# Asymmetric Rules for Claims Problems without Homogeneity

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**Abstract**

We introduce a general class of rules for claims problems, called the *difference rules*, and demonstrate that a rule satisfies *composition down* and *composition up* if and only if it is a difference rule. We show that these rules take a special form when there are two agents. In a variable population framework, we introduce a family of rules satisfying *consistency*, *composition down*, and *composition up*, which we term the *logarithmic-proportional rules*. These rules satisfy neither *symmetry* nor *homogeneity*.

Keywords: claims, bankruptcy, rationing, composition down, composition up, homogeneity, consistency.

JEL classification: D63, D70, D71.

## 1 Introduction

Imagine a situation in which there is some limited amount of an infinitely divisible good to divide amongst a group of agents, each of whom has some verifiable claim to the good. If there is not enough of the good to satisfy all of

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the agents' claims, the question of how to award the good arises. This problem is referred to as the "claims problem" in the literature. However, the model used to study this problem has many other interpretations, *e.g.* taxation, bankruptcy, and rationing. Under the taxation interpretation, the parameters of the model are agents' incomes and total tax to be collected. Important works related to this model include O'Neill [7], Aumann and Maschler [1], Young [10, 11], and Moulin [4].

The focus here is not on solving a specific claims problem; rather, we would like to have a general method of recommending awards for any conceivable problem. Call such a method a "rule." Rules permit a study of conditions pertaining to the structure of specific problems, but more importantly, they allow a discussion of conditions specifying how awards should vary as problems vary. The second type of condition is the focus of analysis here.

Many of these conditions are not specific to the interpretation of the formal model, and are generally compelling. Moulin [4] uncovers the complete implications of four such conditions taken together. His takeoff point is that a general theory of rules should cover environments in which agents are treated asymmetrically. Such a viewpoint amounts to a recognition that the model is often oversimplified. Factors exogenous to the modelling situation can be crucial in determining awards.

However, we find at least one of the properties of rules that Moulin postulates generally questionable–a property which we call *homogeneity.* It states that a proportional change in all of the claims and the amount of good available should be accompanied by an equivalent proportional change in awards. It is interpreted as expressing independence of the rule with respect to changes in unit of measurement of the good. While *homogeneity* implies this independence, it also implies more. It rules out any meaning of numerical values in the model except as relative quantities. For example, *homogeneity* requires a rule to ignore comparisons based on marginal utilities; thus under the taxation interpretation, important concepts such as poverty lines and tax brackets become difficult to discuss. Thus, *homogeneity* has little normative justification and also does not mesh well with the idea that exogenous factors may be relevant in determining awards.

Here, we investigate a large class of rules satisfying two of the independence conditions used by Moulin, *composition down* and *composition up.*[1]

[1]This terminology was introduced by Thomson [8].

We will informally describe *composition down*. Suppose that each agent has been promised an award for some problem. What if it is found that the amount to divide is smaller than initially supposed? Two natural possibilities present themselves. Firstly, each agent is justified in claiming the award promised to him. Applying the rule to the problem with these promised awards as claims and the small amount to divide results in a recommended award for each agent. Secondly, it is just as natural to apply the rule to the problem with the original claims and smaller amount to divide. *Composition down* requires that both possibilities result in the same recommended awards. *Composition up* is a dual to this property, applying when the estate is larger than initially supposed.

Many well-known rules satisfy the two *composition* properties. For example, the proportional rule, which always recommends awards proportional to claims, satisfies the two properties. Another example is a rule often called the "constrained equal awards rule." This rule equalizes the agents' awards whenever possible, subject to the constraint that no agent ever receives more than his claim. In fact, many more interesting rules satisfy the two properties. However, all of the commonly discussed rules satisfying the two properties are *homogeneous*. We believe that this is not due to the fact that non-*homogeneous* rules are unintuitive, but to the fact that as of yet, there has not been a simple way to describe them. As it turns out, these rules are very intuitive, especially for the two-agent case.

We uncover a plethora of previously unstudied rules satisfying the two axioms, which we term the *difference rules*. To understand how these rules work, we introduce a minimal amount of notation. Call the set of agents $N$. Each difference rule is associated with a set of continuous, monotonic paths in $\mathbb{R}^N$. This set of paths has the feature that for any vector in $\mathbb{R}^N_+$, there exists a path containing two points whose difference is the vector (hence the term "difference rule"). Any such path is "unique" in that any other path with this property is a translation of the original path in between the two points in question. To find the awards for a given list of claims and an amount to divide, we simply find a path and two points on this path whose difference is the claims vector. By translating the path so that these two points coincide with the origin and the list of claims respectively, there is a unique vector on the path whose components sum to the amount to divide. This unique vector is the list of awards recommended by the rule.

For two agents, the difference rules can be given much greater structure. Specifically, each difference rule can now be associated with a partition of the positive orthant into rays and open convex cones. Each continuous, monotonic path is identified uniquely with an element of this partition. Each such path is either convex or concave, and the limits of the slopes of each path approach the slopes of the boundaries of the element of the partition with which it is identified.

It is not difficult to see that the difference rules satisfy *composition down* and *composition up*. The main contribution of this paper is to show that these rules are the *only* rules satisfying the two properties. Thus, Theorem 1 shows that a rule satisfies *composition down* and *composition up* if and only if it is a difference rule. Theorem 2 discusses the added structure in the two-agent case.

Our analysis also covers a variable population model. The well-known property of *consistency* is the requirement that recommended awards should not be changed when reapplying the rule to a subgroup of agents, who take the sum of their awards as the amount to divide. In this environment, we introduce a new class of rules for three agent problems. These rules are called the *logarithmic-proportional rules*, owing to their logarithmic and linear structure. For such rules, there is an agent whose claims and awards are treated in a logarithmic fashion, where the other two agents' claims are treated linearly. Theorem 3 establishes that the logarithmic-proportional rules satisfy the three axioms.

Section 2 introduces the formal fixed-population model and definition of difference rules. Section 3 presents the main results for this model. Section 4 is devoted to the variable-population model and the logarithmic-proportional rules. Section 5 concludes.

# 2 The model

## 2.1 Notation and properties

Let $N$ be a finite set of agents. A **claims problem** is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ such that $\sum_N c_i \geq E$. Let $\mathcal{C}$ be the set of all claims problems. A **rule** is a function $r : \mathcal{C} \to \mathbb{R}_+^N$ such that for all $(c, E) \in \mathcal{C}$, $\sum_N r_i(c, E) = E$ and for all $i \in N$, $r_i(c, E) \leq c_i$. For all $c \in \mathbb{R}_+^N$, $\|c\| \equiv \sum_N c_i$.

The following properties were introduced in this model by Moulin [3] and Young [11], respectively. The first, *composition down*, was discussed in the introduction. Suppose a rule $r$ recommends an awards vector $x$ for a problem

$(c, E)$. Suppose that it is found that actually, $E$ is not available to divide; only $E' \leq E$ is. It is not clear how to evaluate the resulting problem. A natural candidate for the claims vector is $x$, as it was promised to the agents. This leads to the problem $(x, E')$. Another candidate is the original claims vector $c$, leading to the problem $(c, E')$. *Composition down* states that the resulting awards should be the same for either vector.

**Composition down:** For all $(c, E), (c', E') \in \mathcal{C}$ such that $c = c'$ and $E' \leq E$, $r(c, E') = r(r(c, E), E')$.

The second property, *composition up*, has a similar interpretation. It is closely related to Kalai's [2] notion of "step-by-step negotiation," introduced in a Nash bargaining framework. Imagine that a rule recommends $x$ for a problem $(c, E)$. Suppose that it is found that actually, $E' \geq E$ is available to divide. Again, we run into a problem of how $E'$ should be divided. One possible argument is the following. As the agents were promised the awards $x$, all that really matters is what is left of their claims $c - x$, and how much extra there is to divide, $E' - E$. The agents should receive the sum of $x$ and the awards recommended for $(c - x, E' - E)$. It is equally as natural to apply the rule to the problem $(c, E)$, as $c$ were the original claims. *Composition up* states that the resulting awards should be the same for either way of determining awards.

**Composition up:** For all $(c, E)$, $(c', E') \in \mathcal{C}$ such that $c = c'$ and $E \leq E'$, $r(c, E') = r(c, E) + r(c - r(c, E), E' - E)$.

We also state the condition of *homogeneity* for future reference:

**Homogeneity:** For all $(c, E) \in \mathcal{C}$ and $\alpha > 0$, $r(\alpha c, \alpha E) = \alpha r(c, E)$.

## 2.2 Difference rules

Our goal is to understand the family of rules satisfying *composition down* and *composition up*. To do so, we need some preliminary definitions.

Let $I \subset \mathbb{R}$ be some interval. Let $f : I \to \mathbb{R}^N$ be continuous and weakly monotonic, and suppose that for all $x, y \in I$ such that $y \geq x$, $\sum_N (f_i(y) - f_i(x)) = y - x$.[2] Call such a pair $(I, f)$ a **difference pair**.

[2] A function $f$ is **weakly monotonic** if $x < y$ implies $f_i(x) \leq f_i(y)$ for all $i$ and $f(x) \neq f(y)$.

For any arbitrary index set $\Lambda$, a family of difference pairs parametrized by $\Lambda$, $\left\{\left(I^{\lambda}, f^{\lambda}\right)\right\}_{\lambda\in\Lambda}$, is a **difference family** if the following two conditions are satisfied:

*i)* for all $c \in \mathbb{R}^N_+$, there exists $\lambda \in \Lambda$, $x \in I^{\lambda}$ such that $c = f^{\lambda}(x + \|c\|) - f^{\lambda}(x)$

*ii)* if there exist $\lambda, \mu \in \Lambda$ such that for some $c \in \mathbb{R}^N_+$, there exist $x \in I^{\lambda}$ and $x' \in I^{\mu}$ for which $f^{\lambda}(x + \|c\|) - f^{\lambda}(x) = f^{\mu}(x' + \|c\|) - f^{\mu}(x')$, then for all $0 \leq y \leq \|c\|$, $f^{\lambda}(x + y) - f^{\lambda}(x) = f^{\mu}(x' + y) - f^{\mu}(x')$.

A rule $r$ is a **difference rule** if there exists a difference family $\left\{\left(I^{\lambda}, f^{\lambda}\right)\right\}_{\lambda\in\Lambda}$ such that for all $(c, E) \in \mathcal{C}$, $r(c, E) = f^{\lambda}(x + E) - f^{\lambda}(x)$, where $\lambda \in \Lambda$, $x \in I^{\lambda}$ are chosen so that $c = f^{\lambda}(x + \|c\|) - f^{\lambda}(x)$. Say that $\left\{\left(\mathbf{I}^{\lambda}, \mathbf{f}^{\lambda}\right)\right\}_{\boldsymbol{\lambda\in\Lambda}}$ **generates r** in this case.

Many prominent rules are difference rules. Here we list some examples of familiar rules satisfying the definition together with difference families generating them.

**Proportional rule:** Let $\left\{\left(I^{\lambda}, f^{\lambda}\right)\right\}_{\lambda\in\Lambda}$ be a difference family where $\Lambda \equiv \left\{x \in \mathbb{R}^N_{++} : \sum_N x_i = 1\right\}$ and for all $\lambda \in \Lambda$, $I^{\lambda} \equiv \mathbb{R}$. For all $\lambda \in \Lambda$, let $f^{\lambda}(x) \equiv \lambda x$.

**Constrained equal awards rule:** For $|N| = 2$, let $\{(I^1, f^1), (I^2, f^2)\}$ be a difference family where $I^j \equiv \mathbb{R}$ for $j = 1, 2$. Let $f^1(x) \equiv \begin{cases} \left(\frac{x}{2}, \frac{x}{2}\right) & \text{for } x < 0 \\ (0, x) & \text{for } x \geq 0 \end{cases}$, $f^2(x) \equiv \begin{cases} \left(\frac{x}{2}, \frac{x}{2}\right) & \text{for } x < 0 \\ (x, 0) & \text{for } x \geq 0 \end{cases}$.

**Priority rule:** For $|N| = 2$, let $\{(I, f)\}$ be a singleton difference family where $I \equiv \mathbb{R}$, and $f(x) \equiv \begin{cases} (x, 0) & \text{for } x < 0 \\ (0, x) & \text{for } x \geq 0 \end{cases}$.

Lastly, we conclude with a rule introduced in Moulin [4]. We describe the rule through the use of a difference family.

**Exponential rule:** For $|N| = 2$, let $\{(I^1, f^1), (I^2, f^2), (I^3, f^3)\}$ be a difference family where $I^i = \mathbb{R}$ for $i = 1, 2, 3$. For all $x \in I^1$, $f^1(x) \equiv (x_1, x_2)$ where $(x_1, x_2)$ is the unique solution to the system $x_1 + x_2 = x$ and $x_2 = \exp(x_1)$. Lastly, $f^2(x) = (0, x)$ and $f^3(x) = (x, 0)$.

It is important to note that many difference families may generate the same difference rule. There are two reasons for this. The first reason is that any difference pair is unique only up to "translation." Thus, if $(I, f)$ is a difference pair, then for a fixed vector $\mu \in \mathbb{R}^N$, $(I, f+\mu)$ is another difference pair which is essentially the same as the first. The second reason is that there may be some "overlap" of difference pairs. Thus, suppose $I \equiv [a, b]$ and suppose $(I, f)$ is a difference pair. Let $x \in I$. Then $\left\{\left([a, x], f_{[a,x]}\right), (I, f)\right\}$ is a pair of difference pairs which is essentially the same as $(I, f)$. An interesting question is to find the most "parsimonious" difference family generating a given difference rule.

# 3 Results

## 3.1 The general case

Our first result is a characterization of all rules satisfying *composition down* and *composition up*. The only rules satisfying these two properties are the difference rules. It is relatively simple to verify that the difference rules satisfy the two properties. The other direction is also simple. Our argument is constructive, and the difference family we construct is in no way "parsimonious."

**Theorem 1** A rule satisfies *composition down* and *composition up* if and only if it is a difference rule.

**Proof.** Let $r$ be a difference rule. Let $\left\{\left(I^\lambda, f^\lambda\right)\right\}_{\lambda\in\Lambda}$ be the difference family that generates $r$. We will show that $r$ satisfies *composition down* and *composition up*.

*Composition down:* Let $(c, E), (c', E') \in \mathcal{C}$ such that $c = c'$ and $E' \leq E$. We need to show that $r(c, E') = r(r(c, E), E')$. By definition, there exists $\lambda \in \Lambda$, $x \in I^\lambda$ such that $c = f^\lambda(x + \|c\|) - f^\lambda(x)$. Then $r(c, E') = f^\lambda(x + E') - f^\lambda(x)$ and $r(c, E) = f^\lambda(x + E) - f^\lambda(x)$. But $r(c, E) = f^\lambda(x + \|r(c, E)\|) - f^\lambda(x)$, so that by definition, $r(r(c, E), E') = f^\lambda(x + E') - f^\lambda(x)$. Thus, $r(c, E') = r(r(c, E), E')$.

*Composition up*: Let $(c, E), (c', E') \in \mathcal{C}$ such that $c = c'$and $E \leq E'$. We need to show that $r(c, E') = r(c, E) + r(c - r(c, E), E' - E)$. Equivalently, we show that $r(c, E') - r(c, E) = r(c - r(c, E), E' - E)$. By definition, there exists $\lambda \in \Lambda$ such that $c = f^\lambda(x + \|c\|) - f^\lambda(x)$. Thus $r(c, E') =$

$f^\lambda(x+E') - f^\lambda(x)$ and $r(c,E) = f^\lambda(x+E) - f^\lambda(x)$, so that $r(c,E') - r(c,E) = f^\lambda(x+E') - f^\lambda(x+E)$. Next, $c - r(c,E) = f^\lambda(x+\|c\|) - f^\lambda(x+E)$, which is equivalent to

$$c - r(c,E) = f^\lambda(x + E + \|c - r(c,E)\|) - f^\lambda(x+E).$$

By definition, $r(c - r(c,E), E' - E) = f^\lambda(x + E + (E' - E)) - f^\lambda(x+E)$, so that $r(c,E') - r(c,E) = r(c - r(c,E), E' - E)$.

Conversely, let $r$ be a rule satisfying *composition down* and *composition up*. Let $\Lambda \equiv \mathbb{R}^N_+$. For all $\lambda \in \Lambda$, let $I^\lambda \equiv [0, \|\lambda\|]$, and for all $x \in I^\lambda$, $f^\lambda(x) \equiv r(\lambda, x)$. It is well-known that *composition down* implies that for all $\lambda$, $r(\lambda, \cdot)$ is continuous and weakly monotonic. Thus, $f^\lambda$ is continuous and weakly monotonic on $I^\lambda$.

**Step 1: For all $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, $(\mathbf{I}^\lambda, \mathbf{f}^\lambda)$ is a difference pair**

Let $\lambda \in \Lambda$. By definition of $f^\lambda$, for all $y, x \in I^\lambda$ such that $y \geq x$, $\sum_N \left(f_i^\lambda(y) - f_i^\lambda(x)\right) = \sum_N (r_i(\lambda, y) - r_i(\lambda, x))$. By definition,

$$\sum_N (r_i(\lambda, y) - r_i(\lambda, x)) = y - x.$$

Thus, $\sum_N \left(f_i^\lambda(y) - f_i^\lambda(x)\right) = y - x$. Thus $(I^\lambda, f^\lambda)$ is a difference pair.

**Step 2: $\left\{(\mathbf{I}^\lambda, \mathbf{f}^\lambda)\right\}_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}}$ is a difference family**

Part *i)*: For all $c \in \mathbb{R}^N_+$, $c = r(c, \|c\|) - r(c, 0)$ by the definition of a rule. By definition of $(I^c, f^c)$, $r(c, \|c\|) - r(c, 0) = f^c(0 + \|c\|) - f^c(0)$. Thus, $c = f^c(0 + \|c\|) - f^c(0)$.

Part *ii)*: Suppose that there exists some $\lambda \neq c$ and $x \in I^\lambda$ such that $c = f^\lambda(x + \|c\|) - f^\lambda(x)$. It is sufficient to show that for all $z \in [0, \|c\|]$, $f^\lambda(x+z) - f^\lambda(x) = f^c(z) - f^c(0)$. By definition of $f^\lambda$, $f^\lambda(x+z) - f^\lambda(x) = r(\lambda, x+z) - r(\lambda, x)$. By *composition down*,

$$r(\lambda, x+z) - r(\lambda, x) = r(r(\lambda, x + \|c\|), x+z) - r(r(\lambda, x+\|c\|), x).$$

By *composition up*,

$$\begin{aligned} & r(r(\lambda, x+\|c\|), x+z) - r(r(\lambda, x+\|c\|), x) \\ = \; & r(r(\lambda, x+\|c\|) - r(r(\lambda, x+\|c\|), x), z) \end{aligned}$$

and by *composition down*,

$$r\left(r\left(\lambda, x+\|c\|\right), x\right) = r\left(\lambda, x\right).$$

Thus,

$$r\left(r\left(\lambda, x+\|c\|\right) - r\left(r\left(\lambda, x+\|c\|\right), x\right), z\right) = r\left(r\left(\lambda, x+\|c\|\right) - r\left(\lambda, x\right), z\right).$$

By definition of $f^{\lambda}$,

$$r\left(r\left(\lambda, x+\|c\|\right) - r\left(\lambda, x\right), z\right) = r\left(f^{\lambda}\left(x+\|c\|\right) - f^{\lambda}\left(x\right), z\right).$$

By the hypothesis of the theorem, $c = f^{\lambda}\left(x+\|c\|\right) - f^{\lambda}\left(x\right)$, so that

$$r\left(f^{\lambda}\left(x+\|c\|\right) - f^{\lambda}\left(x\right), z\right) = r\left(c, z\right).$$

By definition of $f^{c}$, $r\left(c, z\right) = f^{c}\left(z\right) - f^{c}\left(0\right)$. By stringing together the equalities, we obtain $f^{\lambda}\left(x+z\right) - f^{\lambda}\left(x\right) = f^{c}\left(z\right) - f^{c}\left(0\right)$.

**Step 3:** $\left\{\left(\mathbf{I}^{\lambda}, \mathbf{f}^{\lambda}\right)\right\}_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}}$ **generates r**

Let $r'$ be the difference rule generated by $\left\{\left(f^{\lambda}, I^{\lambda}\right)\right\}_{\lambda\in\Lambda}$. We show that $r' = r$. Let $(c, E) \in \mathcal{C}$. By definition of $f^{c}$, $c = f^{c}\left(c\right) - f^{c}\left(0\right)$. Thus $r'\left(c, E\right) = f^{c}\left(E\right) - f^{c}\left(0\right)$. But $f^{c}\left(E\right) - f^{c}\left(0\right) = r\left(c, E\right) - r\left(c, 0\right) = r\left(c, E\right)$. ■

## 3.2 The two-agent case

We now demonstrate that more can be said about the difference rules when $|N| = 2$. A difference rule $r$ is a **relative claims difference rule** if there exists a partition $\mathcal{P}$ of $\mathbb{R}^2_+ \setminus \{0\}$ into rays and open convex cones and a difference family $\left\{\left(I^{\lambda}, f^{\lambda}\right)\right\}_{\lambda\in\Lambda}$ generating $r$ such that the following four conditions hold:

*i)* $\Lambda = \mathcal{P}$

*ii)* For all $\lambda \in \Lambda$, $I^{\lambda} = \mathbb{R}$

*iii)* For all $\lambda \in \Lambda$, $f^{\lambda}\left(I^{\lambda}\right)$ is a curve which is either "concave" or "convex"

*iv)* If $c \in \lambda$, then there exists $x \in I^{\lambda}$ such that $c = f^{\lambda}\left(x+\|c\|\right) - f^{\lambda}\left(x\right)$.

We informally discuss the relative claims difference rules. Any relative claims difference rule is associated with a partition of the positive orthant as described above. Each element of the partition is identified with a difference pair–this is the content of condition *i*). Conditions *ii*), *iii*), and *iv*) relate to the structure of these difference pairs. Condition *ii*) states that any difference

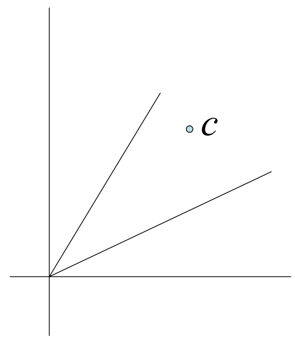


Figure 1: A partition of the orthant into cones

pair in the difference family must have a domain which is the entire real line. Thus, for any difference pair $\left(I^{\lambda}, f^{\lambda}\right)$, the image $f^{\lambda}\left(I^{\lambda}\right)$ is an infinite and monotonic path in $\mathbb{R}^{N}$.

Conditions *iii*) and *iv*) relate to the structure of these infinite paths. Condition *iii*) states that these paths have either a convex or a concave shape. Thus, the "slopes" of each path either increase or decrease monotonically along the path.

The last point relates each infinite path to the element of the partition with which is it associated. Given that the "slopes" of each path either increase or decrease monotonically, they have limits in each direction (which they are permitted to reach). These limits must be exactly the "slopes" of the boundaries of the element of the partition with which the path is associated. If the element of the partition is an open cone, there are two such directions. If it is a singleton, there is only one direction, and thus given the other conditions, the infinite path must be a straight line. This condition on the slope limits is equivalent to condition *iv*) given the other three conditions.

Figures 1, 2, and 3 explain geometrically how the relative claims difference rules work. Given a claims vector $c$, we show how to construct the path

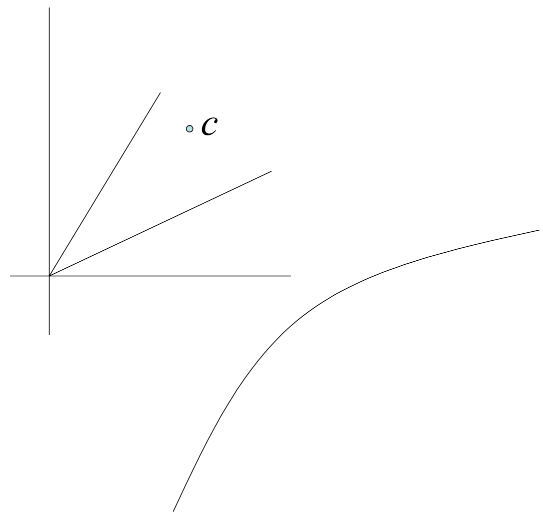


Figure 2: The image of the difference pair associated with the cone containing $c$

of awards for such a rule. Firstly, find the cone which $c$ lies in. This is illustrated in Figure 1. The next step is to find the image of the difference pair associated with this cone; this is depicted in Figure 2. One can see that this image satisfies the properties given in the definition; it is "concave," and the slopes of this path approach the slopes of the boundaries of the cone. Lastly, Figure 3 shows how to find the path of awards for $c$. Given the concave shape of the path, it is easy to see that there exists some $x$ on the path such that $x + c$ also lies on the path. The path of awards for $c$ is simply the translation of the image in between $x$ and $x + c$.

We now return to the examples discussed in section 2. The proportional rule is identified with the finest possible partition of the orthant; a partition into rays. Indeed, the proportional rule is the unique relative claims difference rule associated with this partition. Every path must be a straight line, with slope equivalent to the slope of the element of the partition with which it is identified.

We state an equivalent definition of the constrained equal awards rule, using its relative claims difference rule formulation, in order to point out a

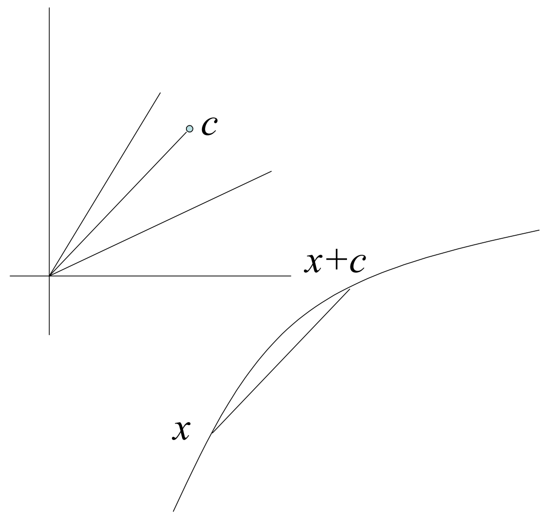


Figure 3: Finding the path of awards for $c$

feature of the definition.

**Constrained equal awards rule (relative claims definition):** Partition the orthant into five elements, including three rays and two open cones. The three rays are given by the directions:

$$\left\{P_1 = (0,1)\,, P_2 = \left(\frac{1}{2}, \frac{1}{2}\right), P_3 = (1,0)\right\},$$

and the open cones are given by the directions:

$$\left\{P_4 = \left((0,1)\,, \left(\frac{1}{2}, \frac{1}{2}\right)\right), P_5 = \left(\left(\frac{1}{2}, \frac{1}{2}\right), (1,0)\right)\right\}.$$

The three rays are each associated with straight lines; that is, $f^1(x) \equiv (0,x)$, $f^2(x) \equiv \left(\frac{x}{2}, \frac{x}{2}\right)$, and $f^3(x) \equiv (x,0)$. The remaining elements of the partition are associated with the following paths:

$$f^4(x) \equiv \begin{cases} \left(\frac{x}{2}, \frac{x}{2}\right) & \text{for } x < 0 \\ (0,x) & \text{for } x \geq 0 \end{cases}$$

and

$$f^5(x) \equiv \begin{cases} \left(\frac{x}{2}, \frac{x}{2}\right) & \text{for } x < 0 \\ (x, 0) & \text{for } x \geq 0 \end{cases}.$$

The definition of the constrained equal awards rule given in section 2 consists of only two paths; whereas the definition given here uses five paths, three of which (those associated with singletons) seem redundant. In fact, these paths *are* redundant, in the sense that the claims vectors spanned by these paths are already spanned by the paths associated with the open intervals. Thus, some justification is needed for the use of the redundant definition.

We choose the latter as the formal definition because we desire a "standard" normalization for the representation of relative claims difference rules. A natural specification is to associate each relative claims difference rule with a *partition* of the positive orthant; this is what is done here. An alternative is to associate each "maximal" path with the set of "differences" spanned by the path. The latter is what is done in Section 2. In many cases, choosing the latter alternative is equivalent to the former; for example, the exponential rule satisfies this property. However, in general, this is not true. Consider again the definition of the constrained equal awards rule given in section two. The sets of differences spanned by the two paths are each closed cones with nonempty intersection. Thus, there is no way to associate these sets of differences with a partition of the positive orthant. If we choose this route, we must opt for a messier definition. Instead, we associate each path with the *relative interior* of the set of differences spanned by the path, and accept the redundancies that arise.

The priority rule is another rule that allows redundancies in its definition:

**Priority rule (relative claims definition):** Partition the orthant into three elements; two rays and one open cone. The two rays are

$$\{P_1 = (0, 1), P_2 = (1, 0)\}.$$

The open cone is

$$\{P_3 = ((0, 1), (1, 0))\}.$$

The paths associated with the rays are $f^1(x) \equiv (0, x)$ and $f^2(x) \equiv (x, 0)$. The path associated with the open cone is

$$f(x) \equiv \begin{cases} (x, 0) & \text{for } x < 0 \\ (0, x) & \text{for } x \geq 0 \end{cases}.$$

It turns out that the relative claims difference rules generalize of a family of rules discussed in Moulin [4], Lemma 2, characterized by adding to our two conditions the condition of *homogeneity*. The rules Moulin studied can be described by adding to conditions *iii*) and *iv*) the condition that each infinite path be piecewise linear in at most two pieces.

Informally speaking, the relative claims difference rules can be discussed by how closely they approximate "proportionality." There are two senses in which such rules can approximate "proportionality." The finer is the partition of the positive orthant generating a rule, the closer a rule is to being "proportional." Moreover, the slower the "slopes" of the paths associated with a difference pair change, the closer a rule is to being "proportional," especially for small claims vectors.

In Theorem 2, we show that all difference rules in two agent environments are relative claims difference rules.

The idea behind the proof of Theorem 2 is not difficult to grasp. The most important step consists in showing that if a rule is a difference rule, then for any claims vector, the path traced out as the amount of good to divide varies is continuous and has a convex shape. The continuity of such paths and the fact that we are working with only two dimensions implies that as we consider larger and larger multiples of a fixed claims vector, the paths associated with these vectors are "nested." By studying the "limit" of such paths, we construct a new difference family. By virtue of the convex shape of these infinite paths, each constructed path may be associated with a cone of claims vectors. If two cones have a nonempty intersection, then there is a claims vector that can be solved by using either of the associated paths. This implies that the difference pairs associated with the cones must be the same, up to translation. Hence we may without loss of generality suppose that these cones form a partition.

**Theorem 2** Suppose $|N| = 2$. If a rule is a difference rule, then it is a relative claims difference rule.

**Proof.** We prove the theorem in steps. Throughout the proof, $\left\{\left(f^{\lambda}, I^{\lambda}\right)\right\}_{\lambda\in\Lambda}$ denotes the difference family which generates $r$. The path of awards generated by a claims vector $c$ is defined as the image of $r\left(c,\cdot\right)$.

Define the function $\Delta : \mathcal{C} \to \mathbb{R}$ by $\Delta\left(c,E\right) \equiv c_1 r_2\left(c,E\right) - c_2 r_1\left(c,E\right)$. It is convenient to discuss facts about $r$ in terms of $\Delta$. This technique was introduced in Moulin [4].

Let $c \in \mathbb{R}_+^N$. Then $\Delta(c, E)$ is continuous in $E$ over $[0, \|c\|]$. This follows as $r(c, E)$ is continuous in $E$.

The first step in the proof consists of showing that for all $c$, $\Delta(c, E)$ is either convex, concave, or both as $E$ ranges over $[0, \|c\|]$.

**Step 1: For all $\mathbf{c} \in \mathbb{R}_+^N$, $\boldsymbol{\Delta}(\mathbf{c}, \cdot)$ is either convex or concave.**

To prove this Step, we assume that the statement is false. Thus, there exists $c \in \mathbb{R}_+^N$ such that $\Delta(c, \cdot)$ is neither convex nor concave.

Define the auxiliary function

$$\psi(x, \varepsilon) \equiv \Delta(c, x + \varepsilon) - \Delta(c, x),$$

where $(x, \varepsilon)$ satisfies $\varepsilon \in [0, \|c\|]$ and $x \in [0, \|c\| - \varepsilon]$. The function $\psi$ evaluated at a particular $\varepsilon$ can be viewed as an "$\varepsilon$-differential." As $\Delta$ is not necessarily differentiable, we need to work with such a structure.

It is obvious by the continuity of $\Delta$ in its second parameter that $\psi$ is continuous in both $x$ and $\varepsilon$. The statement that $\Delta$ is neither convex nor concave is equivalent to the statement that there exists $\varepsilon^*$ such that $\psi(\cdot, \varepsilon^*)$ is not monotone.

So, suppose that there exists $\varepsilon^*$ such that $\psi(\cdot, \varepsilon^*)$ is not monotone. As $\psi$ is continuous, there exist $y, z \in [0, \|c\| - \varepsilon^*]$ and $\eta^* > 0$ such that $\psi(y, \varepsilon^*) = \psi(z, \varepsilon^*)$, and for all $\eta < \eta^*$, $\psi(y + \eta, \varepsilon^*) > \psi(y, \varepsilon^*)$ and $\psi(z + \eta, \varepsilon^*) < \psi(z, \varepsilon^*)$.

In particular, $\psi(y, \varepsilon^*) = \psi(z, \varepsilon^*)$ implies that $r(c, y + \varepsilon^*) - r(c, y) = r(c, z + \varepsilon^*) - r(c, z)$. Thus, as $r$ is a difference rule, the path of awards for the vector $r(c, y + \varepsilon^*) - r(c, y)$ can be determined as the path of awards for $c$ lying in between either $y$ and $y + \varepsilon^*$ or in between $z$ and $z + \varepsilon^*$.

We will now establish that the path of awards for $c$ in between $y$ and $y+\varepsilon^*$ is different from the path of awards for $c$ in between $z$ and $z+\varepsilon^*$, establishing a contradiction. By continuity of $\psi$, we may choose $\varepsilon' < \varepsilon^*$ close enough to $\varepsilon^*$ so that there exists $\mu^*$ such that for all $\mu < \mu^*$, $\psi(y + \mu, \varepsilon') > \psi(y, \varepsilon')$ and $\psi(z + \mu, \varepsilon') < \psi(z, \varepsilon')$. Let $\mu$ satisfy $\mu + \varepsilon' < \varepsilon^*$.

Then

$$\Delta(c, y + \mu + \varepsilon') - \Delta(c, y + \mu) > \Delta(c, y + \varepsilon') - \Delta(c, y)$$

and

$$\Delta(c, z + \mu + \varepsilon') - \Delta(c, z + \mu) < \Delta(c, z + \varepsilon') - \Delta(c, z).$$

Thus, the path of awards for $r$ in between the values $y$ and $y+\varepsilon^*$ is indeed different from the path of awards for $r$ in between the values $z$ and $z+\varepsilon^*$. Hence $r$ cannot be a difference rule and therefore does not satisfy *composition down* and *composition up*.

**Step 2: Construction of the difference family**

Let $c \in \mathbb{R}^2_+$. By hypothesis, there exist $\lambda \in \Lambda$ and $x \in I^\lambda$ such that $c = f^\lambda(x + \|c\|) - f^\lambda(x)$. By a continuity argument, it can be verified that for all $\alpha \leq 1$, there exists $y \in I^\lambda$ such that $x \leq y$ and $\alpha c = f^\lambda(y + \alpha\|c\|) - f^\lambda(y)$.

Using the fact in the preceding paragraph, we now construct a difference pair which contains $c$, called $(J^c, g^c)$. It is constructed as a "maximal" difference pair which can be used to evaluate the path of awards for $c$. Thus, $(J^c, g^c)$ will have the property that for all $\alpha \in \mathbb{R}_+$, there exists $y \in J^c$ such that $\alpha c = g^c(y + \alpha\|c\|) - g^c(y)$.

The first paragraph of Step 2 tells us that difference pairs are "nested" for increasing scalar multiples of a claims vector.

There are two possible cases for which we will construct difference pairs:

Case *i*) Suppose that for all $\alpha \in \mathbb{R}_+$, $\Delta(\alpha c, \cdot)$ is constant and equal to zero. In this case, let $g^c(x) \equiv \frac{c}{\|c\|}x$.

Case *ii*) Suppose that there exists some $\alpha \in \mathbb{R}_+$ such that for some $E$, $\Delta(\alpha c, E) \neq 0$.

Let $J^c \equiv \mathbb{R}$. For all $\alpha \in \mathbb{R}_+$, there exists $\lambda(\alpha) \in \Lambda$ and $x^{\lambda(\alpha)} \in I^{\lambda(\alpha)}$ such that $\alpha c = f^{\lambda(\alpha)}\left(x^{\lambda(\alpha)} + \alpha\|c\|\right) - f^{\lambda(\alpha)}\left(x^{\lambda(\alpha)}\right)$. If $\alpha > 1$, by the first paragraph of Step 2, there also exists $y^{\lambda(\alpha)} \in I^{\lambda(\alpha)}$ such that $c = f^{\lambda(\alpha)}\left(y^{\lambda(\alpha)} + \|c\|\right) - f^{\lambda(\alpha)}\left(y^{(\alpha)}\right)$.

Let $z \in J^c$. Let $\alpha \in \mathbb{R}_+$ be chosen large enough so that $f^{\lambda(\alpha)}\left(y^{\lambda(\alpha)} + z\right)$ is well defined.[3] Such an $\alpha$ can easily be shown to exist, using the the fact that the generated paths are either convex or concave.

Define $g^c(z) \equiv f^{\lambda(\alpha)}\left(y^{\lambda(\alpha)} + z\right) - f^{\lambda(\alpha)}\left(y^{\lambda(\alpha)}\right)$. That $g^c$ is well-defined follows from the definition of a difference rule. By construction, for all $c \in \mathbb{R}^2_+$ $\{(J^c, g^c)\}$ is a difference pair.

By construction, $\{(J^c, g^c)\}_{c \in \mathbb{R}^2_+ \setminus \{0\}}$ is a difference family. Say $(\mathbf{J}^c, \mathbf{g}^c)$ **is equivalent to** $\left(\mathbf{J}^{c'}, \mathbf{g}^{c'}\right)$ if there exists $x \in J^c$, $x' \in J^{c'}$ such that for all $y \in \mathbb{R}$, $g^c(x+y) - g^c(x) = g^{c'}(x'+y) - g^{c'}(x')$. Using the Axiom of Choice, select one element from each equivalence class of this equivalence relation. The resulting selections form a difference family, say $\left\{\left(J^\phi, g^\phi\right)\right\}_{\phi \in \Phi}$.

[3] Note that here we are allowing $z$ to be negative.

By construction, $\left\{\left(J^{\phi},g^{\phi}\right)\right\}_{\phi\in\Phi}$ generates $r$; moreover, by Step 1, each $g^{\phi}$ separates the space into two sets, one of which is convex.

**Step 3: Verification that $\left\{\left(\mathbf{J}^{\phi},\mathbf{g}^{\phi}\right)\right\}_{\phi\in\mathbf{\Phi}}$ can be associated with a partition of $\mathbb{R}^2_+\setminus\{0\}$**

For all $\phi\in\Phi$, let $K^{\phi}\equiv\left\{g^{\phi}(y)-g^{\phi}(x):y>x\right\}$. By construction, $K^{\phi}$ is a convex cone. The set $K^{\phi}$ has a nonempty interior if and only if the image of $g^{\phi}$ under $J^{\phi}$ is not a straight line. For all $\phi\in\Phi$, define $p^{\phi}$ to be the relative interior of $K^{\phi}$. To see that $p^{\phi}$ is a partition, note first that for all $x\in\mathbb{R}^2_+$, there exists $\phi\in\Phi$ such that $x\in K^{\phi}$. We claim that for all $\phi,\phi'\in\Phi$, $K^{\phi}\cap K^{\phi'}=\emptyset$. Otherwise, there exists some $c\in K^{\phi}\cap K^{\phi'}$. Let $\alpha\in\mathbb{R}_+$. As in Step 2, we can find the path of awards for $\alpha c$ by using either the $\phi$ difference pair or by using the $\phi'$ difference pair. These two paths must coincide. But as $\alpha$ was arbitrary, this implies that the difference pairs are equivalent, as defined in Step 2. But we constructed $\left\{\left(J^{\phi},g^{\phi}\right)\right\}_{\phi\in\Phi}$ so that there is only one member of each indifference class. This is a contradiction. ■

# 4 A variable population model

## 4.1 The model

In this section, we describe a variable population model and a familiar requirement, *consistency*. We introduce a large family of rules satisfying *consistency*, *composition down*, and *composition up*.[4]

Let $\mathbb{N}^*\subset\mathbb{N}$ be a set of "potential agents." These are all the possible agents we could ever imagine being involved in a claims problem. Let $\mathcal{N}$ denote the finite subsets of $\mathbb{N}^*$. A variable population claims problem is a tuple $(N,c,E)\in\mathcal{N}\times\left(\bigcup_{N'\in\mathcal{N}}\mathbb{R}^{N'}_+\right)\times\mathbb{R}_{++}$ such that $c\in\mathbb{R}^N_+$ and $\sum_N c_i\geq E$. Let $\mathcal{C}^v$ be the set of all variable population claims problems. A variable population rule is a function $r:\mathcal{C}^v\to\left(\bigcup_{N'\in\mathcal{N}}\mathbb{R}^{N'}_+\right)$ such that for all $(N,c,E)\in\mathcal{C}^v$, $r(N,c,E)\in\mathbb{R}^N_+$.

The next property is well-known in the literature. It relates to changes in the population of agents. Suppose $x$ is recommended by $r$ for the problem $(N,c,E)$. *Consistency* requires that applying the rule to any subset $N'$ of

[4]For a good survey on *consistency*, see Thomson [9].

agents who redivide the sum of their awards according to the rule using their original claims should result in the same recommended awards for $N'$.

For any vector $x \in \mathbb{R}^N$, $x_{N'}$ denotes the projection of $x$ on $\mathbb{R}^{N'}$.

**Consistency:** Let $(N, c, E) \in \mathcal{C}^v$ and $N' \subset N$. Let $x \equiv r(N, c, E)$. Then $x_{N'} = r\left(N', c_{N'}, \sum_{N'} x_i\right)$.

To be complete, we restate the definitions of *composition down* and *composition up* in a variable population framework:

**Composition Down:** For all $(N, c, E), (N', c', E') \in \mathcal{C}^v$ such that $N = N'$, $c = c'$, and $E \leq E'$, $r(N, c, E) = r(r(N, c, E'), E)$.

**Composition Up:** For all $(N, c, E)$, $(N', c', E') \in \mathcal{C}$ such that $N = N'$, $c = c'$, and $E' \leq E$, $r(N, c, E) = r(N, c, E') + r(N, c - r(N, c, E'), E - E')$.

## 4.2 The logarithmic-proportional rules

The only example of a *consistent* difference rule violating *homogeneity* is found in Moulin [4]. He defines the rule only for two agents; however, using the idea of "priority compositions" also introduced in his paper, it is clear how to construct such rules for greater numbers of agents. The example Moulin gives is exactly the exponential rule, studied above. One way to extend the exponential rule to greater populations of agents in a *consistent* manner is simply to divide the class of all agents into two "priority classes." The first priority class consists of only two agents. If there is enough to satisfy these two agents' claims, then they are awarded their claims. Otherwise, we apply an exponential rule to these agents and their claims only. If there is any amount to divide left over, it is awarded to the remaining agents on the basis of the proportional rule. It is clear to see that this rule is *consistent*.

We introduce another family of *consistent* variable population rules satisfying the two composition properties, termed the **logarithmic-proportional rules**. Such rules do not feature the drastic partitioning of agents into priority classes. Thus, a logarithmic-proportional rule always gives each agent with a positive claim a positive award. They are also based on the exponential rules, and are only defined when $|\mathbb{N}^*| = 3$. Without loss of generality, assume that $\mathbb{N}^* = \{1, 2, 3\}$. The logarithmic-proportional rules are parametrized by the set $\mathbb{R}_{++} \times \mathbb{N}^*$. Without loss of generality, we define the logarithmic-proportional rules for the case $i = 1$. The rest of the rules are defined in a similar fashion, by permuting the agents.

We define the logarithmic-proportional rule $(a, 1)$ for each possible group of agents.

For problems involving $\{1, 2, 3\}$, given $(\{1, 2, 3\}, c, E) \in \mathcal{C}^v$, there exist unique $y_2$, $y_3 \in \mathbb{R}$ such that $\log_a(y_2 + c_2) - \log_a(y_2) = c_1$ and $\log_a(y_3 + c_3) - \log_a(y_3) = c_1$. Let $e^{(a,1)}(\{1, 2, 3\}, c, E) = (x_1, x_2, x_3)$, where $x$ satisfies the following system of equations:

$$x_1 + x_2 + x_3 = E \text{ and } x_1 = \log_a(y_2 + x_2) - \log_a(y_2) = \log_a(y_3 + x_3) - \log_a(y_3).$$

For problems involving $\{2, 3\}$, given $(\{2, 3\}, c, E) \in \mathcal{C}^v$, let

$$e^{(a,1)}(\{2, 3\}, c, E) = \lambda c,$$

where $\lambda$ is chosen to achieve feasibility.

For problems involving $\{1, j\}$, where $j \neq 1$, given $(\{1, j\}, c, E) \in \mathcal{C}^v$, there exists a unique $y_j \in \mathbb{R}$ such that $\log_a(y_j + c_j) - \log_a(y_j) = c_1$. Let $e^{(a,1)}(\{1, j\}, c, E) = (x_1, x_j)$, where $x$ satisfies the following system of equations:

$$x_1 + x_j = E \text{ and } x_1 = \log_a(y_j + x_j) - \log_a(y_j).$$

**Theorem 3:** The logarithmic-proportional rules satisfy *consistency*, *composition down*, and *composition up*.

**Proof:** To see that the logarithmic-proportional rules satisfy *consistency*, suppose that $(\{1, 2, 3\}, c, E) \in \mathcal{C}^v$, and let $x \equiv e^{(a,1)}(\{1, 2, 3\}, c, E)$. We claim that there exists some $\lambda$ such that $x_2 = \lambda c_2$, $x_3 = \lambda c_3$. To see this, by definition, there exist $y_2, y_3$ such that

$$\log_a(y_2 + c_2) - \log_a(y_2) = \log_a(y_3 + c_3) - \log_a(y_3) \tag{4}$$

and

$$\log_a(y_2 + x_2) - \log_a(y_2) = \log_a(y_3 + x_3) - \log_a(y_3). \tag{5}$$

By (4), $\frac{c_2}{y_2} = \frac{c_3}{y_2}$ and by (5), $\frac{x_2}{y_2} = \frac{x_3}{y_3}$. Thus, $\frac{c_2}{c_3} = \frac{y_2}{y_2} = \frac{x_2}{x_3}$, so that there exists $\lambda$ such that $x_2 = \lambda c_2$, $x_3 = \lambda c_3$. Next, let $\left(\{2, 3\}, c_{\{2,3\}}, x_2 + x_3\right) \in \mathcal{C}^v$. By definition, $y \equiv e^{(a,1)}\left(\{2, 3\}, c_{\{2,3\}}, x_2 + x_3\right)$ is the unique vector such that $y = \lambda^* c_{\{2,3\}}$, where $\lambda^*$ is chosen to achieve feasibility. As $\lambda c_2 + \lambda c_3 = x_2 + x_3$, feasibility requires $\lambda = \lambda^*$ and $y = \lambda c_{\{2,3\}}$, so $y = x_{\{2,3\}}$. For the problem $\left(\{1,j\}, c_{\{1,j\}}, x_1 + x_j\right)$, it is clear that $x_{\{1,j\}} = e^{(a,1)}\left(\{1,j\}, c_{\{1,j\}}, x_1 + x_j\right)$, simply by definition.

Now, we verify that *composition down* is satisfied. We only need to check for problems involving $\{1,2,3\}$, as $e^{(a,1)}$ restricted to two-agent problems is obviously a difference rule. So, let $(\{1,2,3\}, c, E) \in \mathcal{C}^v$, and $(\{1,2,3\}, c, E') \in \mathcal{C}^v$ such that $E' < E$. Let $x \equiv e^{(a,1)}(\{1,2,3\}, c, E)$ and $z \equiv e^{(a,1)}(\{1,2,3\}, c, E')$. We need to show that $e^{(a,1)}(\{1,2,3\}, x, E') = e^{(a,1)}(\{1,2,3\}, c, E')$. By definition, there exist $y_2, y_3$ so that $\log_a(y_2 + c_2) - \log_a(y_2) = \log_a(y_3 + c_3) - \log_a(y_3) = c_1$, and that $x$ satisfies $x_1 = \log_a(y_2 + x_2) - \log_a(y_2) = \log_a(y_3 + x_3) - \log_a(y_3)$. Hence, treating $x$ as our claims vector, $z$ satisfies $z_1 = \log_a(y_2 + z_2) - \log_a(z_2) = \log_a(z_3 + y_3) - \log_a(z_3)$, further $z_1 + z_2 + z_3 = E'$. But these last three equalities imply that $z = e^{(a,1)}(\{1,2,3\}, x, E')$, so that *composition down* is satisfied.

*Composition up* follows from a similar argument.■

The logarithmic-proportional rules are not *homogeneous*; in fact, they are also not *symmetric*. In other words, two agents with equal claims need not receive equal awards. We do not know of any non-*homogeneous* rules satisfying the remaining three axioms which are *symmetric*. In fact, we conjecture that a characterization of the constrained equal awards, constrained equal losses, and proportional rules obtains from the axioms *consistency*, *composition down*, *composition up*, and *symmetry*.

# 5 Conclusion

The related model in which claims and awards are discrete leads to totally different results. Although a variant of Theorem 1 holds for this model, Moulin and Stong [6] show that *composition down* and *composition up*, together with two mild *monotonicity* properties, characterize a class of priority-like rules. In this discrete model, the complete implications of *consistency*, *composition down*, and *composition up* are known and first established in Moulin [4], later derived as a Corollary in [5]. A rule satisfies these three properties if and only if it is a priority rule. In a discrete model where randomization of awards is permitted, Moulin [5] also establishes a characterization of a much larger class of rules based on the same axioms.

It is still not obvious what the most general class of rules satisfying *consistency*, *composition down*, and *composition up* is in the continuous model, but we conjecture that it is similar to the priority rules introduced in Moulin [4], with the added possibilities of general difference rules for priority classes of size two, and of logarithmic-proportional rules for priority classes of size three.

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