CHOICE AND MATCHING[†]

CHRISTOPHER P. CHAMBERS AND M. BUMIN YENMEZ[*]

Abstract. We study choice functions that guarantee the existence of optimal stable matchings. Stability requires the choice to be *path independent*. First we show that optimality requires it to be *acceptant*. Next we prove that every path independent choice function has an acceptant path independent expansion and we provide an algorithm for doing so. Finally we characterize classes of acceptant and path independent choice functions.

1. Introduction

In this paper, we highlight the connection between the theory of stable matchings and path independent choice functions by utilizing and strengthening both theories. When choice functions are primitives of a model for two-sided matching markets, *substitutability* alone does not guarantee the existence of stable matchings. In addition to substitutability, *irrelevance of rejected partners* is needed (**?**). The conjunction of substitutability and irrelevance of rejected partners is equivalent to *path independence* (Aizerman and Malishevski, 1981). A choice function is path independent if the choice from the union of two sets is the same with the choice of the union of the choices from these two sets (Plott, 1973). Thus far, the theory of path independent choice functions and stable matchings have been developed independently. In contrast, we establish new results on each topic by using the connection between them.

First, using the theory of path independent choice functions, we establish several new results on stable matchings. Imagine a set of workers (or students) and a collection of firms (or schools). Each agent possesses a choice function that selects, from any pool of potential partners, a subset of those partners. Aizerman and Malishevski (1981) showed that every path independent choice function can be decomposed as the union of preference relations. Therefore, a firm can be viewed as the union of different jobs such that the firm's choice over a set of workers is the union of the best worker for each job. Analogously, a worker can be viewed as the union of different personas. Using this result, we show that there is a natural

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[*]Chambers: Department of Economics, University of California, San Diego, 9500 Gilman Drive, #0508, La Jolla, CA 92093. Yenmez: Carnegie Mellon University, Tepper School of Business, 5000 Forbes Ave, Pittsburgh, PA 15213, USA. Chambers: cpchambers@ucsd.edu. Yenmez: byenmez@andrew.cmu.edu. Keywords: Two-Sided Matching, Acceptant, Path Independent. JEL: C78, D47, D71, D78.

generalization of the deferred acceptance algorithm (DA) (Gale and Shapley, 1962) to many-to-many matching markets. In one version of the algorithm, the firm positions make offers to the workers, and in the other one the worker personas make offers to the firms. We show that DA produces a stable matching (Theorem 1). We also show that expanding the choice function of an agent makes every other agent on the same side worse off and every agent on the other side better off (Theorem 2). We rely on the decomposition result of Aizerman and Malishevski to prove these two results.

In a perfect world, firms would admit every worker from any given pool (or schools would admit every student), but they are usually bound by some capacity. Therefore, the capacity acts as a physical constraint on the set of workers, so that choices must be feasible for that capacity. We say that an agent's choice function is $q-$*acceptant* if it always selects q partners when at least q partners are available in the pool, and otherwise selects all of them. A choice function is *acceptant* if it is q-acceptant for some q.

Next we prove several results on path independent choice functions. First, we show that every path independent choice function for which all choices have cardinality at most q can be expanded to a q-acceptant path independent choice function, and we provide an algorithm for doing so (Theorem 3). Thus being acceptant comes essentially for free. This result is important because if the Gale-Shapley deferred acceptance algorithm (DA) is used to assign workers to firms, then all of the workers prefer the outcome with the expanded choices.[1,2] Second, we establish the computationally useful fact that every path independent and q-acceptant choice function is uniquely determined by the choices made on sets of cardinality $q+1$ (Theorem 5). Therefore, the number of sets whose choices must be reported grows only polynomially in the number of partners, and not exponentially.

Aside from this, we offer a characterization of a class of path independent choice functions that are used in practice such as school choice (**?**), without directly imposing path independence. We say that an agent's choice function is *responsive* for value q if there is a preference relation over the partners, and the choice function always selects the q highest-ranked available partners according to this preference relation (Roth and Sotomayor, 1990). It turns out that, together with q-acceptant, it is enough to impose an axiom of Ehlers and Sprumont (2008), the weakened weak axiom of revealed preference (WWARP). This axiom states that for any pair of partners, x, y, if x is chosen when y is available and y is not chosen, then it can never be that y is chosen when x is available and x is not chosen. We show that a choice function is responsive for value q if and only if it is q-acceptant and satisfies WWARP

[1]This follows directly from Theorem 2. DA is used in many such markets; see Roth (2008) and **?**.

[2]Since the law of aggregate demand (LAD) (Alkan and Gale, 2003; Hatfield and Milgrom, 2005) is a weaker condition than being acceptant, this result also implies that every path independent choice function can be expanded to satisfy LAD.

(Theorem 6). This result is a direct generalization of the classical result that a single-valued choice function satisfying the weak axiom of revealed preference is classically rationalizable.

Finally, we investigate a class of choice functions recently described by **?**. This is the class of lexicographic choice functions. A choice function of this type consists of a q-vector of preference relations over the partners; for any group of partners, the choice function first picks the highest-ranked partner with respect to the first preference relation, then the highest-ranked remaining partner with respect to the second preference relation, and so forth. We show that every such choice function is path independent and q-acceptant, but that the converse is not true (Theorem 4). That is, there are path independent and acceptant choice functions not of this form.

Path independent choice functions are a heavily studied class of choice functions, and much is known about the structure of these objects. In depth studies of these objects include Plott (1973) and Aizerman and Malishevski (1981) (see also **?**). These papers established many important properties of these choice functions some of which we use to prove our results. From the matching literature, perhaps the closest paper to ours is Alkan and Gale (2003) who study schedule matching using choice functions. They extend several classic results of matching to their setup such as the existence of stable matchings. In contrast, our focus is on the comparative statics of stable matchings and also on the theory of acceptant choice functions.

Section 2 introduces the model and some preliminary results. Section 3 presents the matching results and Section 4 has the results on choice. Finally, Section 5 concludes.

2. Model and Preliminary Results

Suppose $\mathcal{X}$ is the set of partners and $\mathcal{P}(\mathcal{X}) = 2^{\mathcal{X}}$ is the powerset of $\mathcal{X}$. There exists an agent with a ***choice function*** $C : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ such that

- for every $X \subseteq \mathcal{X}$, $C(X) \subseteq X$ and
- for every $\emptyset \neq X \subseteq \mathcal{X}$, $C(X) \neq \emptyset$.

The interpretation is if X is the set of available partners to the agent, then $C(X)$ is the set of chosen partners. We require the chosen set to be non-empty if there is at least one partner available.[3]

A ***preference relation*** $\succeq$ on $\mathcal{X}$ is a binary relation on $\mathcal{X}$ that is complete, transitive, and antisymmetric.[4] Choice function C is ***lexicographic*** if there exists a sequence of preference relations on $\mathcal{X}$, $\{\succeq_i\}_{i \in I}$, such that for any $X \in \mathcal{X}$

[3]The assumption that the choice is not empty-valued plays a role only in Theorems 4 and 6. For the rest of the results, it is not needed.

[4]Complete: For all $x, y \in \mathcal{X}$, $x \succeq y$ or $y \succeq x$. Antisymmetric: For all $x, y \in \mathcal{X}$, $x \succeq y$ and $y \succeq x$ implies $x = y$. Transitive: For all $x, y, z \in \mathcal{X}$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

$$C(X) = \bigcup_{i \in I} \{x_i^*\},$$

where x_i^* is defined inductively as $x_1^* = \max_X \succeq_1$ and, for $i \geq 2$, $x_i^* = \max_{X \setminus \{x_1^*, \ldots, x_{i-1}^*\}} \succeq_i$ (**?**). Choice function C is ***responsive*** if it is lexicographic with $\{\succeq_i\}_{i \in I}$ such that $\succeq_i = \succeq_j$ for all $i, j \in I$ (Roth and Sotomayor, 1990).

Next we consider two properties of a choice function that guarantee the existence of stable matchings in matching markets.

Definition 1. *Choice function C is **substitutable** if for every $x \in X \subseteq Y$, $x \in C(Y)$ implies $x \in C(X)$.*

Substitutability requires that if a partner is chosen from a set, then it must also be chosen from a subset containing her. Kelso and Crawford (1982); Roth (1984) were the first to study substitutability in a matching context. However, it was first studied in the choice literature and known as Sen's α or Chernoff's axiom (**?**).

Definition 2. *Choice function C satisfies **irrelevance of rejected partners (IRP)** if $C(Y) \subseteq X \subseteq Y$ then $C(X) = C(Y)$.*

Irrelevance of rejected partners implies that excluding partners that are not chosen does not affect the chosen set. IRP together with substitutability guarantee that there exists a stable matching in two-sided markets (Roth, 1984; **?**). They are also necessary, in the maximal domain sense, for the existence (Hatfield and Milgrom, 2005).

The following axiom has been studied extensively in the choice literature. It was first introduced informally in Arrow (1951), and formally in Plott (1973). **?** provides an excellent survey related to the topic. It is important in the study of non-rationalizable choice functions.

Definition 3. *Choice function C is **path independent** if for every X and Y, $C(X \cup Y) = C(C(X) \cup C(Y))$.*

The two necessary and sufficient conditions for the existence of stable matchings are equivalent to path independence.

Lemma 1 (Aizerman and Malishevski (1981))**.** *Choice function C is path independent if and only if it is substitutable and satisfies IRP.*

The structure of path independent choice functions is well-known. In particular, every path independent choice function can be written as the union of choices made by preference relations as follows.

Lemma 2 (Aizerman and Malishevski (1981))**.** *A choice function C is path independent if and only if there exists a finite sequence of preference relations on $\mathcal{X}$, $\{\succeq_i\}_{i \in I}$, such that for any $X \in \mathcal{X}$*

$$C(X) = \bigcup_{i \in I} \{x_i^*\},$$

where x_i^* *is defined as* $x_i^* = \max_X \succeq_i$.

This representation is different from lexicographic representation since every partner is considered in the maximization of the preference relations rather than the set of remaining partners. In the next section, we show that this representation is useful in establishing the theory of stable matchings. If the choice function can be empty-valued, then this result goes through when $\succeq_i$ is over $\mathcal{X} \cup \emptyset$ and $x_i^* = \max_{X \cup \emptyset} \succeq_i$.

A ***partial order*** is a reflexive, antisymmetric, and transitive relation.[5] If $\succeq$ is a partial order on $\mathcal{K}$, we say that the pair $(\mathcal{K}, \succeq)$ is a ***partially ordered set***. A partially ordered set $(\mathcal{K}, \succeq)$ is a ***lattice*** if, for every $k, l \in \mathcal{K}$, the least upper bound and the greatest lower bound of $\{k, l\}$ exist in X with respect to the partial order $\succeq$. We denote the least upper bound of $\{k, l\}$ by $k \vee l$; and the greatest lower bound of $\{k, l\}$ by $k \wedge l$.

The following lemma is important for establishing a lattice structure for a path independent choice function.

Lemma 3 (Koshevoy (1999))**.** *Suppose that* C *is path independent. For any* X*, let* $X_C^{\#} = \bigcup_{C(Y)=C(X)} Y$*. Then* $\{Y : C(Y) = C(X)\} = \{Y : X_C^{\#} \supseteq Y \supseteq C(X)\}$.

Let $\mathcal{I}(C) = \{C(X) : X \in \mathcal{X}\}$ be the image of a choice function C. Define the following partial order: for every $X, Y \in \mathcal{I}(C)$, $X \geq Y$ if and only if $X_C^{\#} \supseteq Y_C^{\#}$.

Lemma 4 (Johnson and Dean (2001))**.** *For any path independent choice function* C*,* $(\mathcal{I}(C), \geq)$ *is a lattice where for any* $X, Y \in \mathcal{I}(C)$*,* $X \vee Y = C(S \cup S')$ *and* $X \wedge Y = C(X_C^{\#} \cap Y_C^{\#})$.

Remark 1. The structure of such lattices has been studied at least since Dilworth (1940). These lattices are isomorphic to the ones studied in Koshevoy (1999), and are surveyed in Edelman and Jamison (1985).

This lattice structure will be useful in establishing one of our results below. Here, we demonstrate how to construct this lattice structure in the following example.

Example 1. Consider a path independent choice function C on $\{1,2,3,4,5,6\}$ with $C(1,2,3,4,5,6) = \{1,2\}$, $C(3,4,5,6) = \{3,4\}$, $C(3,5,6) = \{3,5\}$, $C(4,5,6) = \{4,5\}$, $C(1,5,6) = \{1,6\}$, and $C(2,5,6) = \{2,6\}$.[6] Since $C(1,2,3,4,5,6) = \{1,2\}$, then $\{1,2\} \in \mathcal{I}(C)$ with $\{1,2\}_C^{\#} = \{1,2,3,4,5,6\}$. This is the greatest element of $(\mathcal{I}(C), \geq)$. This choice function is not completely defined yet since there is only partial information. We complete it using path independence and also in a random way to get the choice function in Figure 1. Next to each element $X \in \mathcal{I}(C)$ in the lattice, we write $(X_C^{\#})$ if it is different from X. For example,

[5]Reflexive: For all $x \in \mathcal{X}$, $x \succeq x$.

[6]For ease of notation we denote $C(\{x, \ldots, y\})$ by $C(x, \ldots, y)$.

$C(1,3,4,5,6) = \{1,4\}$, so $\{1,4\} \in \mathcal{I}(C)$ with $\{1,4\}^{\#}_C = \{1,3,4,5,6\}$. Note that since $\{1,2\}^{\#}_C \supseteq \{1,4\}^{\#}_C$, $\{1,2\} \geq \{1,4\}$ in $\mathcal{I}(C)$. Similarly, $\{2,4\} \vee \{1,6\} = C(1,2,4,6) = \{1,2\}$ and $\{2,4\} \wedge \{1,6\} = C(\{2,4,5,6\} \cap \{1,5,6\}) = C(56) = \{5,6\}$.

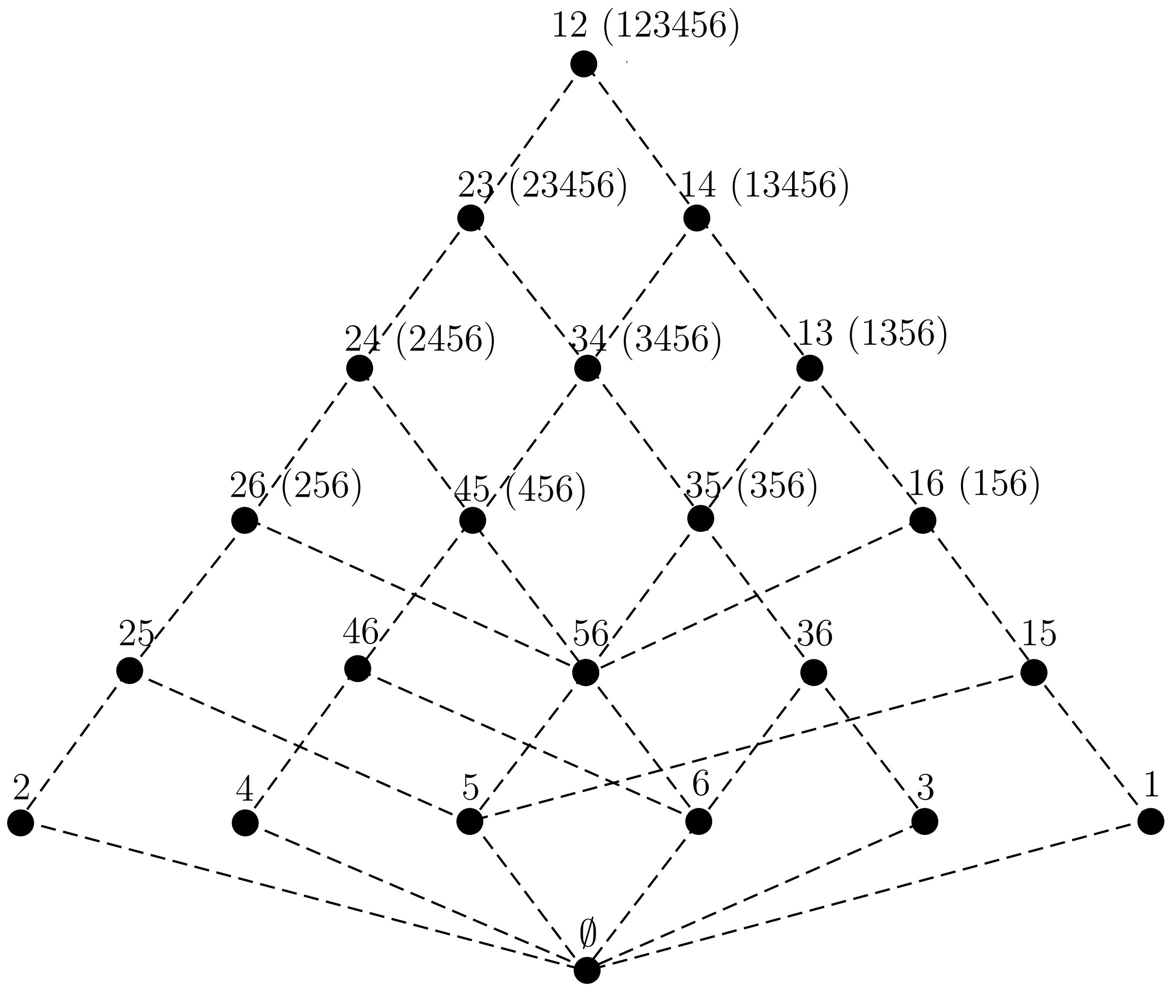

FIGURE 1. Lattice representation of the choice function in Example 1

Since C has such a lattice representation it is path independent. Moreover, it is also 2-acceptant (see the next section).

3. **Matching Markets**

There exist a set of workers $\mathcal{W}$ and a set of firms $\mathcal{F}$. Each worker w has a choice function C_w on $\mathcal{P}(\mathcal{F})$ and each firm f has a choice function C_f on $\mathcal{P}(\mathcal{W})$. The profile of workers' choice functions is denoted by $C_{\mathcal{W}}$ and the profile of firms' choice functions is denoted by $C_{\mathcal{F}}$. A matching market is a tuple $\langle \mathcal{W}, \mathcal{F}, C_{\mathcal{W}}, C_{\mathcal{F}} \rangle$.

A ***matching*** is a function μ on the set of all agents, $\mathcal{F} \cup \mathcal{W}$, such that

(1) $\mu(w) \subseteq \mathcal{F}$,
(2) $\mu(f) \subseteq \mathcal{W}$, and
(3) $w \in \mu(f) \iff f \in \mu(w)$.

In words, (1) each worker is matched with a set of firms, (2) each firm is matched with a set of workers, and (3) worker w is matched with firm f if and only if firm f is matched with worker w.

For two-sided matching markets, stability has proved to be a useful solution concept (Roth and Sotomayor, 1990). Matching μ is ***stable*** if

(1) (individual rationality) for every agent $a \in \mathcal{F} \cup \mathcal{W}$, $C_a(\mu(a)) = \mu(a)$
(2) (no blocking) there exists no pair $(w, f) \in \mathcal{F} \times \mathcal{W}$ such that $w \in C_f(\mu(f) \cup \{w\})$ and $f \in C_w(\mu(w) \cup \{f\})$.

Therefore, a matching μ is individually rational if every agent wants to keep all of her matching partners. The second condition rules out the existence of blocking pairs (w, f) such that worker w would like to be matched with firm f and firm f would like to be matched with worker w.

The existence of stable matchings requires choice functions to be path independent (Roth and Sotomayor, 1990; Hatfield and Milgrom, 2005; **?**). Thus, we assume that all choice functions are path independent. Note that allowing more general blocking coalitions do not make a difference under path independent choice functions (Echenique and Oviedo, 2006).

By Lemma 2, for each worker w, C_w can be viewed as the union of preference relations over individual firms such that for any set of firms F, $C_w(F)$ can be calculated by taking the union of highest ranked firms with respect to the preference relations. Thus, each worker can be viewed as the union of some personas that have different preferences over firms but for each firm all personas of a worker are the same. Analogously, each firm can be thought of the union of some positions each of which has its own preference over individual workers but for each worker all positions in the same firm are the same.

This interpretation allows us to establish some new results as well as provide simple proofs of known results in the literature. First, we generalize the deferred acceptance algorithm of Gale and Shapley (1962) to many-to-many matching markets in which each worker is viewed as a union of many personas.[7]

[7] Note that this approach is different than decomposing each agent to many agents with unit demand with the same preferences, which is done in the case of responsive choice functions (Roth and Sotomayor, 1990) because different personas of the same worker is the same for all firms.

Worker Proposing Deferred Acceptance Algorithm (DA)

Step 1: Each persona of every worker applies to her most preferred firm. Each firm f considers the set of workers that has applied to it, say W_f^1, accepts $C_f(W_f^1)$ and rejects the rest. If there are no rejections, stop.

Step k: Each persona whose offer was rejected at Step $k-1$ applies to her next preferred firm if such a firm exists. Otherwise, this persona does not apply to any firm. Each firm f considers all of the new workers that have applied to it and the tentatively accepted workers at Step $k-1$, say W_f^k, accepts $C_f(W_f^k)$ and rejects the rest. If there are no rejections, stop.

In the worker proposing DA, each persona of a worker applies to firms. Each firm treats different personas of the same worker the same and either rejects all of them or accepts all of them. The firm proposing deferred acceptance algorithm can be defined analogously by viewing each firm as the union of some positions with different preferences over workers. Since there is a finite number of agents and personas of workers, the algorithm ends in finite time. When choice functions can be empty-valued, then DA can be defined analogously where each persona applies to the next preferred firm if the firm is better than being unmatched.

Even though the next result is well-known in the matching literature, we include it for completeness and also to provide a new and simple proof using Lemma 2. See Roth (1984); Sotomayor (1999); Echenique and Oviedo (2006); **?** for the existence of stable matchings in many-to-many matching markets. All of these papers take preferences over groups of agents as primitives rather than their choice functions, which is the more general approach that we provide here.

Theorem 1. *Suppose that each agent has a path independent choice function. Then the worker proposing deferred acceptance algorithm produces a stable matching.*

Proof. Let μ^W be the matching produced by the worker prosing deferred acceptance algorithm. First we show that μ^W is individually rational for all agents. By construction of the algorithm, for each firm f, $C_f(\mu^W(f)) = \mu^W(f)$, so μ is individually rational for firms. For each persona of worker w, the firm held at the end of the algorithm is the best among the set of firms who have not rejected worker w yet, so this firm is also the best among $\mu^W(w)$ according to this persona's preference. Therefore, $C_w(\mu^W(w)) = \mu^W(w)$, so μ is also individually rational for workers.

Second we show that there does not exist a blocking pair. For worker w consider a firm f such that $f \in C_w(\mu(w) \cup \{f\}) \setminus \mu(w)$. Thus, worker w would like to be matched with firm f. But since $f \notin \mu(w)$, all the applications from worker w must have been rejected by firm f. But each position in firm f gets a weakly better worker at every step of the algorithm, so every position in firm f has a better worker than worker w at $\mu(f)$. This means $w \notin C_f(\mu(f) \cup \{w\})$. Therefore, there exists no blocking pair and that μ is stable. □

Even though this result is known in the literature, we provide a surprisingly simple proof by using Lemma 2. Similarly, simpler proofs for many results in the literature can be provided. For example, it can be shown that there exists a lattice structure on the set of stable matchings where the lattice operators are defined by considering the preferences of firm positions or worker personas. This lattice structure is equivalent to the one established in Fleiner (2003) for many-to-many matching markets using choice functions. In particular, the worker proposing deferred acceptance algorithm produces the worker optimal stable matching when agents have path independent choice functions. Instead of providing shorter proofs for more results known in the literature, we establish the following new result. First a definition is in order.

Definition 4. *Choice function C' is an* ***expansion*** *of C if, for every X, $C'(X) \supseteq C(X)$.*

Thus, for an agent, if choice function C' is an expansion of another choice function C, then for any set of partners, every partner chosen by C is also chosen by C'. Below we establish comparative statics when an agent's choice function is expanded.

Theorem 2. *Suppose that μ is a stable matching with respect to $C = (C_{\mathcal{W}}, C_{\mathcal{F}})$ where each choice function is path independent. Fix a firm $\hat{f}$. Suppose that $C'_{\hat{f}}$ is a path independent expansion of $C_{\hat{f}}$. Then there exists a stable matching μ' with respect to $C' = (C_{\mathcal{W}}, (C'_{\hat{f}}, C_{-\hat{f}}))$ such that, for every worker w, $C_w(\mu'(w) \cup \mu(w)) = \mu'(w)$ and, for every firm $f \neq \hat{f}$, $C_f(\mu'(f) \cup \mu(f)) = \mu(f)$.*

Proof. Since we have two profiles of choice functions, we are going to append the choice function profiles to stability, individual rationality, and no blocking to avoid confusion.

Since μ is C-stable, we have $C_a(\mu(a)) = \mu(a)$ for every $a \in \mathcal{W} \cup \mathcal{F}$. In particular, $C_{\hat{f}}(\mu(\hat{f})) = \mu(\hat{f})$. Since $C'_{\hat{f}}$ is an expansion of $C_{\hat{f}}$, we have $C'_{\hat{f}}(\mu(\hat{f})) \supseteq C_{\hat{f}}(\mu(\hat{f})) = \mu(\hat{f})$. This implies $C'_{\hat{f}}(\mu(\hat{f})) = \mu(\hat{f})$. Therefore, μ is C'-individually rational.

If there are no C'-blocking pairs for μ, then μ is C'-stable and we can take $\mu' = \mu$. Suppose otherwise that there are blocking pairs. For each such blocking pair (w, f), we must have $f = \hat{f}$ because $C_a = C'_a$ for every $a \neq \hat{f}$.

Modify μ as follows to get a new matching μ_0. Consider $W = \{w | \hat{f} \in C_w(\mu(w) \cup \{\hat{f}\})\}$, the set of workers who would like to get matched with firm $\hat{f}$. Note that $W \supseteq \mu(\hat{f})$ because μ is individually rational for workers. Let $\mu_0(\hat{f}) = C'_{\hat{f}}(W)$. Since $C'_{\hat{f}}(W) \supseteq C_{\hat{f}}(W) = \mu(\hat{f})$ we get $\mu_0(\hat{f}) \supseteq \mu(\hat{f})$. Define the outcome for workers in μ_0 as follows:

$$\mu_0(w) = \begin{cases} C_w(\mu(w) \cup \{\hat{f}\}) & \text{if } w \in \mu_0(\hat{f}) \setminus \mu(\hat{f}) \\ \mu(w) & \text{otherwise.} \end{cases}$$

For any firm f, define $\mu_0(f)$ as $w \in \mu_0(f) \iff f \in \mu_0(w)$. By construction, μ_0 is a matching. We show that the firm proposing deferred acceptance algorithm starting at μ_0 produces a stable matching with respect to C'.

Step 0: Each worker $w \in \mu_0(\hat{f}) \setminus \mu(\hat{f})$ rejects firms in $\mu(w) \setminus \mu_0(w)$. If there are no rejections then stop. The matching at the end of this step is μ_0.

Step k: Each rejected position applies to the next preferred worker if there is any acceptable worker left. Otherwise, this position remains unmatched. Each worker w that receives a new proposal considers all firms who are accepted tentatively at Step $k-1$ and firms who just applied, say F_w^k, accepts $C_w(F_w^k)$ and rejects the rest. If there are no rejections then stop.

The algorithm ends in finite time since there is a finite number of positions and at least one rejection at every step except the last one. Let μ' be the outcome of this algorithm. Just like the proof of Theorem 1, it is easy to see that μ' is stable with respect to C' since every transition from μ to μ' including the steps of DA makes workers better off and firms worse off.

In addition, each persona of every worker is getting a weakly better firm at every step of the algorithm, so $C_w(\mu'(w) \cup \mu(w)) = \mu'(w)$. Similarly, each position of every firm $f \neq \hat{f}$ is getting a weakly worse worker at every step of the algorithm, so $C_f(\mu'(f) \cup \mu(f)) = \mu'(f)$. □

This result establishes that when an agent's choice function expands, other agents on the same side are worse off and all agents on the other side are better off. For example, when a firm opens a new position then all positions in the other firms get a worse worker and all worker personas get a better firm. Similarly, this result can be applied to the situation in which a new agent joins the market because this can be modeled as the agent's choice function being expanded from the empty choice. Therefore, we get as corollaries the comparative statics results of Kelso and Crawford (1982) and Blum et al. (1997).

Theorem 2 generalizes Theorem 6 of **?** in two dimensions. First, the result in **?** only shows that expanding choice functions on one side makes every agent on the other side better off but does not compare the outcome for agents on the same side. Second their result is for many-to-one matching markets whereas ours is for many-to-many matching markets.

As a corollary to Theorem 2, we establish the following.

Corollary 1. *Suppose that μ is a stable matching with respect to $C = (C_F, C_W)$ and C'_f is an expansion of C_f for every firm f. Then there exists a stable matching μ' with respect to $C' = (C_{\mathcal{W}}, C'_{\mathcal{F}})$ such that for every worker w, $C_w(\mu'(w) \cup \mu(w)) = \mu'(w)$.*

In the school choice context (**?**), when schools are matched with sets of students, this corollary establishes that to maximize welfare of students, schools' choice functions should be maximized as much as possible. However, each school has a limited capacity, so its choice function's cardinality cannot exceed this given capacity. Therefore, a school's choice function can be expanded if the number of admitted students does not exceed the capacity. In the next section, we study when choice rules can be expanded in such a manner.

4. Acceptant Path Independent Choice Functions

We say that a choice function is acceptant if it admits as many partners as it can without violating the capacity.

Definition 5. *Choice function C is* ***q-acceptant*** *if $|C(S)| = \min\{q, |S|\}$. Choice function C is* ***acceptant*** *if it is q-acceptant for some q.*

? used q-acceptant priorities to characterize DA. In an earlier work, Alkan (2001) showed the lattice structure of stable matchings under acceptant substitutable choice functions.

Given a path independent choice function, can we find a path independent expansion, which is also acceptant? We show that this is always possible using the lattice structure of the choice function.

Theorem 3. *Every path independent choice function with maximum cardinality q has a q-acceptant path independent expansion.*

Proof. The proof is by construction. Consider a path independent choice function C. If there exists no $X \in \mathcal{I}(C)$ such that $X^{\#}_{C} \neq X$ and $|X| < q$ then C is q-acceptant and there is nothing to prove because a choice function is trivially an expansion of itself. Otherwise, consider a minimal such set S (minimality is with respect to set inclusion). Take any $x \in X^{\#}_{C} \setminus X$. Then there exists a path independent expansion C' of C where there is an additional element $X \cup \{x\}$ in $\mathcal{I}(C')$, so $\mathcal{I}(C') = \mathcal{I}(C) \cup \{X \cup \{x\}\}$ (Johnson and Dean, 2001, Theorem 6). By construction C' is a path independent expansion of C where $(X \cup \{x\})^{\#}_{C'} = X^{\#}_{C}$ and $X^{\#}_{C'} = X^{\#}_{C} \setminus \{x\}$. If C' is q-acceptant, then we are done. Otherwise, repeat this procedure of expanding the choice function so that there is an additional element in the lattice. If this procedure ends in a finite number of steps, then we have constructed a q-acceptant path independent expansion of choice function C. Next we show that this procedure ends in a finite number of steps.

For any path independent choice function C, we assign an inefficiency number $i(C)$ by $i(C) = \sum_{X \in \mathcal{I}(C), |X| < q} 2^{|X^{\#}_{C} \setminus X|} - 1$. Note that if $i(C) = 0$, then the choice function is q-acceptant. Now suppose that C' is an expansion of C via $X^{\#}_{C}$ and X as constructed above. In $\mathcal{I}(C')$ we have an additional element $\{X \cup \{x\}\}$ such that $(X \cup \{x\})^{\#}_{C'} = X^{\#}_{C}$ and also $X^{\#}_{C'} = X^{\#}_{C} \setminus \{x\}$. Let $|X^{\#} \setminus X| = k$. Then $i(C') - i(C) \leq [(2^{k-1} - 1) + (2^{k-1} - 1)] - (2^{k} - 1) = -1$. Therefore, each iteration of the above procedure reduces the inefficiency of the choice function by at least one. Since $i(C)$ is finite for every path independent C, the above algorithm ends in a finite number of steps. □

In the proof, we show that a path independent choice function can be expanded in such a way that the new choice function is also path independent using the lattice structure. In addition, we show that each path independent choice function can be assigned an inefficiency number such that each expansion reduces this inefficiency number by at least one. The proof is immediate by repeating the procedure since the inefficiency number is finite. Next we provide the following example to illustrate how this expansion algorithm works.

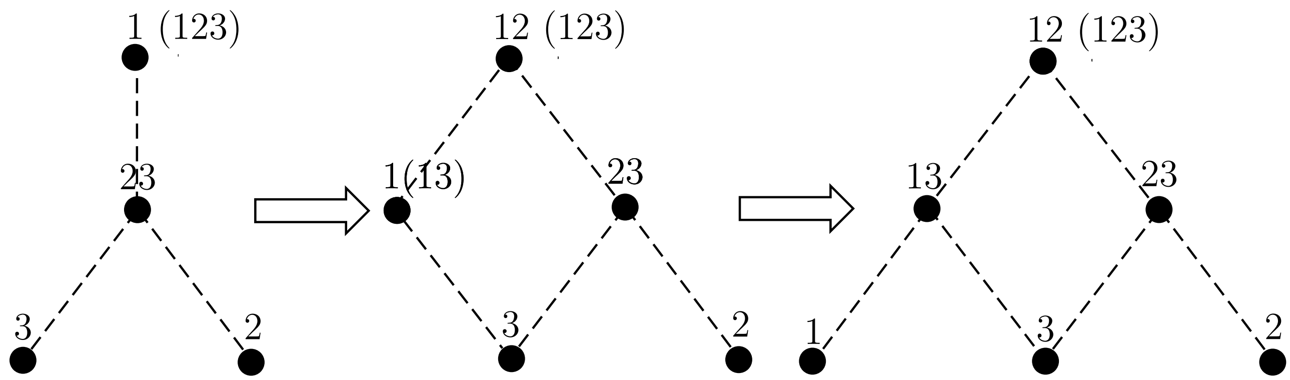

Figure 2. Expansion of the choice function in Example 2

Example 2. Consider the following choice function C on $\{1,2,3\}$: $C(1,2,3)=\{1\}$ and $C(2,3)=\{2,3\}$. There is a unique path independent choice function with these values shown on Figure 2 (the leftmost lattice). The maximum cardinality of C is 2 but C is not 2-acceptant because $|C(123)|=1$. According to the algorithm we pick a minimal set X (with respect to set inclusion) such that $X_C^{\#}\neq X$ and $|X|<q$. The unique such set for C is $\{1\}$ with $\{1\}_C^{\#}=\{1,2,3\}$. Therefore, the inefficiency of C is $2^2-1=3$. We pick any partner x in $\{1,2,3\}\setminus\{1\}=\{2,3\}$. Let $x=2$. In the lattice representation, we add the node $X\cup\{x\}=\{1,2\}$ such that $\{1,2\}^{\#}=\{1,2,3\}$. In other words, we modify the choice function so that $C(1,2,3)=\{1,2\}$. In the new lattice representation $\{1\}^{\#}=X_C^{\#}\setminus\{x\}=\{1,3\}$ (the middle lattice on Figure 2). This new path independent choice function, say C', is still not acceptant and the inefficiency number is $2^1-1=1$, so we find a minimal set Y such that $Y_{C'}^{\#}\neq Y$ and $|Y|<q$. The unique such set is $\{1\}$ with $\{1\}^{\#}=\{1,3\}$. We pick $y\in\{1,3\}\setminus\{1\}=\{3\}$ and create and additional node in the lattice for $\{1,3\}$. The new choice function is an acceptant path-independent expansion of C (the rightmost lattice on Figure 2).

Next we return to lexicographic choice functions, which are important because, in practical school choice, schools may find it easier to report preference relations over individual students rather than a choice function on some sets of students. In addition, this representation can also be used to promote diversity (**???**). We show that these choice functions are acceptant and path independent. However, there can be acceptant and path independent choice functions that are not lexicographic.

Theorem 4. *A lexicographic choice function is acceptant and path independent. There exists an acceptant, path independent choice function that is not lexicographic.*

Proof. First we show that if C is lexicographic with $\{\succeq_i\}_{i\in I}$ then it is acceptant and path independent. It is easy to see that C is $|I|$-acceptant and that it satisfies ORS. We show that C is also substitutable.

Suppose that $x \in X \subseteq Y$ and $x \in C(Y)$. We show that $x \in C(X)$. Let x_i^* and y_i^* be as in the definition of $C(X)$ and $C(Y)$. We show by induction that $X \setminus \{x_1^*, \ldots, x_k^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_k^*\}$. The claim is true for $k = 0$ because $X \subseteq Y$.[8] Suppose that the claim holds for $k = 0, \ldots, n-1$, we prove it for n where $n \leq |I|$. Since $y_n^* = \max_{Y\setminus\{y_1^*,\ldots,y_{n-1}^*\}} \succeq_n$ and $x_n^* = \max_{X\setminus\{x_1^*,\ldots,x_{n-1}^*\}} \succeq_n$. There are two cases. Fist, $y_n^* = x_n^*$. In this case, $X \setminus \{x_1^*, \ldots, x_{n-1}^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_{n-1}^*\}$ implies $X \setminus \{x_1^*, \ldots, x_n^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_n^*\}$. Second, $y_n^* \neq x_n^*$. This implies that $y_n^* \notin X \setminus \{x_1^*, \ldots, x_{n-1}^*\}$, so $X \setminus \{x_1^*, \ldots, x_{n-1}^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_n^*\}$ and $X \setminus \{x_1^*, \ldots, x_n^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_n^*\}$. Since $x \in C(Y)$, there exists k such that $x = y_{k+1}^*$. By the claim above, $X \setminus \{x_1^*, \ldots, x_k^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_k^*\}$. Therefore, if $x \in X \setminus \{x_1^*, \ldots, x_k^*\}$, then $x = x_{k+1}^*$ and so $x \in C(X)$. Otherwise, if $x \notin X \setminus \{x_1^*, \ldots, x_k^*\}$, then $x = x_i^*$ for some $i \leq k$ and so $x \in C(X)$. Thus C is also substitutable.

We prove that not every acceptant path independent choice function is lexicographic by providing an example. Consider choice function C in Example 1, which is a 2-acceptant path independent choice function. We show that C is not lexicographic. In fact, we only use partial knowledge on C that $C(1,2,3,4,5,6) = \{1,2\}$, $C(3,4,5,6) = \{3,4\}$, $C(3,5,6) = \{3,5\}$, $C(4,5,6) = \{4,5\}$, $C(1,5,6) = \{1,6\}$, and $C(2,5,6) = \{2,6\}$ to get a contradiction.

Consider any choice function C with the stated values above. Since 1 and 2 are symmetric for this choice function and that $C(1,2,3,4,5,6) = \{1,2\}$ we can assume without loss of generality that the highest ranked element in $\succ_1$ is 1 and the highest ranked element in $\succ_2$ is 2. Since $C(3,4,5,6) = \{3,4\}$ either $3 \succ_1 4,5,6$, $4 \succ_2 5,6$ or $4 \succ_1 3,5,6$, $3 \succ_2 5,6$.

Case 1 ($\mathbf{3 \succ_1 4,5,6,\ 4 \succ_2 5,6}$): Since $C(3,5,6) = \{3,5\}$ and $3 \succ_1 5,6$, we get $5 \succ_2 6$. But $C(1,5,6) = \{1,6\}$ and $1 \succ_1 5,6$, so we have $6 \succ_2 5$, which is a contradiction.

Case 2 ($\mathbf{4 \succ_1 3,5,6,\ 3 \succ_2 5,6}$): Since $C(4,5,6) = \{4,5\}$ and $4 \succ_1 5,6$, we get $5 \succ_2 6$. But $C(1,5,6) = \{1,6\}$ and $1 \succ_1 5,6$, so we have $6 \succ_2 5$, which is a contradiction. □

Now, we ask whether a path independent and acceptant choice function is uniquely determined by its behavior on some subclass of sets. Indeed, it turns out that a q-acceptant path

[8] With an abuse of notation let $\{x_1^*, \ldots, x_k^*\} = \{y_1^*, \ldots, y_k^*\} = \emptyset$ for $k = 0$.

independent choice function is characterized completely by its behavior on sets of cardinality $q+1$. While this result is not difficult to prove, it can be very useful computationally. Reporting the value of a choice function on every admissible set requires reporting the value of the choice function on $2^{|\mathcal{X}|}-1$ sets–this quantity is exponential in the number of partners. On the other hand, reporting the value of the choice function on sets of cardinality $q+1$ requires a report only for $\binom{|\mathcal{X}|}{q+1}$ sets; this is polynomial of degree $q+1$.

Theorem 5. *Suppose C and C' are two q-acceptant and path independent choice functions such that for all A with $|A| = q+1$, we have $C(A) = C'(A)$. Then $C = C'$.*

Proof. For any A for which $|A| \leq q$, we know that $C(A) = C'(A)$ since each of C and C' are q-acceptant. Let $l > q+1$ and suppose we have shown that for all A such that $|A| < l$, $C(A) = C'(A)$. Let A be a set for which $|A| = l$. Let $s, t \in A$ such that $s \neq t$. Then by path independence, $C(A) = C(C(A\backslash\{s\}) \cup C(A\backslash\{t\}))$. Now, we claim that $|C(A\backslash\{s\}) \cup C(A\backslash\{t\})| < l$. Otherwise, we must have $C(A\backslash\{s\}) = A\backslash\{s\}$ and $C(A\backslash\{t\}) = A\backslash\{t\}$. However, this contradicts the assumption that C is q-acceptant, as $|A\backslash\{s\}| = l-1 > q$. Similarly, $C'(A) = C'(C'(A\backslash\{s\}) \cup C'(A\backslash\{t\}))$ and $|C'(A\backslash\{s\}) \cup C'(A\backslash\{t\})| < l$. By induction $C(A\backslash\{s\}) \cup C(A\backslash\{t\}) = C'(A\backslash\{s\}) \cup C'(A\backslash\{t\})$, which implies $C(A) = C'(A)$. The proof is completed by induction. $\square$

For a more general result, including a characterization of the implications of path independence on the sets of cardinality $q+1$, see Dietrich (1987).

Responsiveness is equivalent to being lexicographic where the same preference relation is used at every step. Therefore, any responsive choice function is also acceptant and path independent. In addition, responsive choice functions satisfy the following rationality axiom.

Definition 6. *Choice function C satisfies the* ***weakened weak axiom of revealed preference (WWARP)*** *if, for any x, y, X, and Y such that $x, y \in X \cap Y$,*

$$x \in C(X) \text{ and } y \in C(Y) \setminus C(X) \text{ imply } x \in C(Y).$$

WWARP was first introduced by Ehlers and Sprumont (2008) in the context of nonrationalizable choice, though it was implicitly discussed in Wilson (1970), who analyzed the class of choice functions satisfying it (calling them "Q cuts"). We show that a choice function is responsive if and only if it is acceptant and satisfies WWARP. Since responsive choice functions are path independent, this result also implies that path independence follows from WWARP and being acceptant.

Theorem 6. *Choice function C is responsive if and only if it is acceptant and satisfies WWARP.*

Proof. One direction is obvious. Conversely, suppose that C satisfies WWARP and it is acceptant. Define $\succ^*$ by $x \succ^* y$ if there exists X for which $\{x, y\} \subseteq X$, $x \in C(X)$ and

$y \notin C(X)$. WWARP is equivalent to asymmetry of the relation $\succ^*$. We claim that $\succ^*$ is transitive.

Suppose that $x \succ^* y \succ^* z$. Associated with being acceptant is $k \in \mathbb{N}_+$ such that $C(X) = \min\{k, |X|\}$. Since $x \succ^* y$, there exists X for which $\{x, y\} \subseteq X$, $x \in C(X)$ and $y \notin C(X)$. Hence, $|C(X)| = k$. Consequently, there exists $\{a_1, \ldots, a_{k-1}\} \subseteq X$ for which $C(X) = \{x, a_1, \ldots, a_{k-1}\}$. This implies $a_i \succ^* y$ for all i. Obviously, $a_i \neq y$ for all i. Now, consider $Y = \{x, y, z, a_1, \ldots, a_{k-1}\}$. Suppose, by means of contradiction, that $z \in C(Y)$. Then $y \in C(Y)$; otherwise, we have $y \succ^* z$ and $z \succ^* y$ contradicting asymmetry. Now, since $y \in C(Y)$, we have $x \in C(Y)$ and $a_i \in C(Y)$ for all i; otherwise, we would have either $x \succ^* y$ and $y \succ^* x$ or $a_i \succ^* y$ and $y \succ^* a_i$ for some i. But $|Y| \geq k+1$, so that $|C(Y)| \geq k+1$, contradicting the assumption that C is acceptant. Similarly we show that $y \notin C(Y)$. Since C is q-acceptant, this implies that $C(Y) = \{x, a_1, \ldots, a_{k-1}\}$. So $x \succ^* z$.

The rest is now standard; by the Szpilrajn Theorem (see for example Duggan (1999)), there is a preference relation $\succeq$ for which $x \succ^* y$ implies $x \succ y$. Clearly, if $x \in C(X)$ and $y \succeq x$ and $y \in X$, we have $y \in C(X)$. Otherwise, we would have $y \succ^* x$, and $y \succ x$, a contradiction. By definition C is responsive with respect to $\succeq$. This kind of construction was first introduced in Aleskerov et al. (2007) or Tyson (2008). □

Responsivity in the case of $|C(X)| = 1$ is equivalent to the standard notion of rationalizability by a preference relation. And moreover, when $|C(X)| = 1$ for all X, WWARP is equivalent to the classical weak axiom of revealed preference (see for example Uzawa (1956) or Arrow (1959)). Thus, Theorem 6 is a generalization of the well-known result that the weak axiom of revealed preference characterizes rationalizability when choice is single-valued and all budgets are available.

A weaker version of WWARP is also useful.

Definition 7. *Choice function C satisfies the **q-weakened weak axiom of revealed preference (WWARP)** if, for any x, y, X, and Y such that $x, y \in X \cap Y$ and $|X| = |Y| = q$,*

$$x \in C(X) \text{ and } y \in C(Y) \setminus C(X) \text{ imply } x \in C(Y).$$

Lemma 5. *If choice function C satisfies substitutability, $q+1$-WWARP, and q-acceptance, then it satisfies WWARP.*

Proof. Suppose by means of contradiction that there are X and Y, x, y for which $x, y \in X \cap Y$, $x \in C(X)$, $y \notin C(X)$, $y \in C(Y)$, $x \notin C(Y)$. By q-acceptance, it follows that $|X| \geq q+1$ and $|Y| \geq q+1$, and by q-WWARP, at least one of these inequalities is strict.

Now, $C(C(X) \cup \{y\}) = C(X)$ by substitutability and q-acceptance; likewise, $C(C(Y) \cup \{x\}) = C(Y)$. But note that $|C(X) \cup \{y\}| = |C(Y) \cup \{x\}| = q+1$, and that $x \in C(C(X) \cup$

$\{y\})$, $y \notin C(C(X) \cup \{y\})$, $y \in C(C(Y) \cup \{x\})$, and $x \notin C(C(Y) \cup \{x\})$, a contradiction to q-WWARP. □

Theorem 7. *Choice function C is q-responsive if and only if it is substitutable, satisfies $q+1$-WWARP, and q-acceptance.*

Proof. Follows immediately from Lemma 5 and Theorem 6. □

5. Conclusion

We have established several results on the theory of acceptant choice functions and stable matchings. First, we have shown how classical characterizations of path independence allow new, simplified proof of a classical existence result for stable matching and a new result on comparative statics of stable matchings. Further, we have developed characterizations of families of acceptant choice functions, as well as shown how to construct acceptant choice functions from ones which are merely path independent.

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