Money metric utilitarianism

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Introduction

- Measuring social welfare in monetary terms
- Simple measure of social welfare: amount of money needed to bring everyone to same welfare level

Whatever the merits of the money-metric utility concept developed here, a warning must be given against its misuse. Since money can be added across people, those obsessed by Pareto-optimality in welfare economics as against interpersonal equity may feel tempted to add money-metric utilities across people and think that there is ethical warrant for maximizing the resulting sum. That would be an illogical perversion, and any such temptation should be resisted.

Paul Samuelson

- Such rule depends on prices, it generates ordinal ranking (which also depends on price)
- Ranking can be discussed without reference to prices
- What are the properties of possible rules that can be generated?
- Rules collectively satisfy an independence condition
- Appropriate for large economies

- Some notions of measuring social welfare in large economies
- Rank efficient allocations according to aggregate monetary value
- Samuelson–exists no welfare indicator depending only on prices and consumption
- "GDP" certainly doesn't work at given prices: can have $p \cdot x > p \cdot y$ and $q \cdot y > q \cdot x$

- Alternatively: for two efficient aggregate allocations (p, x) (q, y); can compare $p \cdot x$ and $p \cdot y$, and $q \cdot x$ and $q \cdot y$
- Gorman–this rule not transitive (obviously not even complete!)
- These rules are easy to compute but discard *too much* info on individual preference
- Resolution: use a base set of prices and compute social welfare using these

- With base prices; however, Pareto property not respected
- That is, each i may prefer x_i to y_i, yet $p \cdot \left(\sum y_i\right) \geq p \cdot \left(\sum x_i\right)$
- Hence the modification: minimal amount of money needed at base prices to bring each individual to their welfare level

- Social choice requires info about individual preferences
- Often runs into impossibilities all the same (eg Arrow)
- Impossibility results from independence which discards information about all alternatives not being compared
- Our result finds boundary between possibility and impossibility

- Rule proposed above (money needed to bring agents to a certain welfare level)
- At prices p, money needed to bring individual to welfare level at x is

$$\inf \{p \cdot y : y \succsim x\}.$$

- For group of individuals, it is

$$\sum_{i \in N} \inf \{p \cdot y_i : y_i \succsim_i x_i\}.$$

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$$= \inf \left\{ p \cdot y : y = \sum_{i\in N} y_i,\ y_i \succsim_i x_i \right\}$$

- Thus, to compute social utility of allocation $(x_i)_{i\in N}$, only need to know aggregate bundles which can be allocated to Pareto dominate

- Set of such bundles generates well known "community indifference curve," or Scitovsky upper contour set
- Suggests independence condition: ranking of two allocations depends only on Scitovsky upper contour sets
- Similar in spirit to axioms proposed in social choice–Hansson, Fleurbaey & Maniquet (etc) stating that ranking should depend only on individual indifference sets
- Requires info about preferences over other alternatives, but not *all info*

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- Critical (but informal) point: In order to generate meaningful ranking compatible with Pareto principle, we *need* to know at least the Scitovsky sets
- If two allocations are Pareto ranked, their Scitovsky sets are nested (set inclusion on Scitovsky sets is an extension of Pareto relation)
- As little information as can possibly required without generating impossibility

- Aggregate information is needed
- But info about *names*, *numbers*, *etc* of individuals is not needed
- Info about individual preference and consumption not needed
- In this sense, such an independence is appropriate for a "large" or aggregate economy

- This independence axiom is almost characteristic
- Exogenous prices, however, may be problematic (any such choice may have nontrivial ethical consequences)
- Instead of fixing prices, may randomize price and take social welfare to be expectation

The model

- Consumption space $X = \mathbb{R}^m_+$
- Preferences $\succsim \in \mathcal{P}$: weak order, convex, monotonic, continuous
- Variable population model, so let $\mathbb{N}$ be potential agents
- $N \subset \mathbb{N}$ finite a society; an economy specifies tuple of N and preference profile in $\mathcal{P}^N$: economies denoted $e \in \mathcal{E}$
- Rankings over X^N denoted $\mathcal{P}_N$
- A rule maps profiles to rankings of allocations: $\succsim^0 : \mathcal{E} \to \bigcup_{N \subset \mathbb{N}} \mathcal{P}_N$ such that if $e = \left(N, (\succsim_i)_{i \in N}\right)$, then $\succsim^0 (e) \in \mathcal{P}_N$

- An “unrestricted” domain: homothetic, quasilinear not required (etc)
- Ranking *all* allocations, efficient or otherwise (though theorem still holds on efficient domain)
- Key point: use as little info as possible *and* retain Pareto property on large domain (no parametric assumptions on prefs)

- Axioms placed on rules
- Require standard axioms–that for all $e \in \mathcal{E}$, $\succsim^0 (e)$ is a continuous weak order satisfying the Pareto property
- Continuity guarantees a utility representation of our rule for all economies
- To define our independence axiom, define the **Scitovsky upper contour set** as follows: for economy $e = \left(N, (\succsim_i)_{i \in N}\right)$ and allocation $x \in X^N$,

$$Sc\left(e, x\right) = \left\{ \sum_{i \in N} y_i : y_i \succsim_i x_i \right\}.$$

- Aggregate bundles which can be allocated to Pareto dominate allocation x

Aggregate independence

- To understand main axiom, we first discuss strengthening:

Strong aggregate independence: For all $e, e' \in \mathcal{E}$ and corresponding allocations x, y, z, w, if $Sc(e, x) = Sc(e', z)$ and $Sc(e, y) = Sc(e', w)$, then $x \succsim^0 (e)\, y$ if and only if $z \succsim^0 (e)\, w$.

- This axiom states that a ranking of allocations can only depend on their Scitovsky sets (changing allocations/economies/preferences only has any effect if Scitovsky sets are changed).

- Main axiom weakens this a bit (to be more intuitive); either axiom will work in characterization

Aggregate independence: For all $e, e' \in \mathcal{E}$ and corresponding allocations x, y, z, w, if $Sc(e, x) = Sc(e', z)$, $\sum_N x_i = \sum_{N'} z_j$, and $Sc(e, y) = Sc(e', w)$, $\sum_N y_i = \sum_{N'} w_j$, then $x \succsim^0 (e)\, y$ if and only if $z \succsim^0 (e)\, w$.

- Dependence only on Scitovsky sets (and aggregate bundles)
- Stronger axioms lead to impossibility
- Completely precludes equity considerations
- Similar in spirit to IIA except for whole indifference surfaces
- Coefficient of resource utilization

Reinforcement

- Version of the Pareto property for groups (Young 1975)

Reinforcement: Suppose $e = (N, (\succsim_i)_N)$ and $e' = (N', (\succsim_i)_{N'})$, where $N \cap N' = \varnothing$. If $x \succsim^0 (e)\, y$ and $z \succsim^0 (e')\, w$, then $(x, z) \succsim^0 (N \cup N', (\succsim_i)_{N \cup N'})\, (y, w)$ (with a corresponding statement for strict preference)

- If each of two groups prefers one allocation to the other, then the larger group does not reverse this preference
- Rules out complementarity (formally a condition of additive separability)

- Fix some $p \in \Delta_{++}(m)$ (interior of price simplex)
- Define the p-money metric utility for preference $\succsim \in \mathcal{P}$

$$U^p_{\succsim}(x) \equiv \inf \{p \cdot y : y \succsim x\}$$

- Under our assumptions, $U^p_{\succsim}$ is a continuous utility representation (Weymark; alternatively use Maximum theorem)
- When prices are p, $U^p_{\succsim}(x)$ is the amount of money required to be at welfare level specified by x
- Closely related to McKenzie's expenditure function
- Also sometimes called equivalent income function

The main theorem

- The following is the main result here:

Theorem: A rule $\succsim^0$ satisfies

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Theorem: A rule $\succsim^0$ satisfies weak order, Pareto, continuity, aggregate independence, and reinforcement *if and only if* there exists a Borel probability measure π over $\Delta_{++}(m)$ for which for all $e \in \mathcal{E}$,

$$
\begin{aligned}
x & \succsim^0 (e)\, y \\
& \Leftrightarrow \int_{\Delta_{++}(m)} \sum_N U^p_{\succsim_i}(x_i)\, d\pi(p) \\
& \geq \int_{\Delta_{++}(m)} \sum_N U^p_{\succsim_i}(y_i)\, d\pi(p).
\end{aligned}
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- If p places high value on luxury goods, potentially unfair to wealthy (same if places high value on inferior goods)–so allow for randomization according to π

- Can be proved with either strong or aggregate independence
- Money metric utilitarian rule gives continuous function (as well as preference)
- For any rule with utility representation, can take Bergson Samuelson form: $W\left(\left(U^p_{\succsim_i}(x_i)\right)_{i\in N}\right)$; likewise any separable rule has representation $\sum U_i(x_i)$; content is joint restriction on functional form and social welfare functional form

Proof sketch

- Verification that axioms satisfied is trivial, with exception of continuity
- Continuity verified using dominated convergence theorem argument

- Converse divided into several parts
- First is to show that Pareto and agg. ind. imply strong agg. ind.
- Only need to rank Scitovsky upper contour sets (aggregate independence) or their support functions
- To rank functions, only need to know their difference (reinforcement)
- Differences which are ranked at least as good dyadically convex cone (reinforcement)
- Cone is disjoint from “negative orthant” (Pareto), so is its convex hull
- Separation argument gives π which is weakly respected

- To ensure it is strictly respected use continuity axiom
- Continuity axiom also forces no mass on boundary of price simplex

- Define $U(\succsim, x)$ weak upper contour set of x for $\succsim$
- Important fact: $Sc(e,x) = \sum_{i \in N} U(\succsim_i, x_i)$
- Look now at support function generated by this:

$$H(Sc(e,x), \lambda) = \inf_{y \in Sc(e,x)} \lambda \cdot y$$

- Note

$$\begin{aligned} H(Sc(e,x), \lambda) &= \inf_{y_i \succsim x_i} \lambda \cdot \left(\sum_{i \in N} y_i \right) \\ &= \sum_N \inf_{y_i \succsim x_i} \lambda \cdot y_i \\ &= \sum_N U^{\lambda}_{\succsim_i}(x_i) \end{aligned}$$

- Aggregate independence implies transitive ranking $\succeq^c$ on support functions: $H\left(Sc\left(e,x\right),\cdot\right)$ (there is something to show here)
- Define

$$\mathcal{F} \equiv \left\{f - g : f \succeq^c g\right\}.$$

- Key observation: for all $e \in \mathcal{E}$, $H\left(Sc\left(e,x\right),\cdot\right) - H\left(Sc\left(e,y\right),\cdot\right) \in \mathcal{F}$ if and only if $x \succsim^0 \left(e\right) y$
- This is the same as establishing that for all $e, e' \in \mathcal{E}$, if $x \succsim^0 \left(e\right) y$ and

$$\begin{aligned} & H\left(Sc\left(e,x\right),\cdot\right) - H\left(Sc\left(e,y\right),\cdot\right) \\ = \; & H\left(Sc\left(e',x'\right),\cdot\right) - H\left(Sc\left(e',y'\right),\cdot\right), \end{aligned}$$

then $x' \succsim^0 \left(e'\right) y'$

- By means of contradiction, suppose $x \succsim^0 (e)\, y$,

$$\begin{aligned} & H\left(Sc\left(e,x\right),\cdot\right) - H\left(Sc\left(e,y\right),\cdot\right) \\ = \; & H\left(Sc\left(e',x'\right),\cdot\right) - H\left(Sc\left(e',y'\right),\cdot\right), \end{aligned}$$

  and $y' \succ^0 (e')\, x'$.
- Therefore

$$\begin{aligned} & H\left(Sc\left(e',x'\right),\cdot\right) + H\left(Sc\left(e,y\right),\cdot\right) \\ = \; & H\left(Sc\left(e',y'\right),\cdot\right) + H\left(Sc\left(e,x\right),\cdot\right) \end{aligned}$$

- Put together (wlog) disjoint economies e', e and allocations (x', y), (y', x). Reinforcement implies former strictly preferred to latter; contradicting equality of Scitovsky sets

- Similar techniques can be applied to show $\mathcal{F}$ closed under addition and dyadic scalar multiplication
- It is also disjoint from $C_{--}(\Delta(m))$ (strictly negative functions)
- Otherwise could construct one-agent economy with $x \succ y$, yet $y \succsim^{0}(e)\, x$ (always exists an agent with pair of nested indifference sets)

- Special structure of $C(\Delta(m))$ allows us to conclude convex hull of $\mathcal{F}$ also disjoint from $C_{--}(\Delta(m))$ (in other models this is more difficult)
- Get a separating hyperplane which is monotonic... and normalize it to be a probability measure

- In theory; proof could be done using mixture space theorem on domain of continuous functions–but would require a variant allowing only rational convex combinations
- Implicitly this is what we are doing–the special structure of the space of continuous functions as separated from the negative orthant allows full convex combinations after separation argument has been applied

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- Samuelson was obviously right about lack of ethical properties
- Tradeoffs–independence in aggregate, versus equity illuminated
- Is this rule useful? In many senses better than GDP (complete, transitive)
- Hard to compute Scitovsky sets