relations, and functions")

Consequence of our result is a sufficient condition for first order axiomatizability.

Karl Popper and universality

Why is this different from Popper?

Popper said universal: we say UNCAF. Closely related, but not identical.

Consider "all swans are white." Our language specifies relations we can observe.

If we can't observe or talk about non-white swans, we cannot falsify this theory. Partial observability key here.

Distinguish: "All swans are white" vs. "All swans are not non-white." Second is UNCAF when "non-white" is a relation in the language.

Compare with completeness: "All pairs are ranked."

To falsify this, we must have a way of observing when pairs are unranked.

When such observation is possible, "All pairs are ranked" becomes "All pairs are not unranked." An observation of an unranked pair falsifies the theory.

Precedents

Mathematical precedents: Tarski's theorem on "universal theories" (closure under substructure, isomorphism, and unions of chains)
Also, Łoś-Tarski Theorem (first order axiomatizability and closure under substructure)

Economic precedents: Results of Herbert Simon using Tarski's theorem (datasets as substructures)

Psychological precedents: Foundations of measurement.

Conclusion

General method for discussing empirical content (SARP is a well-known special case).

Can be used to establish empirical content of new theories (different theories described in different versions of our paper)