Consistent Representative Democracy

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Abstract

We study axioms which define "representative democracy" in an environment in which agents vote over a finite set of alternatives. We focus on a property that states that whether votes are aggregated directly or indirectly makes no difference. We call this property *representative consistency*. *Representative consistency* can also be understood as a "gerrymandering-proofness" requirement for voting rules. We characterize the class of rules satisfying *unanimity*, *anonymity*, and *representative consistency*. We call these rules "partial priority rules." A partial priority rule can be interpreted as a rule in which each agent can "veto" certain alternatives. We investigate the implications of imposing other axioms to the list specified above. We also study the partial priority rules in the context of specific economic models.

1   Introduction

Our objective is to discuss an interpretation of "representative democracy" in a voting model. In so doing, we provide a meaningful axiomatic definition of the meaning of representative.

In binary social choice, the only voting rule which is *anonymous*, *neutral*, and *positively responsive* is majority rule (May's Theorem, [15]). Majority rule is not compatible with "representative democracy," in an important sense. The alternatives selected by this rule are generally not the same as the alternatives selected when agents are first grouped into "districts," where each district of agents selects a "representative vote" for its members according to the rule. Such a system is referred to a "single-member district" system. In common language, we say that "gerrymandering" is possible.

A natural question is whether or not there *exist* voting rules which can never be manipulated in the sense described above. Our aim is to formally study this question, providing a workable and sufficiently general notion of a "representative alternative" for a group of agents. We assume that the districts into which agents may be grouped are not prespecified.

Society's collective goal is to select one element out of a *finite* set of social alternatives. We suppose that each agent in society submits a vote for one of the alternatives. We assume that a social alternative must be selected based only on the observed votes, not taking into account strategic behavior on the part of voters. We study "rules," which associate a selected social alternative with every possible society of voters and list of votes that they may submit.

When we say a voting rule is democratic, we mean that it *i*) selects any alternative which is unanimously voted for and *ii*) treats all agents identically. These requirements are captured by the properties *unanimity* and *anonymity*, respectively. Unlike May, we make no monotonicity or neutrality assumptions on rules.

We specify notions of "direct democracy" and "indirect democracy." These two notions are discussed with respect to a given rule. In a direct democracy, each member of society votes for an alternative, and an alternative is selected by aggregating these votes according to the rule. In an indirect democracy, agents are partitioned into districts, and each agent votes for an alternative. The rule is then applied within each *district*, resulting in a "representative vote" for each district. The winning alternative in each district takes all of the votes for that district. Thus, each agent in society

is treated as having voted for his district's representative vote. The rule is then applied to the representative votes, leading to a social alternative. We say a rule is a "representative democracy" if direct democracy and indirect democracy lead to the same chosen alternative, independently of the partitioning of agents. The equivalence of direct and indirect vote aggregation is logically independent from the notion of "democracy," so we give it an independent name: *representative consistency*.

The notion of indirect vote aggregation in social choice is familiar from the works of Murakami [17, 18], Fishburn [12, 13], and Fine [11], although these papers are not concerned with the consistency of indirect vote aggregation and direct vote aggregation. The idea behind these papers is that a given "representative system" is fixed; the goal is to understand the general structure of social choice using these representative systems. In our work, we are modelling a situation in which districts are not fixed. Some agent, who is implicitly present, may be in charge of defining these districts. In his famous paper on *strategy-proof* voting schemes, Moulin [16] also briefly touches on an idea related to *representative consistency*. However; he is also not interested in a situation when districts may change. He mentions the following two-stage procedure; whereby in the first stage, a *strategy-proof* rule is used to select a representative vote for each district, and in the second stage, a *strategy-proof* rule is used to aggregate the votes of the representatives. His claim is that the resulting rule is also *strategy-proof*.

Representative consistency is a strong condition of "gerrymandering-proofness."[1] With *almost any* voting rule one can think of, some form of gerrymandering is sometimes possible. However, there are exceptions to this rule.

The conditions *unanimity*, *anonymity*, and *representative consistency* are compatible, and characterize a class of rules that we believe has not been studied before. These rules are easiest to understand when there are only two alternatives. In this case, one of the alternatives is "special," and the rule chooses the special alternative unless all agents vote for the other alternative. Thus, each agent has a "veto" and can force society to choose the special alternative by voting for it.

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[1]By ignoring strategic effects on the part of voters, we are ruling out the possibility that agents may behave differently when placed in different districts. Such behavior may have the effect of "cancelling out" attempts at gerrymandering. A weaker notion of *representative consistency* may be desired in this case; such a notion will depend on the assumed equilibrium concept for voter behavior.

These rules can be described in a different way. Think of the two alternatives as being *ordered*, where the special alternative comes first in the order. Among the alternatives receiving votes, the rule selects the one that comes first in the order. Our main contribution is to show that, for any finite number of alternatives, each rule satisfying our conditions can be identified with a *partial* order over the set of alternatives. If two alternatives are ranked according to this partial order, then we say that the two alternatives are prioritized. Of these two alternatives, the alternative which comes first according to the partial order has a higher priority. The rule selects the lowest priority alternative which has a priority (weakly) higher than all of the alternatives which receive votes. This alternative need not have been voted for by any of the agents. We call such a rule a "partial priority rule."

A typical example of a partial priority rule is found in jury trials. A jury must decide between the guilt or the innocence of a suspect. A suspect is considered innocent unless the jurors unanimously agree that she is guilty. The partial order corresponding to this partial priority rule ranks "guilty" after "innocent." Partial priority rules with three or more alternatives are similar to jury trials with the option of abstention which is communicated by voting for an alternative called "hung jury." Here, if all agents agree on guilty, the suspect is guilty. If all agents agree on innocent, the suspect is innocent. If there is not unanimous agreement for either guilt or innocence, or if some voter chooses to abstain (by voting "hung jury"); however, we say that there is a "hung jury." This corresponds to a partial priority rule which ranks both "guilty" and "innocent" after "hung jury," but leaves "guilty" and "innocent" incomparable.

The class of partial priority rules is nonempty, and there is room to impose other interesting properties on rules. For example, the *positive vote-share* condition requires that any selected alternative must receive at least one vote. When coupled with our other axioms, this condition implies that the partial order corresponding to a partial priority rule is a linear order.

One advantage of our model is its generality. We assume no structure on the set of alternatives except that it is finite. In specific examples, however, restrictions on the set of alternatives are warranted. To this end, we study the partial priority rules in a spatial model. Here, the alternatives are exogenously ordered according to some linear order.

We postulate conditions relating to this exogenous order. *Betweenness* is the requirement that the selected alternative lie between the minimal and maximal votes according to the linear order. If it is implicitly understood

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that agents have single-peaked preferences over the alternatives and votes are the agents' peaks, *betweenness* is equivalent to *Pareto efficiency*. *Vote monotonicity* states that if the votes of all agents move in a certain direction, then so should the selected alternative. *Vote monotonicity* can be understood as a simple *positive responsiveness* condition. Lastly, we study *strategy-proofness*, assuming that agents have preferences over the alternatives which are single-peaked. *Strategy-proofness* states that in a strategic situation in which agents are required to vote for an alternative, it is a dominant strategy for each agent to vote for her peak. When coupled with our other axioms, these axioms lead to three progressively more restrictive classes of rules, which we characterize.

Section 2 discusses the model and the main axioms. Section 3 proves our main theorem and several simple corollaries. Section 4 discusses our theorem in a spatial model. Section 5 concludes.

2 The model

2.1 Preliminaries

Let $\mathbb{N}$ be a set of "potential agents." Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$. Let X be a finite set. Elements of X are "alternatives," to be interpreted as policies, candidates, etc. Each agent submits a "vote," or an element of X. A **rule** is a function $f : \bigcup_{N \in \mathcal{N}} X^N \to X$. Thus, a rule specifies a "representative" alternative for any population of agents and any list of votes these agents may submit.

2.2 Democracy

The first property we discuss in this section states that a rule should respect the "will of the people" when this "will" is unambiguous. It is an extremely weak axiom.

For all $N \in \mathcal{N}$ and all $x \in X$, let x^N be a vector in X^N such that for all $i \in N$, $x_i^N = x$. For all $N \in \mathcal{N}$, all $x \in X^N$, and all $M \subset N$, let x_M be the restriction of x to X^M.

Unanimity: For all $N \in \mathcal{N}$ and all $x \in X$, $f\left(x^N\right) = x$.

The next axiom states that a rule should be ignorant of the names of agents. This is a mild principle of equality, capturing the "one-man, one-vote" principle.

Anonymity: For all $N, N' \in \mathcal{N}$ such that $|N| = |N'|$, all bijections $\sigma : N \to N'$, and all $x \in X^N$ and $x' \in X^{N'}$ such that for all $i \in N$, $x_i = x'_{\sigma(i)}$, $f(x) = f(x')$.

One would expect most democratic voting rules to satisfy these two properties.

2.3   Representative consistency

Informally, *representative consistency* states that for any population of agents and any collection of votes, it is without loss of generality to partition the set of agents into districts, find the choice for each district, and then treat each district as if each agent in the district had voted for the outcome selected for the district.

Representative consistency: For all $N \in \mathcal{N}$, all partitions $\{N_1, ..., N_K\}$ of N, and all $x \in X^N$, $f(x) = f\left(f(x_{N_1})^{N_1}, ..., f(x_{N_K})^{N_K} \right)$.

Under the *unanimity* principle, *representative consistency* is equivalent to the stronger statement that for all $N \in \mathcal{N}$, all $M \subset N$, and all $x \in X^N$, $f(x) = f\left(f(x_M)^M, x_{N \setminus M} \right)$. This latter version of *representative consistency* is more useful in the proofs of theorems.

3   Results

3.1   Preliminaries

Any *anonymous* rule can be specified without reference to the specific names of agents. In the proofs of results in which *anonymity* plays a role, we often exploit this fact without mention, disregarding the variable N.

The following axiom can be interpreted as meaning that only the *proportions* of votes received for each alternative are used in determining the social alternative.

Let m be an integer, let $N \in \mathcal{N}$, and let $x \in X^N$. Let $N' \in \mathcal{N}$ be such that $|N'| = m|N|$. A vector $x' \in X^{N'}$ is an **m-replica of x** if there exists a partition of N' into m sets of size $|N|$, say $\{N_1, ..., N_m\}$ such that for all N_i, there exists a bijection $\sigma_i : N \to N_i$ so that for all $j \in N$, $x_i = x'_{\sigma(i)}$. For all $N \in \mathcal{N}$, $x \in X^N$, $m \in \mathbb{N}$, x^m denotes an m-replica of x.[2]

Replication invariance: Let m be an integer. Let $N \in \mathcal{N}$ and let $x \in X^N$. Let x' be an m-replica of x. Then $f(x') = f(x)$.

The following trivial observation is useful:

Lemma 1: If a rule satisfies *unanimity, anonymity*, and *representative consistency*, then it satisfies *replication invariance*.

Proof: Let $N \in \mathcal{N}$ and let $x \in X^N$. Let x' be an m-replica of x. Then by definition of x', $f(x') = f\left(\underbrace{x, ..., x}_{m}\right)$. By *representative consistency*,

$$f(x') = f\left(\underbrace{f(x), ..., f(x)}_{m|N|}\right). \quad \text{By } unanimity, \ f\left(\underbrace{f(x), ..., f(x)}_{m|N|}\right) = f(x).$$

Thus $f(x') = f(x)$. ∎

Our main result is a characterization of the class of rules satisfying *unanimity, anonymity*, and *representative consistency*. The following example illustrates this class in the two-alternative case. The description of the general class follows.

Example 1: Let $X \equiv \{y, z\}$. Then there are only two rules satisfying *unanimity, anonymity*, and *representative consistency*. One such rule selects $f(x) = y$ unless all agents vote for z. The other such rule always selects z, unless all agents vote for y.

We now generalize Example 1 to the case of an arbitrary (finite) number of alternatives. A **partial order** is a binary relation which is i) reflexive, ii) transitive, and iii) anti-symmetric. A partial order need not be complete.[3]

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[2] An m-replica of a given vector is not uniquely defined.

[3] Reflexive: For all $x \in X$, $x \succeq x$.
Transitive: For all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$. then $x \succeq z$.
Anti-symmetric: For all $x, y \in X$, if $x \succeq y$ and $y \succeq x$, then $x = y$.
Complete: For all $x, y \in X$, either $x \succeq y$ or $y \succeq x$.

For a partial order $\preceq$, $\prec$ denotes the asymmetric part.[4]   A pair $(Y, \preceq)$ is a **partially ordered set** if Y is a set and $\preceq$ is a partial order on Y. For all $x, y \in Y$, where $(Y, \preceq)$ is a partially ordered set, $x \wedge y \in Y$ is the **meet of x and y** if it is the unique greatest lower bound for x and y according to $\preceq$. Generally, two elements $x, y \in Y$ need not possess a meet. A **meet-semilattice** is a partially ordered set such that any pair of elements possesses a meet. For more on these definitions, see Birkhoff [5].

Figure 1 displays a typical meet-semilattice. Each meet-semilattice can be pictured as a directed graph with a unique root. Here, a is the root of the graph. An alternative precedes another alternative in terms of the partial order if one can construct a path emanating from the first alternative up the graph to the other alternative. Thus, in Figure 1, $a \preceq c$ as there exists a path going up the graph starting at a and ending at c. Moreover, $a \preceq e$. However, the alternatives b and c are unrelated according to $\preceq$, as there exists no directed path between them. The meet of two alternatives is easily found as the highest common predecessor of the two alternatives. Thus, $b \wedge e = a$ in the example, whereas $c \wedge e = c$.

A rule f is a **partial priority rule** if there exists a partial order $\preceq$ over X so that $(X, \preceq)$ is a meet-semilattice, and for all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) = \bigwedge_{i \in N} x_i$.[5,6]

Thus, with any partial priority rule, the *proportion* of votes received for each alternative does not matter. To select an alternative, one only needs to determine which alternatives receive a positive number of votes.

3.2   Interpreting partial priority rules

Let f be a partial priority rule with partial order $\preceq$. Let $x, y \in X$ be ordered by $\preceq$, so that $x \preceq y$. We may interpret this statement as reading "x has a higher priority than y." In other words, x is "more important" than y. This interpretation is possible because y can be selected in place of x only if all

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[4]That is, $x \succ y$ if and only if $x \succeq y$ is true and $y \succeq x$ is false.

[5]The notation $\bigwedge_{i \in N} x_i$ refers to the meet of the set of elements $\{x_i\}_{i \in N}$, which exists for any meet-semilattice.

[6]There is no significance to the fact that partial priority rules select the *meet* of the alternatives receiving a positive number of votes. Every partial priority rule can equivalently be described through the use of a partially ordered set in which every pair of elements has a **join** (unique least upper bound), where the rule selects the join of the alternatives receiving a positive number of votes.
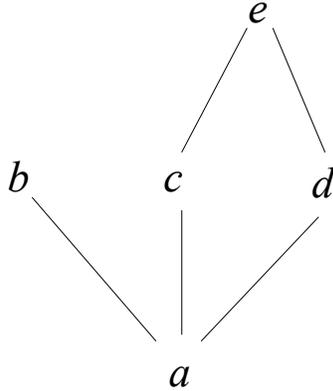
Figure 1: A meet-semilattice

agents vote for y in lieu of x. Thus, each agent can "veto" y by voting for x.

However, not all alternatives need be related by $\preceq$. Suppose in fact that x and y are unrelated according to $\preceq$. Then we may interpret this as meaning that neither x nor y can be deemed a more important alternative. In this case, some "compromise" must be struck between the two. The compromise between x and y is decided using the partial priority. It is the lowest priority alternative which has priority higher than both x and y.

3.3   The main theorem

The main theorem states that a voting rule is a "representative democracy" if and only if it is a partial priority rule.   The proof is constructive and is divided into five steps. Given is a rule satisfying the axioms describing "representative democracy." In the first step, an order on the alternatives is defined from the rule, and it is verified that this order is a partial order. This construction is always valid–even if X is not finite. The most significant obstacle is in showing that this partial order forms a meet-semilattice when coupled with X, which need not be true if X is not finite. We establish this

fact in Steps 2, 3, and 4. Step 2 shows that the rule is "monotonic" in the partial order. Step 3 shows that for two-agent environments, the selected social alternative lies below the two alternatives voted for in terms of the partial order. Step 4 uses Steps 2 and 3 to show that the partial order is actually a meet-semilattice and that for two-agent environments, the rule selects the meet of the alternatives which receive votes. Step 5 establishes that the rule always selects the meet of the alternatives which receive votes.

Theorem 1: A rule satisfies *unanimity*, *anonymity*, and *representative consistency* if and only if it is a partial priority rule.

Proof: If f is a partial priority rule, then it clearly satisfies *unanimity*, *anonymity*, and *representative consistency*. For the other direction, let f be a rule which satisfies *unanimity*, *anonymity*, and *representative consistency*.

Step 1: Construction of the order and verification of its properties.

For all $x,y \in X$, define $x \preceq y$ if $f(x,y) = x$. Let $x \in X$. By *unanimity*, $f(x,x) = x$, so that $x \preceq x$. Thus, $\preceq$ is reflexive. Let $x, y \in X$. Suppose $x \preceq y$ and $y \preceq x$. Then by definition, $f(x,y) = x$ and $f(x,y) = y$. Thus, $x = y$ and $\preceq$ is anti-symmetric.

We claim that $\preceq$ is transitive. Let $x, y, z \in X$. Suppose $x \preceq y$ and $y \preceq z$. By *replication invariance*,

$$f(x,z) = f(x,x,z,z).$$

As $x \preceq y$, by definition,

$$f(x,x,z,z) = f(f(x,y), f(x,y), z, z).$$

By *representative consistency*,

$$f(f(x,y), f(x,y), z, z) = f(x,y,z,z).$$

By *representative consistency*,

$$f(x,y,z,z) = f(x, f(y,z), f(y,z), z).$$

As $y \preceq z$, by definition,

$$f(x, f(y,z), f(y,z), z) = f(x,y,y,z).$$

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By *representative consistency,*

$$f\left(x, y, y, z\right) = f\left(x, y, f\left(y, z\right), f\left(y, z\right)\right).$$

As $y \preceq z$, by definition,

$$f\left(x, y, f\left(y, z\right), f\left(y, z\right)\right) = f\left(x, y, y, y\right).$$

By *representative consistency,*

$$f\left(x, y, y, y\right) = f\left(f\left(x, y\right), f\left(x, y\right), y, y\right).$$

As $x \preceq y$, by definition,

$$f\left(f\left(x, y\right), f\left(x, y\right), y, y\right) = f\left(x, x, y, y\right).$$

By *replication invariance,*

$$f\left(x, x, y, y\right) = f\left(x, y\right).$$

As $x \preceq y$, by definition,

$$f\left(x, y\right) = x.$$

Thus, $f\left(x, z\right) = x$, so that by definition, $x \preceq z$. Therefore, $\preceq$ is transitive.

Step 2: The rule is monotonic in the partial order.

We claim that for all $x, y, z \in X$, if $x \preceq y$, then $f\left(x, z\right) \preceq f\left(y, z\right)$. Thus, let $x, y, z \in X$ and suppose $x \preceq y$. By *replication invariance,*

$$f\left(f\left(x, z\right), f\left(y, z\right)\right) = f\left(f\left(x, z\right), f\left(x, z\right), f\left(y, z\right), f\left(y, z\right)\right).$$

By *representative consistency,*

$$f\left(f\left(x, z\right), f\left(x, z\right), f\left(y, z\right), f\left(y, z\right)\right) = f\left(x, z, y, z\right).$$

By *anonymity,* we may rewrite

$$f\left(x, z, y, z\right) = f\left(x, y, z, z\right).$$

By *representative consistency,*

$$f\left(x, y, z, z\right) = f\left(f\left(x, y\right), f\left(x, y\right), z, z\right).$$

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By *replication invariance*,

$$f\left(f\left(x,y\right),f\left(x,y\right),z,z\right)=f\left(f\left(x,y\right),z\right).$$

As $x \preceq y$, by definition,

$$f\left(f\left(x,y\right),z\right)=f\left(x,z\right).$$

Therefore,

$$f\left(f\left(x,z\right),f\left(y,z\right)\right)=f\left(x,z\right),$$

so that, by definition, $f\left(x,z\right) \preceq f\left(y,z\right)$.

Step 3: The rule maps every two element set into an element which precedes them in the partial order.

We claim that for all $x,y \in X$, $f\left(x,y\right) \preceq x$. Let $x,y \in X$. We define an auxiliary function F derived from f. Let $P \equiv \mathbb{Q} \cap (0,1)$ (this will be the domain of F).

For all $p \in P$, p can be written as $p \equiv \frac{m(p)}{n(p)}$, where $m\left(p\right), n\left(p\right) \in \mathbb{N}$.[7] For all $p \in P$, define $F\left(p\right) = f\left(x^{m(p)}, y^{n(p)-m(p)}\right)$. By *replication invariance*, which follows from Lemma 1, F is well-defined.

We claim that for all $p, p' \in P$ and $z \in X$ such that $p < p'$, if $F\left(p\right) = z$ and $F\left(p'\right) = z$, then for all $p'' \in (p,p') \cap P$, $F\left(p''\right) = z$.

Thus, let $p, p' \in P$ be such that $F\left(p\right) = F\left(p'\right) = z$. Let $p'' \in (p,p') \cap P$. Then there exists $\alpha \in \mathbb{Q}$ such that $p'' = \alpha p + \left(1 - \alpha\right)p'$. Since $\alpha \in \mathbb{Q}$, $\alpha = \frac{m(\alpha)}{n(\alpha)}$ for $m\left(\alpha\right), n\left(\alpha\right) \in \mathbb{N}$. Thus $p'' = \frac{m(\alpha)}{n(\alpha)}\frac{m(p)}{n(p)} + \frac{(n(\alpha)-m(\alpha))}{n(\alpha)}\frac{m(p')}{n(p')}$. We can write $p'' = \frac{m(\alpha)m(p)n(p')+(n(\alpha)-m(\alpha))m(p')n(p)}{n(\alpha)n(p)n(p')}$. Thus, by appropriately arranging terms, $F\left(p''\right)$ is equal to:

$$f\left(\left(x^{m(p)}, y^{n(p)-m(p)}\right)^{n(p')m(\alpha)}, \left(x^{m(p')}, y^{n(p')-m(p')}\right)^{n(p)(n(\alpha)-m(\alpha))}\right) \quad (1)$$

By assumption, $f\left(x^{m(p)}, y^{n(p)-m(p)}\right) = f\left(x^{m(p')}, y^{n(p')-m(p')}\right) = z$, so that by *representative consistency*, (1) is equal to $f\left(z^{n(p)n(p')m(\alpha)}, z^{n(p')n(p)(n(\alpha)-m(\alpha))}\right)$. By *unanimity*, $f\left(z^{n(p)n(p')m(\alpha)}, z^{n(p')n(p)(n(\alpha)-m(\alpha))}\right) = z$. Therefore, $F\left(p''\right) = z$. Hence, F is "convex."[8]

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[7]This representation of p is obviously not unique.

[8]In the proof of an important theorem, Young [23] uses a similar technique of defining an auxiliary function based on proportions, and showing it is "convex."

We next show that F is constant on P. Since X is finite, there exists $z \in X$ and a sequence $\{p^n\}_{n=1}^{\infty}$ of elements of P such that $p^n \to 1$, for which $F(p^n) = z$ for all n. Conclude by the previous paragraph (using the "convexity" of F), that there exists $p^* < 1$ such that for all $p \in (p^*, 1) \cap P$, $F(p) = z$.

We claim that for all $p, q \in P$ and all $\alpha \in (0, 1) \cap \mathbb{Q}$, if $F(p) = F(q)$, then $F(\alpha p) = F(\alpha q)$. Thus, let p, q, α satisfy these hypotheses and suppose $F(p) = F(q) = z$. Write $p = \frac{m(p)}{n(p)}$, $q = \frac{m(q)}{n(q)}$, and $\alpha = \frac{m(\alpha)}{n(\alpha)}$, where $m(p), n(p), m(q), n(q), m(\alpha)$, and $n(\alpha) \in \mathbb{N}$. By definition, $F(p) = f\left(x^{m(p)}, y^{n(p)-m(p)}\right)$ and $F(q) = f\left(x^{m(q)}, y^{n(q)-m(q)}\right)$. By appropriately arranging terms,

$$F(\alpha p) = f\left(\left(x^{m(p)}, y^{n(p)-m(p)}\right)^{m(\alpha)}, y^{n(p)(n(\alpha)-m(\alpha))}\right) \tag{2}$$

and

$$F(\alpha q) = f\left(\left(x^{m(q)}, y^{n(q)-m(q)}\right)^{m(\alpha)}, y^{n(q)(n(\alpha)-m(\alpha))}\right). \tag{3}$$

By *replication invariance*, we obtain (by replicating $n(q)$ times the inside of (2))

$$F(\alpha p) = f\left(\left(x^{m(p)}, y^{n(p)-m(p)}\right)^{n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right)$$

and (by replicating $n(p)$ the inside of (3))

$$F(\alpha q) = f\left(\left(x^{m(q)}, y^{n(q)-m(q)}\right)^{n(p)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right).$$

By *representative consistency*,

$$F(\alpha p) = f\left(f\left(x^{m(p)}, y^{n(p)-n(p)}\right)^{n(p)n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right)$$

and

$$F(\alpha q) = f\left(f\left(x^{m(q)}, y^{n(q)-m(q)}\right)^{n(p)n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right).$$

As $f\left(x^{m(p)}, y^{n(p)-m(p)}\right) = f\left(x^{m(q)}, y^{n(q)-m(q)}\right) = z$,

$$F(\alpha p) = f\left(z^{n(p)n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right)$$

and

$$F(\alpha q) = f\left(z^{n(p)n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right).$$

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Thus $F(\alpha p) = F(\alpha q)$.

Let $q^* \in P$ satisfy $q^* \in (p^*, 1)$. Then $F(q^*) = z$ and $F(p^*) = z$. Moreover, there exists $\alpha^* \in (0, 1) \cap \mathbb{Q}$ such that $p^* = \alpha^* q^*$. For all $k = 1, ..., \infty$, let $q_k \equiv (\alpha^*)^k q^*$. For all k, $q_k > 0$. As $\alpha^* < 1$, $q_k \to 0$.

We claim that for all $k = 1, ..., \infty$, $F(q_k) = z$. By definition, $F(q_0) = F(q_1) = z$. Proceed by induction. Let $K \in \mathbb{N}$ such that $K \geq 2$. Suppose that for all $k < K$, $F(q_k) = z$. We show that $F(q_K) = z$. By definition, $F(q_K) = F(\alpha^* q_{K-1})$. By the induction hypothesis, $F(q_{K-1}) = F(q_{K-2}) = z$, so that by the result in the previous paragraph, $F(\alpha^* q_{K-1}) = F(\alpha^* q_{K-2})$. By definition, $F(\alpha^* q_{K-2}) = F(q_{K-1}) = z$, so that $F(q_K) = F(\alpha^* q_{K-1}) = z$. Thus, for all k, $F(q_k) = z$. By the "convexity" of F, we conclude that F is constant and equal to z on $(0, 1)$.

Lastly, we establish that $f(x, y) \preceq x$. By the symmetry of this statement, we also conclude that $f(x, y) \preceq y$. By *replication invariance*,

$$f(f(x, y), x) = f(f(x, y), f(x, y), x, x).$$

By *representative consistency*,

$$f(f(x, y), f(x, y), x, x) = f(y, x, x, x).$$

By definition,
$$f(y, x, x, x) = F(3/4).$$

By the fact that F is constant on P,

$$F(3/4) = F(1/2).$$

By definition,
$$F(1/2) = f(x, y).$$

Thus, $f(f(x, y), x) = f(x, y)$, so that by definition, $f(x, y) \preceq x$.

Step 4: For two agents, the rule chooses the meet of the alternatives receiving votes

Let $x, y \in X$. We claim that $f(x, y)$ is the unique greatest lower bound for x and y according to $\preceq$. By Step 3, $f(x, y)$ is a lower bound for x and y. Suppose there exists another lower bound for x and y, say, z. Then, by definition, $z \preceq x$ and $z \preceq y$. By Step 2, as $z \preceq y$, $f(x, z) \preceq f(x, y)$. As $z \preceq x$, by definition, $f(x, z) = z$, so that $z \preceq f(x, y)$. Thus, by the

anti-symmetry of $\preceq$, $f(x, y)$ is the unique greatest lower bound for x and y. Therefore, $f(x, y) = x \wedge y$. Thus, $(X, \preceq)$ is a meet-semilattice. Therefore, for all $N \in \mathcal{N}$ and all $x \in X^N$, $\bigwedge_{i \in N} x_i$ is well-defined (as $\wedge$ is associative–see Birkhoff [5], p. 8-10).

Step 5: Extending the result to arbitrary finite numbers of agents

We establish that for all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) = \bigwedge_{i \in N} x_i$. The proof proceeds by induction on the cardinality of the set of agents N. Suppose that $|N| = 1$. By *unanimity*, $f(x) = x = \bigwedge_{i \in N} x_i$. Suppose that $|N| = 2$. Then by Step 4, $f(x) = \bigwedge_{i \in N} x_i$.

Let $K \in \mathbb{N}$ such that $K > 2$. Suppose that for all $N \in \mathcal{N}$ such that $|N| < K$ and all $x \in X^N$, $f(x) = \bigwedge_{i \in N} x_i$. Let $N^* \in \mathcal{N}$ such that $|N^*| = K$. Without loss of generality, write $N^* \equiv \{1, ..., K\}$. Let $x \in X^{N^*}$. By *representative consistency*,

$$f(x_1, ..., x_K) = f\left(f(x_1, ..., x_{K-1}), ..., f(x_1, ..., x_{K-1}), x_K\right).$$

By the induction hypothesis,

$$f\left(f(x_1, ..., x_{K-1}), ..., f(x_1, ..., x_{K-1}), x_K\right) = f\left(\bigwedge_{i=1}^{K-1} x_i, ..., \bigwedge_{i=1}^{K-1} x_i, x_K\right).$$

By *representative consistency* (using the fact that $K > 3$),

$$f\left(\bigwedge_{i=1}^{K-1} x_i, ..., \bigwedge_{i=1}^{K-1} x_i, x_K\right) = f\left(\begin{array}{c}\bigwedge_{i=1}^{K-1} x_i, f\left(\bigwedge_{i=1}^{K-1} x_i, ..., \bigwedge_{i=1}^{K-1} x_i, x_K\right), ..., \\ f\left(\bigwedge_{i=1}^{K-1} x_i, ..., \bigwedge_{i=1}^{K-1} x_i, x_K\right)\end{array}\right).$$

By the induction hypothesis, the previous expression is equal to

$$f\left(\bigwedge_{i=1}^{K-1} x_i, \bigwedge_{i=1}^{K} x_i, ..., \bigwedge_{i=1}^{K} x_i\right).$$

By *representative consistency*,

$$f\left(\bigwedge_{i=1}^{K-1} x_i, \bigwedge_{i=1}^{K} x_i, ..., \bigwedge_{i=1}^{K} x_i\right) = f\left(\begin{array}{c}f\left(\bigwedge_{i=1}^{K-1} x_i, \bigwedge_{i=1}^{K} x_i\right), f\left(\bigwedge_{i=1}^{K-1} x_i, \bigwedge_{i=1}^{K} x_i\right), \\ \bigwedge_{i=1}^{K} x_i, ..., \bigwedge_{i=1}^{K} x_i\end{array}\right).$$

By the induction hypothesis, the previous expression is equal to

$$f\left(\bigwedge_{i=1}^{K} x_i, ..., \bigwedge_{i=1}^{K} x_i\right).$$

By *unanimity*, $f\left(\bigwedge_{i=1}^{K} x_i, ..., \bigwedge_{i=1}^{K} x_i\right) = \bigwedge_{i=1}^{K} x_i$. Therefore, our induction hypothesis is true. $\blacksquare$

The following is a natural example of a partial priority rule.

Example 2: Sets of alternatives Let A be a finite set. We study the power set of A. Thus, 2^A is naturally ordered by set inclusion, and it is clear that $\left(2^A, \subset\right)$ forms a meet-semilattice, where for all $B, C \in 2^A$, $B \wedge C = B \cap C$. Define f so that, for all $N \in \mathcal{N}$ and all $B \in \left(2^A\right)^N$, $f\left((B_i)_{i \in N}\right) \equiv \bigcap_{i \in N} B_i$. Thus, f selects all "alternatives" which are voted for by all agents.

The partial priority rules satisfy the three axioms no matter what the cardinality of X is. But in order for them to be the *only* rules satisfying the three axioms, it is necessary that X be finite. Here is an example of a set X which is countably infinite, and a rule f on X satisfying the three axioms which is *not* a partial priority rule.

Example 3: A countably infinite set of alternatives Let X be countably infinite. Let $r : X \to \mathbb{Q}$ be a bijection (such a bijection exists as X and $\mathbb{Q}$ have the same cardinality). Define $f : \bigcup_{N \in \mathcal{N}} X^N \to X$ as

$$f\left((x_i)_{i \in N}\right) \equiv r\left(\frac{\sum_{i \in N} r^{-1}(x_i)}{|N|}\right).$$

It is simple to verify that f satisfies the three axioms listed in the theorem. To see that f is not a partial priority rule, let N be a two-agent set and N' be a three agent set. Let $x, y \in X$ such that $x \neq y$. Then $(x, y) \in X^N$ and $(x, x, y) \in X^{N'}$. Suppose f is a partial priority rule. Then $f(x, y) = x \wedge y = x \wedge x \wedge y = f(x, x, y)$. By definition, $f(x, y) = r\left(\frac{r^{-1}(x)+r^{-1}(y)}{2}\right)$ and $f(x, x, y) = r\left(\frac{2r^{-1}(x)+r^{-1}(y)}{3}\right)$. As r is a bijection, conclude $\frac{r^{-1}(x)+r^{-1}(y)}{2} = \frac{2r^{-1}(x)+r^{-1}(y)}{3}$. Simple algebra allows us to conclude $r^{-1}(x) = r^{-1}(y)$. As r is a bijection, $x = y$. But we

supposed $x \neq y$, a contradiction. A partial order $\preceq$ can be defined from f as in the proof of Theorem 1, Step 1. However, it is immediate that if $x \neq y$, then $f(x,y) \notin \{x,y\}$. This means that no pairs of alternatives are comparable, so that $\preceq$ is the trivial partial order in which all alternatives are only comparable to themselves. Thus, $(X, \preceq)$ does not form a meet-semilattice. [9][10]

3.4   Positive vote-share and priority orderings

A partial priority rule cannot generally be interpreted as prioritizing all alternatives. Here, we introduce a condition which implies that the partial order underlying a partial priority rule is a linear order. It states that an alternative cannot be selected without receiving any votes. It is natural to impose when X is a set of candidates, for example.

Positive vote-share: For all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) \in \{x_i\}_{i \in N}$.

The following corollary states that partial priority rules which satisfy *positive vote-share* are exactly those partial priority rules which prioritize all alternatives.[11]

Corollary 1: A rule f satisfies *unanimity, anonymity, representative consistency,* and *positive vote-share* if and only if there exists a linear order $\preceq$ over X such that for all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) \equiv \min_{\preceq} \{x_i\}_{i \in N}$.

Proof: It is simple to verify that the axioms are satisfied by any such rule.

For the other direction, let $\preceq$ be the partial order constructed from f in Theorem 1. For all $x, y \in X$, *positive vote-share* implies that $f(x,y) \in \{x,y\}$. By definition of $\preceq$, either $x \preceq y$ or $y \preceq x$. Hence $\preceq$ is complete, so that it is a linear order. ■

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[9]We are able to describe a large class of rules on infinite sets of alternatives which satisfy our three axioms. Essentially, these rules work by composing the partial priority rules with rules similar to those described in the example.

[10]Infinite sets of alternatives are useful in modelling "proportional representation." Instead of voting for alternatives, agents vote for lotteries over the alternatives. The interpretation of a lottery is that the governing body is composed of proportions of agents who support each particular alternative.

[11]A **linear order** is any binary relation which is *i)* complete, *ii)* transitive, and *iii)* anti-symmetric.

The preceding corollary can be proved directly without referring to Theorem 1. In fact, a crucial step in the proof of Theorem 1 relies on the fact that X is finite. However, the preceding corollary can be proved even without the finiteness of X. The reasoning is simple. In the proof of the main theorem, we define the partial order over X by $x \preceq y$ if $f(x, y) = x$. However, we had to determine that $(X, \preceq)$ is a meet-semilattice (Step 2, Step 3, and Step 4). When *positive vote-share* is satisfied, $(X, \preceq)$ is easily seen to be a meet-semilattice, as $\preceq$ is complete.

4   The spatial model

As noted, the partial order corresponding to a partial priority rule is interpreted as a *priority* of alternatives and not as a way of relating the similarity of alternatives. However, there is often a natural, exogenous way of ordering the set of alternatives. In the remainder of this section, we suppose X is exogenously endowed with a linear order $\leq^*$. We interpret $\leq^*$ as ordering the alternatives by some attribute in a fashion agreed upon by all potential agents. Thus, for all $x, y, z \in X$, $x \leq^* y \leq^* z$ is to be read as "y is more similar to x than z is" in terms of the attribute.

The analysis of this section leads to three progressively more restrictive classes of rules.

4.1   Formal notions of compromise

The first condition states that the selected social alternative should be a "compromise" amongst the agents' votes, in the weak sense that it lies between the minimal and maximal votes according to $\leq^*$. As mentioned in the Introduction, if agents possess single-peaked preferences over X (to be defined formally below), *betweenness* is equivalent to requiring that a rule select *Pareto efficient* alternatives.

Betweenness: For all $N \in \mathcal{N}$ and all $x \in X^N$, $\min_{\leq^*} \{x_i\}_{i \in N} \leq^* f(x) \leq^* \max_{\leq^*} \{x_i\}_{i \in N}$.

The next condition is a weak monotonicity condition. If all agents' votes move in a certain direction, so should the selected social alternative. This is a simple condition reflecting the fact that the selected social alternative is "representative" of the agents' votes.

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Vote monotonicity: For all $N \in \mathcal{N}$ and all $x, y \in X^N$, if for all $i \in N$, $x_i \leq^* y_i$, then $f(x) \leq^* f(y)$.

The following lemmas will be useful.

Lemma 2: If a rule satisfies *betweenness*, then it satisfies *unanimity*.

Proof: Let f be a rule that satisfies *betweenness*. Let $N \in \mathcal{N}$ and let $x \in X$. Then $\min_{\leq^*} \left\{x_i^N\right\}_{i \in N} = x$ and $\max_{\leq^*} \left\{x_i^N\right\}_{i \in N} = x$. By *betweenness*, $x \leq^* f\left(x^N\right) \leq^* x$. As $\leq^*$ is anti-symmetric, $f\left(x^N\right) = x$. Therefore, f satisfies *unanimity*. ■

Lemma 3: If a rule satisfies *unanimity* and *vote monotonicity*, then it satisfies *betweenness*.

Proof: Let f be a rule that satisfies *unanimity* and *vote monotonicity*. Let $N \in \mathcal{N}$ and let $x \in X^N$. Then for all $i \in N$, $\min_{\leq^*} \left\{x_i\right\}_{i \in N} \leq^* x_i \leq^* \max_{\leq^*} \left\{x_i\right\}_{i \in N}$. By *unanimity*, $f\left(\left(\min_{\leq^*} \left\{x_i\right\}_{i \in N}\right)^N\right) = \min_{\leq^*} \left\{x_i\right\}_{i \in N}$ and $f\left(\left(\max_{\leq^*} \left\{x_i\right\}_{i \in N}\right)^N\right) = \max_{\leq^*} \left\{x_i\right\}_{i \in N}$. Thus, by *vote monotonicity*, $\min_{\leq^*} \left\{x_i\right\}_{i \in N} \leq^* f(x)$ and $f(x) \leq^* \max_{\leq^*} \left\{x_i\right\}_{i \in N}$. Thus f satisfies *betweenness*. ■

4.2   Results on compromise

We begin this section by defining two subclasses of the partial priority rules, which depend on the linear order $\leq^*$.

Say that a rule f is an **interval partial priority rule** if it is a partial priority rule and for all $x, y, z \in X$, if $y, z \geq^* x$, then $y \wedge z \geq^* x$, and if $y, z \leq^* x$, then $y \wedge z \leq^* x$. Thus, a partial priority rule is an interval partial priority rule if the weak lower and upper contour sets of any alternative according to $\leq^*$ are themselves meet-semilattices under the partial order induced by $\preceq$. The term "interval" refers to the fact that restricted to any interval according to $\leq^*$, the rule is still a partial priority rule. Say that a rule f is a **separating partial priority rule** if it is an interval partial priority rule such that for all $x, y, z \in X$, if $x \leq^* y \leq^* z$ and $x, z \succeq y$, then $x \wedge z = y$. Thus, an interval partial priority rule is a separating partial priority rule if the weak lower and upper contour sets of any alternative according to
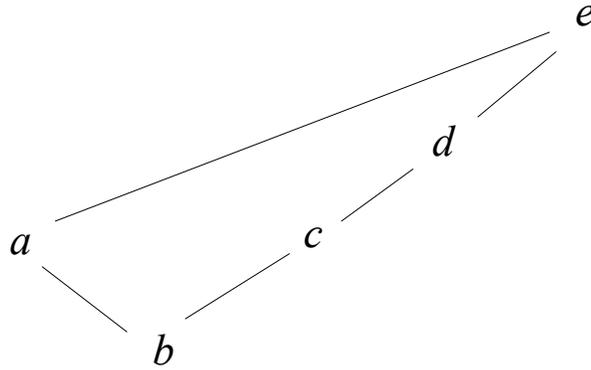
Figure 2: An interval partial priority rule

$\leq^*$ never "intersect" at any element greater than the alternative itself. The term "separating" refers to the fact that these contour sets never intersect.

Figure 2 displays the meet-semilattice corresponding to a typical interval partial priority rule under the presumption that $a \leq^* b \leq^* c \leq^* d \leq^* e$. The general structure of such rules can be grasped from this diagram. The main characteristic of such a rule is that there are at most two branches emanating from any alternative. If there are two branches, one must branch into the upper contour set of the alternative, and the other must branch into the lower contour set (as in the diagram, the two branches emanating from b branch into the two different contour sets). These two branches are allowed to rejoin. It is clear that the meet-semilattice in Figure 2 does not correspond to a separating partial priority rule, as $a \leq^* b \leq^* e$ and $a,e \succeq b$, yet $a \wedge e = a$.

Figure 3 displays the meet-semilattice corresponding to a typical separating partial priority rule. The general structure of separating partial priority rules can be understood from this diagram. Note that the alternatives $a, b, c,$ d, and e are ordered from "left" to "right." Moreover, at most two branches emanate from any given alternative. The last important characteristic is
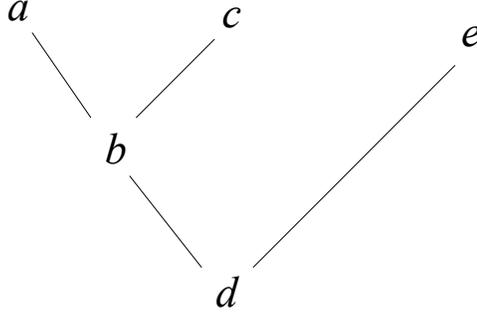
Figure 3: A separating partial priority rule

that for any alternative, one can draw an imaginary vertical line branching up from the alternative which never crosses the graph. Thus, the graph corresponding to a separating partial priority rule is a system of interlocking "y"-shapes. Such a graph looks like an upside-down game tree in which at any given point, there are only two possible moves.

Theorem 2: A rule satisfies *betweenness*, *anonymity*, and *representative consistency* if and only if it is an interval partial priority rule.

Proof: Suppose f is an interval partial priority rule. We already know that it satisfies *anonymity* and *representative consistency*, so we will show that it satisfies *betweenness*. Thus, let $N \in \mathcal{N}$ and $x \in X^N$. As f is an interval partial priority rule, and since for all $i \in N$, $\min_{\leq^*} \{x_i\}_{i \in N} \leq^* x_i$, we obtain $\min_{\leq^*} \{x_i\}_{i \in N} \leq^* \bigwedge_{i \in N} x_i$. A similar expression holds for $\max_{\leq^*} \{x_i\}_{i \in N}$, so that f satisfies *betweenness*.

Now suppose that f is a rule satisfying the axioms. By Lemma 2, we know that it is a partial priority rule. We will verify that f is an interval partial priority rule. Let $x, y, z \in X$, and suppose that $y, z \geq^* x$. Without loss of generality, assume that $y \geq^* z \geq^* x$. By *betweenness*, $f(y, z) \geq^* z$.

By the transitivity of $\leq^*$, $f(y,z) \geq^* x$. By definition, $f(y,z) = y \wedge z$. Thus, $y \wedge z \geq^* x$. A similar statement holds for the lower contour set of x.∎

Theorem 3 demonstrates that a partial priority rule is *vote monotonic* if and only if it is a separating partial priority rule.

Theorem 3: A rule satisfies *unanimity, anonymity, representative consistency,* and *vote monotonicity* if and only if it is a separating partial priority rule.

Proof: By Lemma 3 and Theorem 2, if f satisfies the axioms, it is an interval partial priority rule. Let $x, y, z \in X$ satisfy $x \leq^* y \leq^* z$ and $x, z \succeq y$. By *vote monotonicity*, $x \wedge z = f(x,z) \leq^* f(y,z) = y$ and $x \wedge z = f(x,z) \geq^* f(x,y) = y$. By the anti-symmetry of $\leq^*$, $x \wedge z = y$. Thus, f is a separating partial priority rule.

For the other direction, let f be a separating partial priority rule. We will show that it is *vote monotonic*.

Let $\preceq$ be the partial order associated with f. Say a partial priority rule f satisfies condition $(*)$ if for all $x, y, z \in X$, if $x \leq^* y$, then $f(x,z) \leq^* f(y,z)$. We claim that if a partial priority rule satisfies condition $(*)$, then it is *vote monotonic*. To see why, suppose $(*)$ is true. Let $N \in \mathcal{N}$, and let $x, y \in X^N$. Suppose that for all $i \in N$, $x_i \leq^* y_i$. Suppose, without loss of generality, that N can be ordered as $N \equiv \{1, ..., n\}$. For all $m = 1, ..., n$, let $y_m x \equiv (y_1, ..., y_m, x_{m+1}, ..., x_n)$. We show by induction that for all $m = 1, ..., n$, $f(x) \leq^* f(y_m x)$. Let $m = 1$. Then $x_1 \leq^* y_1$. By condition $(*)$, $x_1 \wedge (\bigwedge_{i=2}^n x_i) \leq^* y_1 \wedge (\bigwedge_{i=2}^n x_i)$. Thus, $f(x) \leq^* f(y_1 x)$. Let $M \leq n$ be some integer. Now, suppose that for all $m < M$, $f(x) \leq^* f(y_m x)$. By assumption, $x_M \leq^* y_M$. By condition $(*)$,

$$
x_M \wedge \left(\left(\bigwedge_{i=1}^{M-1} y_i \right) \wedge \left(\bigwedge_{i=M+1}^{n} x_i \right) \right)
$$
$$
\leq^* y_M \wedge \left(\left(\bigwedge_{i=1}^{M-1} y_i \right) \wedge \left(\bigwedge_{i=M+1}^{n} x_i \right) \right).
$$

The left-hand side of this expression is $f(y_{M-1} x)$, and the right-hand side is $f(y_M x)$. Thus, $f(y_{M-1} x) \leq^* f(y_M x)$. By transitivity of $\leq^*$, $f(x) \leq^* f(y_{M-1} x) \leq^* f(y_M x)$ implies $f(x) \leq^* f(y_M x)$. Thus, for all $m = 1, ..., n$, $f(x) \leq^* f(y_m x)$. In particular, if $m = n$, then $y_n x = y$ and $f(x) \leq^* f(y)$.

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We now show that if a rule is a separating partial priority rule, then it satisfies condition $(*)$. Let $x, y, z \in X$, and suppose that $x \leq^* y$. There are three possible cases.

a) $y \leq^* z$: In this case, as f is a separating partial priority rule, it is an interval partial priority rule. Thus, $y \leq^* y \wedge z \leq^* z$. If $x \wedge z \leq^* y \wedge z$, we are done. Otherwise, $y \wedge z \leq^* x \wedge z$. As f is an interval partial priority rule, $y \wedge z \leq^* (y \wedge z) \wedge (x \wedge z) \leq^* x \wedge z$. As $(y \wedge z) \wedge (x \wedge z) = x \wedge y \wedge z$, $y \leq^* x \wedge y \wedge z \leq^* z$. By definition, $y, z \succeq x \wedge y \wedge z$. As f is a separating partial priority rule, conclude $y \wedge z = x \wedge (y \wedge z)$. Thus, by definition, $y \wedge z \preceq x$. Thus, $x, z \succeq y \wedge z$ and $x \leq^* y \wedge z \leq^* z$. As f is a separating partial priority rule, conclude $x \wedge z = y \wedge z$. Thus, $x \wedge z \leq^* y \wedge z$.

b) $x \leq^* z \leq^* y$: In this case, as f is an interval partial priority rule, $x \leq^* x \wedge z \leq^* z \leq^* y \wedge z \leq^* y$, so that $x \wedge z \leq^* y \wedge z$.

c) $z \leq^* x$: In this case, as f is a separating partial priority rule, it is an interval partial priority rule. Thus, $z \leq^* x \wedge z \leq^* x$. If $x \wedge z \leq^* y \wedge z$, we are done. Otherwise, $y \wedge z \leq^* x \wedge z$. As f is an interval partial priority rule, $y \wedge z \leq^* (y \wedge z) \wedge (x \wedge z) \leq^* x \wedge z$. As $(y \wedge z) \wedge (x \wedge z) = x \wedge y \wedge z$, $z \leq^* x \wedge y \wedge z \leq^* x$. By definition, $x, z \succeq x \wedge y \wedge z$. As f is a separating partial priority rule, conclude $x \wedge z = y \wedge (x \wedge z)$. Thus, by definition, $x \wedge z \preceq y$. Thus, $y, z \succeq x \wedge z$ and $z \leq^* x \wedge z \leq^* y$. As f is a separating partial priority rule, conclude $y \wedge z = x \wedge z$. Thus, $x \wedge z \leq^* y \wedge z$.∎

4.3   Strategic considerations in the spatial model

We now study a domain of *preferences* over X. Say a binary relation R on X is **single-peaked** if there exists $x(R) \in X$ such that for all $x, y \in X$, if $x(R) \leq^* x <^* y$ or if $x(R) \geq^* x >^* y$, xPy.[12] Let $SP(X, \leq^*)$ be the set of all single-peaked binary relations for the linearly ordered set $(X, \leq^*)$.[13]

We ask which partial priority rules satisfy the following condition, which states that if votes are taken to be agents' peaks, then no agent can ever benefit by lying about his peak. This condition was first studied in this model by Moulin [16].

Strategy-proofness: For all $N \in \mathcal{N}$, all $R \in SP(X, \leq^*)^N$, all $j \in N$, and

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[12]As usual, for a binary relation R, P denotes the asymmetric part of R. The alternative $x(R)$ is called the **peak** of R.

[13]In this environment, a rule selects only *Pareto efficient* alternatives if and only if it satisfies *betweenness* with respect to the peaks.
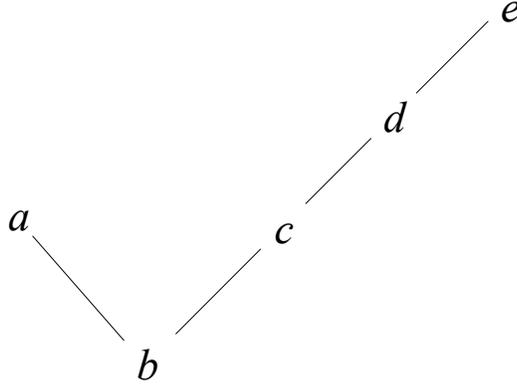
Figure 4: A target rule

all $R_j' \in SP(X, \leq^*)$, $f\left((x(R_i))_{i \in N}\right) R_j f\left((x(R_i))_{i \in N \setminus \{j\}}, x(R_j')\right)$.

Say a partial priority rule f is a **target rule** if there exists $x^* \in X$ such that for all $y, z \in X$, *i)* $z \leq^* y \leq^* x^*$ implies $z \succeq y$, *ii)* $x^* \leq^* y \leq^* z$ implies $z \succeq y$, and *iii)* $y \leq^* x^* \leq^* z$ implies $y \wedge z = x^*$. Thus, a target rule is a partial priority rule with two branches, rooted at x^*, one of which agrees with the order $\leq^*$, the other of which is the opposite of the order $\leq^*$. Target rules were first discussed by Thomson and Thomson and Ching [21, 22] using a different set of axioms. The terminology belongs to them. The word "target" refers to the element $x^* \in X$ discussed in the definition. A target rule selects the alternative in between the agents votes which lies closest (in terms of $\leq^*$) to the target.

In Figure 4, we display the meet-semilattice corresponding to a typical target rule. Here, the alternative x^* corresponds to b. The partial orders corresponding to target rules all have the "v"-shape as depicted here.

Theorem 4: A rule satisfies *unanimity, anonymity, representative consistency,* and *strategy-proofness* if and only if it is a target rule.[14]

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[14]For a fixed population $N \in \mathcal{N}$, it follows that the target rules must be generalized

Proof: Clearly, the target rules are partial priority rules, and thus satisfy *unanimity*, *anonymity*, and *representative consistency*. It is well-known that they also satisfy *strategy-proofness*. Thus, we will show that any rule satisfying the axioms must be a target rule.

First, we show that for all $x, y, z \in X$, if $x \preceq y \preceq z$, then either $x \leq^* y \leq^* z$ or $z \leq^* y \leq^* x$. Call this property the **chain property**. Suppose that this statement is false. There are four alternative orderings of x, y, z according to $\leq^*$. We will only show that one such ordering leads to an impossibility. The remaining three cases can be similarly proved. Thus, suppose $y \leq^* z \leq^* x$. Suppose, without loss of generality, that $y <^* z$ (this is without loss of generality, as otherwise, $z \leq^* y \leq^* x$ holds). Let $R \in SP(X, \leq^*)$ such that $x(R) = z$ and xPy. Then as $y \preceq z$, $f(x(R), y) = y$. However, $f(x, y) = xPy = f(x(R), y)$, in contradiction to *strategy-proofness*.

Next, we show that f is an interval partial priority rule. Thus, let $x, y, z \in X$ such that $x, y \geq^* z$. If $x = y$, then clearly $x \wedge x = x$. So suppose that $x \neq y$, and suppose $x \wedge y <^* z$. Without loss of generality, suppose that $y >^* x$. Let $R \in SP(X, \leq^*)$ satisfy $x(R) = y$. Then $f(x, x) = xP(x \wedge y) = f(x(R), x)$, a contradiction to *strategy-proofness*. The case in which $x, y \leq^* z$ is proved similarly.

Let $x^* \equiv \bigwedge_{x \in X} x$. By definition of x^*, for all $x \in X$, $x^* \preceq x$. We prove that $i)$ in the definition of target rule is true; $ii)$ follows from a symmetric argument.

Let $y, z \in X$ such that $z \leq^* y \leq^* x^*$. We show that $z \succeq y$. As f is an interval partial priority rule, $z \leq^* y \wedge z \leq^* y$. Moreover, by definition, $y \succeq y \wedge z \succeq x^*$, so that by the chain property and the fact that $x^* \geq^* y$, $x^* \geq^* y \wedge z \geq^* y$. Thus, $y \wedge z \leq^* y$ and $y \wedge z \geq^* y$, so that by the anti-symmetry of $\leq^*$, $y \wedge z = y$. Thus, by definition, $z \succeq y$.

To verify $iii)$ in the definition of target rule, let $y, z \in X$ satisfy $y \leq^* x^* \leq^* z$. As $y \succeq y \wedge z \succeq x^*$, by the chain property, $x^* \geq^* y \wedge z \geq^* y$. As $z \succeq y \wedge z \succeq x^*$, by the chain property, $x^* \leq^* y \wedge z \leq^* z$. Therefore, $x^* \geq^* y \wedge z$ and $x^* \leq^* y \wedge z$. By the anti-symmetry of $\leq^*$, $y \wedge z = x^*$. ∎

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median voter rules, as defined in Moulin [16].

5 Discussion and conclusion

5.1 Previous literature related to representative consistency

Representative consistency is mathematically not a new condition; however, we believe that the interpretation here is. Blackorby and Donaldson [6] study a condition called the "population substitution principle," which is the same as *representative consistency*. Their interest in the condition is in comparing the welfares of varying groups of agents. The literature on functional equations has long studied this (dating back to 1930–see [14, 19], as well as [7]) and related notions. These works use the notion to characterize the quasi-arithmetic means [1, 2]. For a good introduction, see Diewert [8]. By establishing that *representative consistency* is an "averaging" condition, these works add further normative justification to the property. The main theoretical distinction between our work and these works is that the cited works require the domain of a rule to be a continuum, usually the unit interval. To our knowledge, no other work has allowed the domain to be a finite set.

The condition of *path independence*, first formalized by Plott [20], bears a striking resemblance to *representative consistency*. However, *path independence* is a condition which is well-defined only for choice correspondences. Hence, *representative consistency* and *path independence* are not logically related. However, Theorem 5 of Plott [20] is closely related to our Corollary 1. Characterizations of the family of choice rules satisfying *path independence* are provided by Aizerman and Aleskerov [3], pp. 167-168. See also Asah and Sanver [4].

Finally, the first works to study indirect vote aggregation in social choice models are those which study representative systems, mentioned in the introduction. This literature was introduced by Murakami [17, 18], and later studied by Fishburn [12, 13] and Fine [11]. These works introduce the distinction between direct and indirect voting in formal social choice models. Their interest is in understanding which voting rules can arise as representative systems when taking plurality rule as primitive and by allowing different voters and districts to have different "weights." The distinction is that they require no consistency between indirect and direct voting. Rather, they study which rules can be thought of as arising from indirect voting, where majority rule is applied at each level. Moulin [16], as also mentioned in

the introduction, discusses the related idea of showing that a "representative system" of *strategy-proof* rules is itself *strategy-proof.*

5.2   Other principles of democracy

An upsetting aspect of the partial priority rules is that they violate the principle of *neutrality.* Informally speaking, *neutrality* requires that the "names" of alternatives should not matter for a voting rule. Unfortunately, in a pure voting model, the properties *anonymity* and *neutrality* are incompatible when rules must be *single-valued* and environments with even populations of agents are permissible. Thus, a natural extension of our model would include rules which are allowed to be *multi-valued.* Such an extension would require redefining *representative consistency* for this case. Whether or not similar results obtain depends on this notion. *Neutrality* can also be captured when lotteries are permitted.

5.3   General comments

Another negative aspect of the partial priority rules is the amount of power they give to each individual. *Any* individual can "veto" an alternative with low priority simply by voting for an alternative with a higher priority. Similar voting rules have been studied in strategic models where information about the social alternatives is incomplete [9, 10]. It is generally found that these voting rules fare worse than other standard *anonymous* rules in terms of their ability to correctly aggregate information.

In representative democracies with single-member districts, the usual practice is to select an actual agent to represent a district of voters. We implicitly identify this representative agent with the alternative she supports. To give each representative the same voting power, and to preserve *anonymity* of a voting rule, districts of agents must be equipopulous. We ask what happens to Theorem 1 when the hypothesis of *representative consistency* is strengthened to require that districts be equipopulous. We show that weakening *representative consistency* as such results in no new rules. This point is discussed in Appendix B.

Another possible objection to the model is that the same voting rule is applied at *all* levels of government. A natural extension of the model would study *two* different voting rules, say f and g. The rule f would be applied at the district level. The rule g would be used for aggregating *representative*

votes at the national level. In this scenario, *representative consistency* would say that the composition of g with f should always select the same alternative, independently of the partitioning of agents into districts. It turns out that if *unanimity* and *anonymity* are required for both voting rules, then the two voting rules coincide and Theorem 1 applies. To see why, let $N \in \mathcal{N}$ and let $x \in X^N$. By putting each agent into her own district, the outcome of the rule is $g\left((f(x_i))_{i \in N}\right)$. By putting all agents in the same district, the outcome of the rule is $g(f(x), ..., f(x))$. By *unanimity* of f, $g\left(f(x_i)_{i \in N}\right) = g(x)$, and by *unanimity* of g, $g(f(x), ..., f(x)) = f(x)$. By *representative consistency*, $g\left(f(x_i)_{i \in N}\right) = g(f(x), ..., f(x))$, so that $g(x) = f(x)$. Hence $g = f$.

A natural question is whether Theorem 1 holds when the set of potential agents is finite. While we have no formal proof, we conjecture that if the cardinality of the set of potential agents is at least three, then a result akin to Theorem 1 holds. A potential approach to solving this problem is presented in Aczél [1] in his characterization of the quasi-arithmetic means based on the bisymmetry equation.

6   Appendix A–Independence of the axioms

The following are examples establishing the independence of our axioms. Each axiom is followed by a rule which violates it and satisfies the other two axioms of Theorem 1.

Unanimity (constant rule): Let f be a constant rule–*i.e.* there exists $z \in X$ such that for all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) \equiv z$.

Anonymity (dictatorship): Let R be a linear order on the set of agents $\mathbb{N}$. For all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) \equiv x_{\arg\max_R N}$.

Representative consistency (median voter): Let $\leq^*$ be a linear order on X. For all $N \in \mathcal{N}$ and all $x \in X^N$,

$$f(x) \equiv \min_{\leq^*} \left\{ x \in \{x_i\}_{i \in N} : |i : x_i \leq^* x| \geq \frac{|N|}{2} \right\}.$$

7   Appendix B–Equipopulous districts

This appendix discusses a restricted version of *representative consistency.*

Weak representative consistency: For all $N \in \mathcal{N}$, all partitions $\{N_1, ..., N_m\}$ of N such that for all i, j, $|N_i| = |N_j|$, and all $x \in X^N$, $f(x) = f\left(f(x_{N_1})^{N_1}, ..., f(x_{N_m})^{N_m}\right)$.

As the name suggests, *weak representative consistency* is a weaker condition than *representative consistency*. Under *unanimity*, *weak representative consistency* has no bite in environments in which the set of agents cannot be partitioned into nontrivial districts of equal cardinalities. This is the case if the set of agents has a cardinality which is a prime number.

We establish that no new rules emerge when requiring that districts be equipopulous. Thus, even if all districts are required to contain the same number of agents, any rule which is not a partial priority rule presents some opportunities for gerrymandering.

Theorem 5 generalizes Theorem 1 using the *weak representative consistency* condition.

Theorem 5: A rule satisfies *unanimity, anonymity,* and *weak representative consistency* if and only if it is a partial priority rule.

The theorem relies on the following lemma, whose proof is identical to Lemma 1. Thus, we state it without proof.

Lemma 2: If a rule satisfies *unanimity, anonymity,* and *weak representative consistency*, then it satisfies *replication invariance*.

The next lemma establishes that if a rule satisfies *unanimity, weak representative consistency,* and *replication invariance*, then it satisfies *representative consistency*. Taken together, Lemmas 2 and 3 will directly imply Theorem 5.

Lemma 3: If a rule satisfies *unanimity, weak representative consistency,* and *replication invariance*, then it satisfies *representative consistency*.

Proof: Let f be a rule that satisfies *unanimity, weak representative consistency,* and *replication invariance*. Let $N \in \mathcal{N}$ and let $x \in X^N$. Let $\{N_1, ..., N_m\}$ be a partition of N. We claim that $f(x) = f\left(f(x_{N_1})^{N_1}, ..., f(x_{N_m})^{N_m}\right)$.

By *replication invariance*,

$$f(x) = f\left(x^{\prod\limits_{i=1}^{m}|N_i|}\right).$$

Rewrite

$$f\left(x^{\prod\limits_{i=1}^{m}|N_i|}\right) = f\left(\left((x_{N_k})^{\prod\limits_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right).$$

Again, rewrite

$$f\left(\left((x_{N_k})^{\prod\limits_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right) = f\left(\left(\left(x_{N_k}^{\prod\limits_{i\neq k}|N_i|}\right)^{|N_k|}\right)_{k=1}^{m}\right).$$

For all $k = 1, ..., m$, $x_{N_k}^{\prod\limits_{i\neq k}|N_i|}$ has cardinality $\prod\limits_{i=1}^{m}|N_i|$. By *weak representative consistency*,

$$f\left(\left(\left(x_{N_k}^{\prod\limits_{i\neq k}|N_i|}\right)^{|N_k|}\right)_{k=1}^{m}\right) = f\left(\left(f\left(x_{N_k}^{\prod\limits_{i\neq k}|N_i|}\right)^{|N_k|\prod\limits_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right).$$

By *replication invariance*, for all $k = 1, ..., m$,

$$f\left(x_{N_k}^{\prod\limits_{i\neq k}|N_i|}\right) = f(x_{N_k}).$$

Thus,

$$f\left(\left(\left(x_{N_k}^{\prod_{i\neq k}^{i\neq k}|N_i|}\right)^{|N_k|\prod_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right) = f\left(\left(f\left(x_{N_k}\right)^{|N_k|\prod_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right).$$

By *replication invariance*,

$$f\left(\left(f\left(x_{N_k}\right)^{|N_k|\prod_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right) = f\left(\left(f\left(x_{N_k}\right)^{|N_k|}\right)_{k=1}^{m}\right).$$

By *anonymity*, which is implied by *replication invariance*,

$$f\left(\left(f\left(x_{N_k}\right)^{|N_k|}\right)_{k=1}^{m}\right) = f\left(f\left(x_{N_1}\right)^{N_1}, ..., f\left(x_{N_m}\right)^{N_m}\right).$$

Thus, $f\left(x\right) = f\left(f\left(x_{N_1}\right)^{N_1}, ..., f\left(x_{N_m}\right)^{N_m}\right)$ and *representative consistency* is satisfied.∎

The proof of Theorem 5 is now simple.

Proof: Let f be a partial priority rule. Then it clearly satisfies *unanimity*, *anonymity*, and *weak representative consistency*.

Let f be a rule that satisfies *unanimity*, *anonymity*, and *weak representative consistency*. By Lemma 2, f satisfies *replication invariance*. By Lemma 3, f satisfies *representative consistency*. Thus, f satisfies *unanimity*, *anonymity*, and *representative consistency*. By Theorem 1, f is a partial priority rule. ∎

References

[1] Aczél, J. (1966): *Lectures on Functional Equations and their Applications*, Academic Press, New York

[2] Aczél, J. (1997): "Bisymmetry and consistent aggregation: Historical review and recent results," *in Choice, Decision, and Measurement: Essays in Honor of R. Duncan Luce*, (A.A.J. Marley, Ed.), Lawrence Erlbaum Associates, New Jersey, 225-233.

[3] Aizerman, M., and F. Aleskerov (1995): *Theory of Choice*, Elsevier Science, Amsterdam.

[4] Asah, G. and M.R. Sanver (2002): "Another characterization of the majority rule," *Economics Letters*, 75, 409-413.

[5] Birkhoff, G. (1967): *Lattice Theory, third edition*, American Mathematical Society, Providence.

[6] Blackorby, C., and D. Donaldson (1984): "Social criteria for evaluating population change," *Journal of Public Economics* 25, 13-33.

[7] de Finetti, B. (1931): "Sul concetto di media," *Giornale dell' Instituto Italiano degli Attuari* 2, 369-396.

[8] Diewert, W.E., (1993): "Symmetric means and choice under uncertainty," *in Essays in Index Number Theory, Vol. 1,* (W.E. Diewert and A.O. Nakamura, Eds.), pp. 355-433, Elsevier Science Publishers, B.V.

[9] Duggan, J., and C. Martinelli (2001): "A Bayesian model of voting in juries," *Games and Economic Behavior* 37, 259-294.

[10] Feddersen, T. and W. Pesendorfer, (1998): "Convicting the innocent: the inferiority of unanimous jury verdicts," *American Political Science Review* 92, 23-35.

[11] K. Fine, (1972): "Some necessary and sufficient conditions for representative decision on two alternatives," *Econometrica* 40, 1083-1090.

[12] Fishburn, P.C., (1971): "The theory of representative majority decision," *Econometrica* 39, 273-284.

[13] Fishburn, P.C., (1973): *The Theory of Social Choice*, Princeton University Press, New Jersey.

[14] Kolmogorov, A., (1930): "Sur la notion de la moyenne," *Rendiconti Accademia dei Lincei (6)* 12, 388-391.

[15] May, K.O., (1952): "A set of independent necessary and sufficient conditions for simple majority decision," *Econometrica* 20, 680-684.

[16] Moulin, H., (1980): "On strategy-proofness and single peakedness," *Public Choice* 35, 437-455.

[17] Murakami, Y., (1966): "Formal structure of majority decision," *Econometrica* 34, 709-718.

[18] Murakami, Y., (1968): *Logic and Social Choice*, Dover, New York.

[19] Nagumo, M., (1930): "Uber eine Klasse der Mittelwerte," *Japan Journal of Mathematics* 7, 71-79.

[20] Plott, C. (1973): "Path independence, rationality, and social choice," *Econometrica* 41, 1075-1091.

[21] Thomson, W., (1993): "The replacement principle in public good economies with single-peaked preferences," *Economics Letters* 42, 31-36.

[22] Thomson, W., and S. Ching, (1993): "Population-monotonic solutions in public good economies with single-peaked preferences," mimeo, University of Rochester.

[23] Young, H.P., (1975): "Social choice scoring functions," *SIAM Journal of Applied Mathematics* 28, 824-838.