SPHERICAL PREFERENCES

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ABSTRACT. We introduce and study the property of orthogonal independence, a restricted additivity axiom applying when alternatives are orthogonal. The axiom requires that the preference for one marginal change over another should be preferred after each marginal change has been shifted in a direction that is orthogonal to both.

We show that continuous preferences satisfy orthogonal independence if and only if they are spherical: their indifference curves are spheres with the same center, with preference being "monotone" moving away from the center. Spherical preferences include linear preferences as a special (limiting) case. We discuss different applications to economic and political environments. Our result delivers Euclidean preferences in models of spatial voting, quadratic welfare aggregation in social choice, and expected utility in models of choice under uncertainty. As an extension, we consider an endogenous notion of orthogonality.

1. INTRODUCTION

We introduce and study the property of *orthogonal additivity,* or *orthogonal independence,* in choice theory, and find that it characterizes a class of preferences with spherical indifference curves. Imagine an agent choosing among consumption bundles: vectors in $\mathbf{R}^n$. The vectors can be interpreted in different ways to capture various economic environments. The property says, loosely speaking, that an agent who prefers to move her consumption

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in direction x to moving her consumption in direction y must preserve this preference when x and y are both shifted in an orthogonal direction.

Our orthogonal independence property, or axiom, is simple to state, and uses ideas familiar to any student of economics. Suppose an agent starts from an endowment, or status quo point, of w. An agent is choosing to either shift her consumption from w to $w+x$, or from w to $w+y$. The axiom, which we term **Origin-independent orthogonal additivity (OIOI)** says that

$$w + x \succeq w + y \text{ and } z \perp x, y \Longrightarrow w + (x + z) \succeq w + (y + z).$$

The direction z is orthogonal to both x and y. In a sense, it both complements and substitutes x and y equally. The axiom says that the comparison of x and y should not be affected by the addition of the orthogonal direction z. The axiom is required to hold for every w, x, y, and z satisfying the hypotheses.

Our main result is that OIOI has strong implications, though much weaker than the analogous unqualified version of independence would have. Together with continuity, OIOI implies ***spherical preferences***: preferences with linear or spherical indifference curves. If the preference has spherical indifference curves, each sphere must have the same center, and the preference must be monotone along any ray emanating from that center. Examples of spherical preferences include perfect substitutes in consumption theory, expected utility in choice under uncertainty, and Euclidean preferences in voting theory.

We now outline several different economic environments where either OIOI has a natural meaning, or the spherical representation has particular interest.

- Net trades. A consumer chooses among consumption bundles $x \in \mathbf{R}^n$, which can be thought of as net trades as they involve negative quantities. Orthogonality has an intuitive geometric meaning.
- Spatial choice. A voter chooses among policy proposals. There are n issues in question, and each policy proposal takes a stand on

each issue, so that proposals can be represented as vectors in $\mathbf{R}^n$. Spherical preferences are closely related to Euclidean preferences, which have received a lot of attention in the literature on voting (Downs, 1957; Stokes, 1963). In fact, Euclidean preferences are the special case of spherical preferences where there is an "ideal point," and the individual is worse off the further away from the idea point.

- Choice under uncertainty. An agent chooses among uncertain monetary payoffs (monetary acts). There are n states of the world and each vector $x \in \mathbf{R}^n$ represents a stage-contingent payoff. When x, y, and z are non-negative, then $z \perp x$ and $z \perp y$ means that z complements x and y in the same states. Thus z's relation to the uncertainty inherent in x is the same as its relation to the uncertainty inherent in y, and we may infer that z is as good as hedge for x as for y. In terms of the representation, monotone spherical preferences coincide with (risk neutral) subjective expected utility.
- Social choice. Consider a society of n agents, and interpret vectors in $\mathbf{R}^n$ as reflecting the level of welfare of each individual agent. Linear preferences embody a form of utilitarianism (Harsanyi, 1955), and more general spherical preferences have been studied by several authors (Epstein and Segal, 1992).
- Dispersion. Consider a finite set of states of the world, with a uniform probability measure over them. The set of vectors which sum to zero are now mean-zero random variables—or monetary acts, and they form a well-defined finite-dimensional vector space. Since all acts have mean zero, we can interpret a ranking as a measure or riskiness or dispersion. Orthogonality now becomes the statement that two random variables are uncorrelated. So the axiom then requires that the addition of an act which is uncorrelated with each of two additionally present acts will not reverse their ranking. In this environment, our axiom becomes related to Pomatto et al. (2019)

and Mu et al. (2019), except that we explore the stronger condition of zero correlation rather than statistical independence.

A key property here is that OIOI is a universal property: it claims a relationship to hold for all collections satisfying certain hypotheses. As such, and according to Chambers et al. (2014), it is falsifiable. On the other hand, the model described by the axiom is apparently existential, relying on the existence of a sphere's center, or a linear direction.

An extension of our work (see Section 2.3) establishes how one might endogenize an orthogonality operator. A more general notion of orthogonality would permit more general quadratic transformations. For example, instead of $x \cdot x = 0$, we could ask whether $x \cdot Ax = 0$ holds for some symmetric A. In Section 2.3, we do exactly this. Observe that A need not be positive semidefinite, so "orthogonality" could be of a hyperbolic form. Importanly, the notion of orthogonality is derived from a utility function, so that orthogonality is obtained as instances where a conditional linearity property, like OIOI, holds.

Final results establish that the set of preferences satisfying OIOI is homeomorphic to a sphere (with the topology of closed convergence), and a finite test for preferences satisfying our property.

1.1. **Related literature.** Many authors have studied Euclidean preferences and quadratic utility. We give a very brief overview of the literature, but it is fair to say that our result is quite different from the existing work. Bogomolnaia and Laslier (2007) consider a profile of preferences over a finite set of alternatives, and study numbers n for which these can be embedded into $\mathbf{R}^n$ so that preferences are Euclidean. Eguia (2011) and Eguia (2013) also studies the embedding problem, and considers expected utility preferences where the von-Neumann Morgenstern function has the Euclidean form for the chosen embedding. Azrieli (2011) considers Euclidean preferences when there is a valence dimension and considers families of voters indexed by their ideal point. Knoblauch (2010) and Peters (2017) study the algorithmic problem of recognizing whether preferences are Euclidean. Degan and Merlo (2009) looks at the empirical implications of Euclidean preferences for voter data. Henry and Mourifié (2013) follows up on the paper by Degan and Merlo by providing a formal statistical test, and an identification strategy for Euclidean preferences.

General polynomial (expected) utility was studied by Müller and Machina (1987), who connects an m-order polynomial to preferences that only care about the first m moments of the relevant uncertain act. In the social choice context, quadratic utility was introduced in a generalization of Harsanyi's theory of utilitarian aggregation by Epstein and Segal (1992), who consider a sort of betweenness axiom.

2. Model and main result

2.1. **Model and notation.** The objects of choice, or ***alternatives*** are vectors in $\mathbf{R}^n$. The inner product between two vectors is denoted by $x \cdot y = \sum_{i=1}^n x_i y_i$. Two alternatives x and y are ***orthogonal*** (or ***perpendicular***) if $x \cdot y = 0$. In this case we write $x \perp y$. The ***norm*** of a vector x is defined as, and denoted by, $\|x\| = \sqrt{x \cdot x}$.

Choice behavior is modeled through a binary relation $\succeq$ on $\mathbf{R}^n$, which dictates choice among pairs of alternatives in $\mathbf{R}^n$.

2.2. **Axioms and main result.** Suppose that $w \in \mathbf{R}^n$ is given as a starting, or endowment, point, and consider two alternative marginal changes x and y from w. Ultimately, the choice is between $w + x$ and $w + y$. Suppose further that $w + x$ is deemed at least as good as $w + y$; we ask what happens when an additional marginal change z, *orthogonal to both x and y* is additionally appended. Our axiom requires that $w + x + z$ be at least as good as $w + y + z$. In other words, the ranking of the two marginal changes should not be affected when we shift those changes in an orthogonal direction. Since it imposes additivity, our axiom is similar in spirit to the independence axiom of von Neumann-Morgenstern, but it restricts the set of marginal changes to be those qualified by orthogonality.

Origin independent orthogonal independence (OIOI): For all $w, x, y, z \in \mathbf{R}^n$, if $z \perp x$ and $z \perp y$, then $w+x \succeq w+y$ iff $w+x+z \succeq w+y+z$.

The other two axioms are standard.

Continuity: For all $x \in \mathbf{R}^n$, the sets $\{y \in \mathbf{R}^n : y \succeq x\}$ and $\{y \in \mathbf{R}^n : x \succeq y\}$ are closed.

Weak order: $\succeq$ is complete and transitive.[1]

Our main theorem says that continuous weak orders satisfy OIOI if and only if they can be represented by one of three classes of utility functions.

Theorem 1. *Suppose that $n \geq 3$. Then a preference $\succeq$ satisfies OIOI, continuity, and weak order if and only if one of the following is true:*

(1) *There is $u \in \mathbf{R}^n$ for which $x \succeq y$ iff $u \cdot x \geq u \cdot y$*
(2) *There is $x^* \in \mathbf{R}^n$ for which $x \succeq y$ iff $\|x - x^*\| \leq \|y - x^*\|$*
(3) *There is $x^* \in \mathbf{R}^n$ for which $x \succeq y$ iff $\|x - x^*\| \geq \|y - x^*\|$.*

Remark. *We may replace OIOI by an axiom requiring that for any $x, y \in \mathbf{R}^n$ and any $d \perp (x - y)$, $x \succeq y$ iff $x + d \succeq y + d$. In fact, this is the axiom we utilize in our proof.*

Thus, OIOI essentially requires that a preference take one of these three forms. The first representation, in (1) of Theorem 1, is a standard linear preference. In fact, we have as a simple consequence of the theorem that:

Corollary 2. *Suppose that $n \geq 3$ and that a preference $\succeq$ satisfies OIOI, continuity, and weak order, and that there is $z \in \mathbf{R}^n$ such that for all $x \in \mathbf{R}^n$, $x + z \succeq x$. Then there is $u \in \mathbf{R}^n$ for which $x \succeq y$ iff $u \cdot x \geq u \cdot y$.*

[1] Complete: For every $x, y \in \mathbf{R}^n$, $x \succeq y$ or $y \succeq x$. Transitive: For all $x, y, z \in \mathbf{R}^n$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

In particular, if $\succeq$ satisfies a standard monotonicity axiom ($x \succeq y$ whenever $x \geq y$), then OIOI and continuity implies the existence of a linear representation. Actually, the condition in Corollary 2 can be significantly weakened. It is enough to postulate that there is no point that is either a strict local maximum, or a strict local minimum. Other sufficient additional conditions for linearity can similarly be based off of the non-compactness of weak lower and upper contour sets.

The second representation, statement (2) of Theorem 1, implies that preferences are ***Euclidean:*** there is an ideal point x^*, whereby preference is maximized. All other points (consumption bundles, or acts) are compared with respect to the distance to the ideal point. The further away is a point, the worse it is. As discussed in the introduction, Euclidean preferences are heavily used in spatial models in political science, but they have applications elsewhere as well. The next result says that if we add a property of "strict convexity," then our axioms pin down Euclidean preferences.

Corollary 3. *Suppose that* $n \geq 3$ *and that a preference* $\succeq$ *satisfies OIOI, continuity, and weak order, and that* $x \succeq y$ *and* $x \neq y$ *implies that* $(1/2)x + (1/2)y \succ y$*. Then there is* $x^* \in \mathbf{R}^n$ *for which* $x \succeq y$ *iff* $\|x - x^*\| \leq \|y - x^*\|$

The axiom of strict convexity is well known and used in many areas of economics.

The last possibility in statement (3) is a kind of dual to the Euclidean idea. Instead of an ideal point, there is a worst point x^*. The further away from the worst point, the better. This is a model which might explain "NIMBY" style-preferences.

We term these preferences ***spherical*** because they have spherical indifference curves, where we understand the linear preferences in (1) as spherical because a line is like a limit of spheres with larger and larger radii. Corollary 2 says that linear preferences are the only spherical preferences that satisfy a basic monotonicity axiom. Corollary 3 makes the obvious point that the only strictly convex spherical preference is Euclidean.

2.3. **A cardinal approach: endogenous orthogonality.** One drawback of the previous approach is that it can be hard to ascribe meaning to the notion of orthogonal vectors. One may instead want orthogonality to be an endogenous condition that triggers additivity. Here we turn to a cardinal version of our exercise where we start from a utility, or social welfare function, as the primitive object. This primitive is harder to justify and reason about than the ordinal approach in our main theorem, but it has the advantage that we do not need an exogenous notion of orthogonality. Instead, orthogonality will be endogenous.

To fix ideas, think of a social choice framework. Consider a society of n individuals that chooses among vectors that represent individual agents' welfare. Let $U : \mathbf{R}^n \to \mathbf{R}$ be a social welfare function.

Our main axiom asks us to think of outcomes $w+x$ obtained by starting from a ***status quo*** w, and modifies it in the direction of x. Then we can use U to evaluate a change in the direction of x, or a change in the opposite direction, $-x$. The axiom requires that this evaluation has to be the same regardless of the status quo.

Status quo independence:

$$\frac{1}{2}\left[U(w+x)-U(w)\right]+\frac{1}{2}\left[U(w-x)-U(w)\right]$$

is independent of (or constant in) w.

Status quo independence says that a lottery that "shorts" and "longs" x with equal probability has cardinal gain that is independent of the status quo w.

Eventual linearity: For any x and y there is w such that

$$U(w+(x+y))-U(w-(x+y)) = U(w+x)-U(w-x)+U(w+y)-U(w-y)$$

This means that U acts linearly on x and y for *some* status quo.

Theorem 4. *Let U be continuous and satisfy $U(0) = 0$. Then U satisfies status quo independence and eventual linearity iff $U = f + g$, where f is quadratic and g is linear. Moreover, f and g are uniquely identified from U.*

That f is quadratic means that there is a symmetric and bilinear function S such that $f(x) = S(x, x)$. Observe that, as a consequence, we obtain an endogenous notion of orthogonality. We say that x and z are U-orthogonal whenever $S(x, z) = S(z, x) = 0$. As a special case we have the conventional definition of orthogonality used in OIOI, with $S(x, z) = x \cdot z$.

This means that if x and z and U-orthogonal, then $U(x + z) = S(x + z, x + z) + g(x + z) = S(x, x + z) + S(z, x + z) + g(x) + g(z) = U(x) + U(y)$. Thus U satisfies a *conditional* linearity property, in the same spirit as OIOI. Linearity is conditional on U-orthogonality. For any x and y, if z is U-orthogonal to both x and y, then $U(x) - U(y) = U(x + z) - U(y + z)$.

2.4. **Discussion.** Theorem 1 relies on a functional equation (the orthogonal Cauchy equation), which is a restricted type of additivity. Axioms of this type are relatively common in decision theory. Perhaps most closely related to our notion is the notion of *comonotonic additivity*, due to Schmeidler (1986) (an ordinal counterpart first appears in Schmeidler (1989)). This axiom states that for $x, y \in \mathbf{R}^n$ which are *comonotone* (*i.e.* for which for all $i, j \in \{1, \ldots, n\}$, we have $(x_i - x_j)(y_i - y_j) \geq 0$), it follows that $u(x + y) = u(x) + u(y)$.

The functional equation we describe can be rewritten as: $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ implies $u(x + y) = u(x) + u(y)$. We can write comonotonic additivity similarly. Let $x^* \in \mathbf{R}^n$ be any vector for which for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we have $x_i \neq x_j$. Let Σ denote the set of permutations on $\{1, \ldots, n\}$. Define the function $f : \mathbf{R}^n \to \mathbf{R}$ by $f(z) = \sup_{\sigma \in \Sigma} z \cdot (x \circ \sigma)$, where $x \circ \sigma$ is the member of $\mathbf{R}^n$ for which $(x \circ \sigma)_i = x_{\sigma(i)}$.

Then, it is easy to see that x and y are comonotonic if and only if $f(x+y) = f(x)+f(y)$; for example, this follows from the classic rearrangement inequality (see Hardy et al. (1952), Theorem 368 on p. 261). Hence, comonotonic additivity then reads: $f(x+y) = f(x)+f(y)$ implies $u(x+y) = u(x)+u(y)$. Thus, there is a kind of general structure common to both the orthogonal additivity property and the comonotonic additivity property.

3. On the topological structure of the set of OIOI preferences

Consider the set of preferences axiomatized in Theorem 1. We wish here to claim that for $\mathbf{R}^n$, upon removing the preference that is total indifference, the set of such preferences becomes homeomorphic to $S^n \equiv \{y \in \mathbf{R}^{n+1} : \|y\| = 1\}$. To do so, we discuss a particular topology, the topology of closed convergence. This topology is the smallest one for which the set $\{(x, y, \succeq) : x \succ y\}$ is open.

Let the set Π denote the set of all preferences axiomatized in Theorem 1, endowed with the topology of closed convergence. The preference $\mathcal{I}$ represents complete indifference.

An important consequence of the following is that the set of OIOI preferences forms a compact set.

Theorem 5. *$\Pi \setminus \{\mathcal{I}\}$ is homeomorphic to S^n.*

Proof. Observe that each $\succeq \in \Pi \setminus \{\mathcal{I}\}$ has a unique representation via:

$$u_{\succeq}(x) = c(x \cdot x) + d \cdot x,$$

where $(c, d) \in S^n$, and that this map is one-to-one.

Further observe that S^n is compact, and that the topology of closed convergence is Hausdorff, compact, metrizable (Corollary 3.81 of Aliprantis and Border (1999)).

Finally, we show that the map $\pi : S^n \to \Pi$ whereby $\pi(x, d)$ is the preference represented by $c(x \cdot x) + d \cdot x$ is continuous, and then apply Theorem 2.33 of Aliprantis and Border (1999). Continuity of the map π follows easily from Theorem 8 of Border and Segal (1994), using the fact that each $\succeq \in \Pi$ is locally strict. □

4. Finite data and testing

Here, we imagine we have two binary relations R and P, each of which are *finite*, in the sense that $|P|, |R| < +\infty$. We ask when there is a preference $\succeq$ of the form described in Theorem 1 for which

(1) If $x \, R \, y$, then $x \succeq y$
(2) If $x \, P \, y$, then $x \succ y$.

In case there is such a $\succeq$, we say that (R, P) are *rationalizable*. In the following, $\Delta(R \cup P) \equiv \{\delta \in \mathbf{R}_+^{R \cup P} : \sum_{(x,y) \in R \cup P} \lambda(x, y) = 1\}$.

The following is a counterpart of Chambers and Echenique (2016), Theorem 11.11.

Proposition 6. *(R, P) are rationalizable if and only if for any $\lambda \in \Delta(R \cup P)$ for which $\sum_{(x,y) \in P} \lambda(x, y) > 0$, we have either*

(1) $\sum_{(x,y) \in R \cup P} \lambda(x, y)(x \cdot x) \neq \sum_{(x,y) \in R \cup P} \lambda(x, y)(y \cdot y)$
(2) $\sum_{(x,y) \in R \cup P} \lambda(x, y)x \neq \sum_{(x,y) \in R \cup P} \lambda(x, y)y$.

Proof. Rationalizability is equivalent to the existence of $c \in \mathbf{R}$ and $u \in \mathbf{R}^n$ for which:

$$x \, R \, y \to c(x \cdot x - y \cdot y) + u \cdot (x - y) \geq 0$$
$$x \, P \, y \to c(x \cdot x - y \cdot y) + u \cdot (x - y) > 0.$$

This is a finite system of linear inequalities, whose consistency is equivalent to the condition in the statement of the Proposition. See Chambers and Echenique (2016), Lemma 1.12. □

5. Intuition behind Theorem 1

We give a simple geometric intuition behind our main theorem. Specifically, we illustrate how the main force of the axiom implies linear indifference curves *on spheres.* Specifically, for each w there is a vector p_w such that for any sphere S centered at w, if $x, y \in S$, then $x \sim y$ if and only if $p_w \cdot x = p_w \cdot y$. This is not quite enough to prove the theorem, but it serves to illustrate some of the forces behind it.[2]

One piece of notation we shall use is that, for $x, y \in \mathbf{R}^n$, $l(x, y) = \{\lambda x + (1-\lambda)y : \lambda \in \mathbf{R}\}$ denotes the line passing through x and y.

We shall use a seemingly stronger property than OIOI, namely:

Strong origin independent orthogonal independence (SOIOI): For all $x, y, a, b, w \in \mathbf{R}^n$, if $x \perp y$, $a \perp b$, $(w+x) \succeq (w+a)$ and $(w+y) \succeq (w+b)$, then $(w+x+y) \succeq (w+a+b)$, with strict preference if either of the antecedent rankings are strict.

One implication of Theorem 1 is that SOIOI is not actually stronger than OIOI. For the purpose of the arguments developed in this section, we use SOIOI because it implies a kind of homotheticity:

Proposition 7. *If $\succeq$ satisfies weak order, continuity, SOIOA, and $n \geq 3$, then for any $w, x, y \in \mathbf{R}^n$ for which $\|x\| = \|y\|$, and any $\beta > 0$, $w+x \succeq w+y$ iff $w + \beta x \succeq w + \beta y$.*

The proof of Proposition 7 is in Section 8.

5.1. The case of $n = 2$.

The first bit of intuition can be seen on the plane, that is with $n = 2$. The preference $\succeq$ is a continuous weak order, so it has a continuous utility representation U. Let S be the sphere with center 0 and radius $r > 0$ on the plane. Write the sphere in polar coordinates, as

$$S = \{(\theta, r) : 0 \leq \theta \leq 2\pi\}.$$

[2]That said, the actual proof relies on a completely different argument.

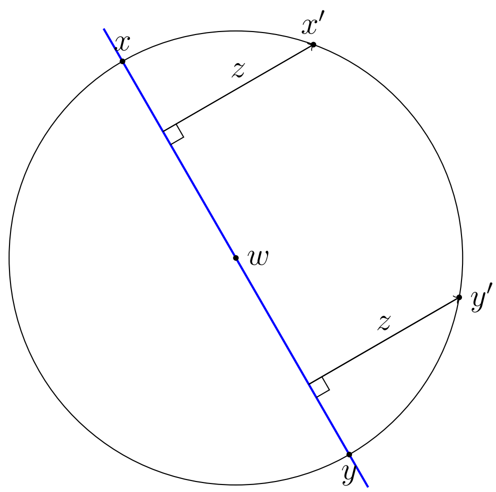
We use addition mod 2π for angles.

First notice that there must exist two points $x = (\theta_x, r)$ and $y = (\theta_y, r)$ that are ***antipodal*** in the sense that $\theta_x = \theta_y + \pi$, and for which $U(x) = U(y)$. To see this suppose (without loss of generality) that $U(0, r) > U(\pi, r)$ and consider the function

$$g(t) = U(t, r) - U(t + \pi, r) : [0, \pi] \to \mathbf{R}.$$

Then $g(0) > 0 > g(\pi)$. Hence, by the intermediate value theorem, there is $\theta \in [0, \pi]$ with $U(\theta, r) = U(\theta + \pi, r)$.

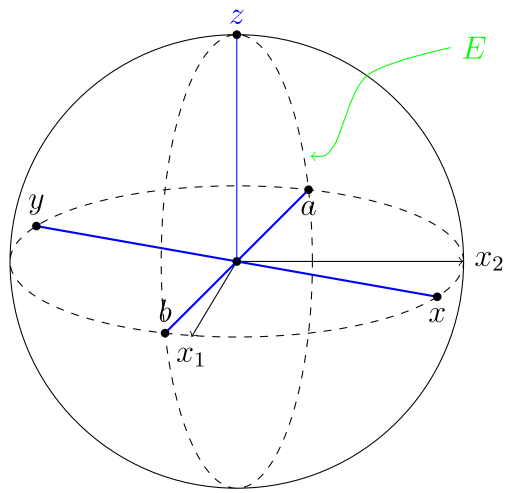
Consider the figure:



We show that indifference curves on $w + S$ are linear. By the previous argument, there exist x and y, antipodal points on S, with the property that $w + x \sim w + y$. Consider the points x', y' that lie on a line parallel to $l(x, y)$. Then there is $z \perp l(x, y)$ for which $x' - z$ and $y' - z$ are on $l(x, y)$.

Now, since $x' \perp z$ and $y' \perp z$, $\|x' - z\| = \|y' - z\|$. So there is $\beta \in \mathbf{R}$ with $x' - z = \beta x$ and $y' - z = \beta y$. Hence Proposition 7 implies that $x' - z \sim y' - z$. Then by OIOI, $x' \sim y'$.

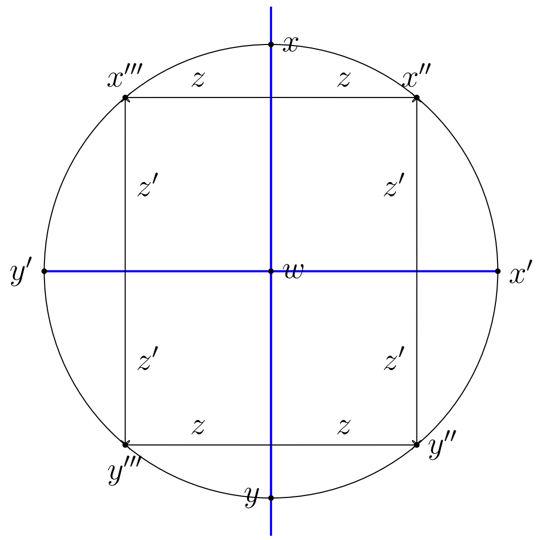
5.2. $n \geq 3$**.** Consider a sphere S with center w and radius r. Choose x_1 and x_2, orthogonal vectors with $\|x_1\| = \|x_2\| = r$. Consider the *equator* defined by x_1 and x_2 on S: the set of points on the linear span of $\{x_1, x_2\}$ that have norm r. By the argument for $n = 2$ there exists a pair of antipodal vectors x and y on the equator such that $w + x \sim w + y$.



Choose a and b on the equator such that a and b are antipodal, and perpendicular to x and y. This is possible because the equator has dimension 2, and x and y are antipodal. Moreover, choose a vector z that is orthogonal to the span of $\{x_1, x_2\}$. Consider the equator E on S defined by a, b and z.

On E we must have, by the argument for $n = 2$, two antipodal points x' and y' with $w + x' \sim w + y'$. Importantly, x' and y' are perpendicular to x and y. Let E' be the equator defined by x and x'. E' is two dimensional and generated by the orthogonal lines $l(x, y)$ and $l(x', y')$.

The equator E' is represented in the following figure. We shall prove that all the points on E' are indifferent.

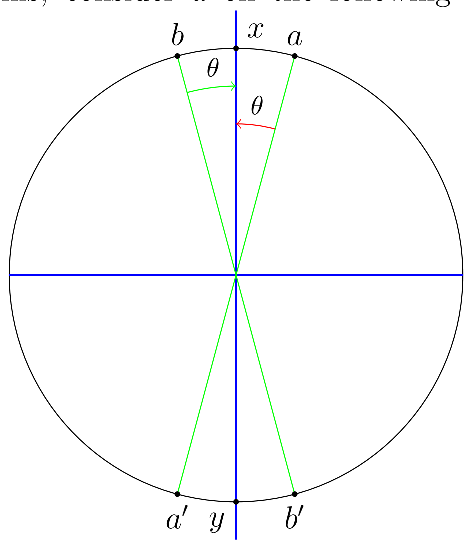


So consider first x'' and y'' that lie on a line parallel to $l(x, y)$. By the argument for $n = 2$, $w + x'' \sim w + y''$. Reflect x'' and y'' across the $l(x, y)$ line and consider the points x''' and y''' on E. Again we obtain that $w + x''' \sim w + y'''$.

Note now that y'' and y''' are the reflection of (respectively) x'' and x''' across the $l(x', y')$ line. Then $w + x' \sim w + y'$ means that $w + x'' \sim w + x'''$ and $w + y'' \sim w + y'''$. Hence we obtain that

$$w + x'' \sim w + y'' \sim w + y''' \sim w + x'''.$$

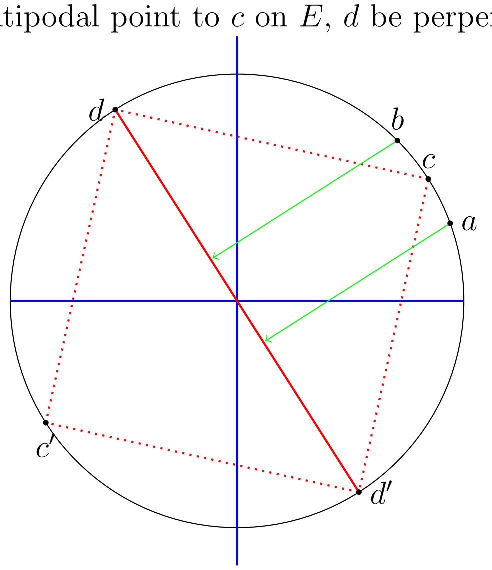
This implies that for any point on E, it is indifferent to its antipodal point. To see this, consider a on the following figure and let a' be its



antipodal point.

Let b be the reflection of a across $l(x, y)$. By the previous argument $w + a \sim w + b \sim w + a'$. So any point is indifferent to its antipodal point.

Finally consider any two points on the same orthant of E': Say a and b. Let c be the vector $\frac{1}{2}(a + b)$, scaled to have norm r. Let c' be the antipodal point to c on E, d be perpendicular to c, and d' be antipodal to



d.

Then $w + d \sim w + d'$ as we have shown that antipodal points are indifferent. This implies that $w + a \sim w + b$ by the same projection argument as before.

Since a and b were arbitrary on the same orthant, we have that $w + a \sim w + b$ for all $a, b \in E'$.

The previous arguments establish the following:

Proposition 8. *For each w and r there is $p \in \mathbf{R}^n$ such that for $x, y \in S(w, r)$, $x \sim y$ iff $p \cdot x = p \cdot y$.*

Now it is easy to show

Proposition 9. *For each w and r there is $p \in \mathbf{R}^n$ such that, for any $r' \leq r$ and $x, y \in S(w, r')$, $x \sim y$ iff $p \cdot x = p \cdot y$.*

Proof. Let p be as in the previous claim and $r' \leq r$ and $\beta = r'/r$. Then $x, y \in S(w, r)$ iff $\beta x, \beta y \in S(w, r')$. Then $p \cdot x = p \cdot y$ iff $w + x \sim w + y$ iff $w + \beta x \sim w + \beta y$ (an implication of Proposition 7). $\square$

6. Proof of Theorem 1

6.1. **Necessity.** We demonstrate that the three types of preferences satisfy OIOI. It is obvious that they are continuous weak orders.

So observe that any preference in the class has a representation as $u(x) = cx \cdot x + v \cdot x$, for some $c \in \mathbf{R}$ and $v \in \mathbf{R}^n$. Then $(w + x) \succeq (w + y)$ implies that

$$c(w + x) \cdot (w + x) + v \cdot (w + x) \geq c(w + y) \cdot (w + y) + v \cdot (w + y).$$

Add $c(w + z) \cdot (w + z) + v \cdot (w + z)$ to both sides to obtain that

$$\begin{aligned} &c(2w \cdot w + 2w \cdot (x + z) + x \cdot x + z \cdot z) + v \cdot (2w + x + z) \\ &\geq c(2w \cdot w + 2w \cdot (y + z) + y \cdot y + z \cdot z) + v \cdot (2w + y + z). \end{aligned}$$

Subtract, from each side, $cw \cdot w + v \cdot w$, obtaining:

$$\begin{aligned} &c(w \cdot w + 2w \cdot (x+z) + x \cdot x + z \cdot z) + v \cdot (w + x + z) \\ &\geq c(w \cdot w + 2 \cdot (y+z) + y \cdot y + z \cdot z) + v \cdot (w + y + z). \end{aligned}$$

Simplify and obtain:

$$c(w+x+z) \cdot (w+x+z) + v \cdot (w+x+z) \geq c(w+y+z) + v \cdot (w+y+z),$$

using the fact that $x \perp z$ and $y \perp z$ (hence $x \cdot x + z \cdot z = (x+z) \cdot (x+z)$ and $y \cdot y + z \cdot z = (y+z) \cdot (y+z)$). Therefore, $w + x + z \succeq w + y + z$.

6.2. **Sufficiency.**

Proposition 10. *For a weak order satisfying OIOI, if $d \perp (x - y)$, then $x \succeq y$ iff $x + d \succeq y + d$.*

Proof. Observe that $x+(0) \succeq x+(y-x)$. Further, $d \perp (0)$ and $d \perp (y-x)$. So by OIOI, $x+(0)+d \succeq x+(y-x)+d$, or $x+d \succeq y+d$, and conversely. □

Say a vector subspace D of $\mathbf{R}^n$ is *inessential* if for any $d \in D$ and any $x \in \mathbf{R}^n$, $x + d \sim x$.

Proposition 11. *If $\succeq$ satisfies OIOI, weak order, and continuity, then if it has a nontrivial inessential subspace, it is a linear preference.*

Proof. To ease notation, suppose that the nontrivial inessential subspace is the subspace spanned by $(0, \ldots, 0, 1)$.

By definition of inessential, there is a preference $\succeq^*$ on $\mathbf{R}^{n-1}$ for which $(x, c) \succeq (y, d)$ iff $x \succeq^* y$. We claim that $\succeq^*$ is a linear preference, that is, for any $x, y, z \in \mathbf{R}^{n-1}$, we have $x \succeq^* y$ iff $x + z \succeq^* y + z$.

Thus, suppose that $x \succeq^* y$ and let $z \in \mathbf{R}^{n-1}$ be arbitrary. Then for any $a \in \mathbf{R}$, $(x, 0) \succeq (y, a)$. In particular, let $a = z \cdot (x - y)$. Observe that $(z, 1) \perp (y-x, a) = (y, a)-(x, 0)$. Consequently by Proposition 10 we have $(x + z, 1) \succeq (y + z, a + 1)$, establishing that $x + z \succeq^* y + z$.

The remainder is now standard. □

Corollary 12. *Suppose that $\succeq$ satisfies OIOI, weak order, and continuity. Suppose $n \geq 3$ and let $\{f_1, \ldots, f_n\}$ be an orthonormal basis for $\mathbf{R}^n$. For any $a, b \in \mathbf{R}^n$ and any subset $G \subseteq \{1, \ldots, n\}$, we have:*

$$\sum_{i \in G} a_i f_i + \sum_{i \notin G} a_i f_i \succeq \sum_{i \in G} b_i f_i + \sum_{i \notin G} a_i f_i$$

iff

$$\sum_{i \in G} a_i f_i + \sum_{i \notin G} b_i f_i \succeq \sum_{i \in G} b_i f_i + \sum_{i \notin G} b_i f_i.$$

Proof. Follows from Proposition 10 by taking $x = \sum_{i \in G} a_i f_i + \sum_{i \notin G} a_i f_i$, $y = \sum_{i \in G} b_i f_i + \sum_{i \notin G} a_i f_i$, and $d = \sum_{i \notin G} (b_i - a_i) f_i$. □

For our final step we need some additional notational conventions. For any subspace T of $\mathbf{R}^n$ and any $x \in \mathbf{R}^n$, let $\alpha_T(x)$ be the orthogonal projection of x onto T. If $T = \text{span}\{f\}$ for some vector $f \in \mathbf{R}^n$, we abuse notation and write $\alpha_f(x)$ as the norm of $\alpha_{\text{span}\{f\}}$.

Remark. *The final steps establish the following. Let S^{n-1} denote the unit sphere. There is a utility representation u of $\succeq$, and for each $f \in S^{n-1}$ a function $u_f : \mathbf{R} \to \mathbf{R}$, satisfying the following properties:*

1. *For any orthonormal basis $\{f_1, \ldots, f_n\}$, if $x = \sum_{i=1}^n \alpha_{f_i} f_i$, then $u(x) = \sum_{i=1}^n u_{f_i}(\alpha_{f_i})$.*
2. *$u(0) = 0$.*

It is then easy to show that $u : \mathbf{R}^n \to \mathbf{R}$ satisfies the property that if $x \perp y$, then $u(x + y) = u(x) + u(y)$.

Remark. *The proof proceeds in Lemma 13 and Proposition 14 by establishing the result for $n = 3$. Then Proposition 15 extends the result to all $n \geq 3$.*

Lemma 13. *Suppose that $n = 3$, and that $\succeq$ is a continuous weak order satisfying OIOI. Suppose that $\succeq$ has no non-trivial inessential subspaces. Let $\{f_1, f_2, f_3\}$ be an orthonormal basis for $\mathbf{R}^3$ and suppose that $u(z) = \sum_{i=1}^3 u_{f_i}(\alpha_{f_i}(z))$ is an additive representation for $\succeq$ for which $u_{f_i}(0) = 0$ for* *each* $i = 1, 2, 3$. *If* $\{e_2, e_3\}$ *is any other orthonormal basis for span*$(\{f_2, f_3\})$, *then there is an additive representation for* $\succeq$,

$$v(z) = v_{f_1}(\alpha_{f_1}(z)) + v_{e_2}(\alpha_{e_2}(z)) + v_{e_3}(\alpha_{e_3}(z)),$$

such that

$$v(z) = u(z) = u_{f_1}(\alpha_{f_1}(z)) + v_{e_2}(\alpha_{e_2}(z)) + v_{e_3}(\alpha_{e_3}(z)),$$

and $v_{e_2}(0) = v_{e_3}(0) = 0$.

Proof. First observe that by Corollary 12 and Debreu (1959), since $\{f_1, e_2, e_3\}$ is an orthonormal basis, and there are no non-trivial inessential subspaces, $\succeq$ has an additive representation $v(z) = v_{f_1}(\alpha_{f_1}(z)) + v_{e_2}(\alpha_{e_2}(z)) + v_{e_3}(\alpha_{e_3}(z))$. We shall prove that we can choose this representation so that $v_{f_1} = u_{f_1}$, $u = v$, $v_{e_2}(0) = 0$, and $v_{e_3}(0) = 0$.

Define $T \equiv \text{span}(\{e_2, e_3\}) = \text{span}(\{f_2, f_3\})$. We have two additive representations u, v of $\succeq$:

$$\begin{aligned} u(z) = u(\alpha_{f_1}(z) + \alpha_T(z)) &= u_{f_1}(\alpha_{f_1}(z)) \\ &\quad + [u_{f_2}(\alpha_{f_2}(\alpha_T(z)) + u_{f_3}(\alpha_{f_3}(\alpha_T(z)))] \\ v(z) = v(\alpha_{f_1}(z) + \alpha_T(z)) &= v_{f_1}(\alpha_{f_1}(z)) \\ &\quad + [v_{e_2}(\alpha_{e_2}(\alpha_T(z)) + v_{e_3}(\alpha_{e_3}(\alpha_T(z)))]. \end{aligned}$$

The pair $(\text{span}(\{f_1\}) \times T, \succeq)$ constitutes an additive conjoint measurement structure in the sense of Krantz et al. (1971) (see Chapter 6.2.4).[3] By their Theorem 2, there exists $\beta > 0$, and γ, γ' with $u_{f_1} = \beta v_{f_1} + \gamma$ and $u_{f_2} + u_{f_3} = \beta(v_{e_2} + v_{e_3}) + \gamma'$. So define $v'_{f_1} = \beta v_{f_1} + \gamma$, $v'_{e_2} = \beta v_{e_2} + \theta_2$ and $v'_{e_3} = \beta v_{e_3} + \theta_3$, where $\theta_2 = -\beta v_{e_2}(0)$ and $\theta_3 = -\beta v_{e_2}(0)$. Note that $0 = (u_{f_2} + u_{f_3})(0) = \beta(v_{e_2} + v_{e_3})(0) + \gamma'$ implies that $\gamma' = \theta_2 + \theta_3$. Hence,

[3]See also Fishburn (1970), Theorem 5.2 or 5.4

we obtain that

$$\begin{aligned} v'(z) &= v'_{f_1}(\alpha_{f_1}(z)) + v'_{e_2}(\alpha_{e_2}(z)) + v'_{e_3}(\alpha_{e_3}(z)) \\ &= u_{f_1}(\alpha_{f_1}(z)) + v'_{e_2}(\alpha_{e_2}(z)) + v'_{e_3}(\alpha_{e_3}(z)) \\ &= u_{f_1}(\alpha_{f_1}(z)) + u_{f_2}(\alpha_{f_2}(z)) + u_{f_3}(\alpha_{f_3}(z)) = u(z). \end{aligned}$$

while $v'_{f_1}(0) = v'_{e_2}(0) = v'_{e_3}(0) = 0$. Thus v' has the desired properties. □

Proposition 14. *Suppose that $n = 3$, that $\succeq$ is a continuous weak order satisfying OIOI, and has no non-trivial inessential subspaces. Then there is a continuous utility representation $u : \mathbf{R}^n \to \mathbf{R}$ for which for any $x, y \in \mathbf{R}^n$ with $x \perp y$, we have $u(x + y) = u(x) + u(y)$.*

Proof. We say that x and y are parallel, or collinear, if there is a scalar $\lambda \in \mathbf{R}$ with $y = \lambda x$.

Let $\{f_1, f_2, f_3\}$ be a given orthonormal basis of $\mathbf{R}^3$. By Corollary 12 and Debreu (1959), since there are no non-trivial inessential subspaces, there exists a representation $u : \mathbf{R}^n \to \mathbf{R}$ of $\succeq$ for which $u(z) = \sum_{i=1}^{3} u_{f_i}(\alpha_{f_i}(z))$. Suppose without loss of generality that $u_{f_i}(0) = 0$, as additive representations are preserved by an additive translation of each component utility.

Now, fix arbitrary $x, y \in \mathbf{R}^n$ for which $x \perp y$. If either x or y is 0, then we know that $u(x + y) = u(x) + u(y)$ because $u(0) = 0$. So lets suppose that $x, y \neq 0$. Now, we have three possible cases to consider:

(1) x is parallel to some f_i and y is parallel to some f_j,
(2) Either x or y is parallel to some f_i, and the other one is not parallel to some f_j
(3) Neither x nor y are parallel to any f_i.

We shall prove that case 3 reduces to case 2, and that case 2 reduces to case 1.

Let us first consider case 3. Note $\text{span}\{f_2, f_3\} \cap \text{span}\{x, f_1\}$ is nonempty, as x is not collinear with f_1. So choose $e_2 \in \text{span}\{f_2, f_3\} \cap \text{span}\{x, f_1\}$, scaled so that $\|e_2\| = 1$. Let $e_3 \in \text{span}\{f_2, f_3\}$ then be a unit vector with

$e_3 \perp e_2$. Thus $x \in \text{span}\{f_1, e_2\}$ and $\{f_1, e_2, e_3\}$ is an orthonormal basis of $\mathbf{R}^3$.

By Lemma 13, there exists v_{e_2} and v_{e_3} such that

$$u(z) = u_{f_1}(\alpha_{f_1}(z)) + v_{e_2}(\alpha_{e_2}(z)) + v_{e_3}(\alpha_{e_3}(z))$$

while $v_{e_2}(0) = 0$ and $v_{e_3}(0) = 0$.

Since $x \in \text{span}(\{f_1, e_2\})$ and $e_3 \perp \text{span}(\{f_1, e_2\})$, there exists e_1 such that $\{e_1, \frac{1}{\|x\|}x\}$ is an orthonormal basis for $\text{span}(\{f_1, e_2\})$ and $\{e_1, \frac{1}{\|x\|}x, e_3\}$ is an orthonormal basis for $\mathbf{R}^n$.

By Lemma 13 again, there exists u_{e_1} and $u_{\frac{x}{\|x\|}}$ such that

$$u(z) = u_{e_1}(\alpha_{e_1}(z)) + u_{\frac{x}{\|x\|}}(\alpha_{\frac{x}{\|x\|}}(z)) + v_{e_3}(\alpha_{e_3}(z)),$$

$u_{e_1}(0) = 0$ and $u_{\frac{x}{\|x\|}}(0) = 0$. We are now in the situation of Case 2, as x is parallel to $\frac{x}{\|x\|}$.

So consider x and y in the configuration of Case 2. In particular suppose that x is parallel to f_2. Since $x \perp y$, $y \in \text{span}(\{f_1, f_3\})$. So there exists a unit vector w such that $\text{span}(\{y, w\}) = \text{span}(\{f_1, f_3\})$. By Lemma 13 there exists v_w and $v_{\frac{y}{\|y\|}}$ such that

$$u(z) = u_{\frac{x}{\|x\|}}(\alpha_{\frac{x}{\|x\|}}(z)) + v_{\frac{y}{\|y\|}}(\alpha_{\frac{y}{\|y\|}}(z)) + v_w(\alpha_w(z)),$$

with $u_{\frac{x}{\|x\|}}(0) = v_{\frac{y}{\|y\|}}(0) = v_w(0) = 0$. Thus, we are now in the situation of Case 1.

So consider x and y in the configuration of Case 1. In particular, suppose that x is parallel to f_1 while y is parallel to f_2. Recall that x and y are each nonzero. Observe that

$$\begin{aligned} u(x+y) &= u_{\frac{x}{\|x\|}}(\|x\|) + u_{\frac{y}{\|y\|}}(\|y\|) + u_{f_3}(0) \\ &= \Big(u_{\frac{x}{\|x\|}}(\|x\|) + u_{\frac{y}{\|y\|}}(0) + u_{f_3}(0)\Big) + \Big(u_{\frac{x}{\|x\|}}(0) + u_{\frac{y}{\|y\|}}(\|y\|) + u_{f_3}(0)\Big) \\ &= u(x) + u(y), \end{aligned}$$

where the penultimate equality follows from the fact that $0 = u(0) = u_{\frac{x}{\|x\|}}(0) + u_{\frac{y}{\|y\|}}(0) + u_z(0)$. □

Proposition 15. *Suppose that $n \geq 3$, and that $\succeq$ is a continuous weak order satisfying OIOI, and has no non-trivial inessential subspaces. Then there exists $v \in \mathbf{R}^n$ and a scalar c such that $u(x) = c\|x\|^2 + v \cdot x$ is a utility representation of $\succeq$.*

Proof. For $n = 3$ we have shown that there exists a utility representation that satisfies $u(x+y) = u(x) + u(y)$ for any $x, y \in \mathbf{R}^n$ with $x \perp y$. Then, by Theorem 1 of Sundaresan (1972), $u(x) = cx \cdot x + v \cdot x$ for some $c \in \mathbf{R}$ and $v \in \mathbf{R}^n$.

So consider $n \geq 3$. By Corollary 12 and Debreu (1959), and since there are no inessential non-trivial subspaces, there exists a utility representation

$$U(x) = \sum_i w_i(x_i)$$

of $\succeq$.

By the preceding argument, for any subset $\{i, j, k\} \subseteq \{1, \ldots, n\}$ of cardinality 3, the representation restricted to $\mathbf{R}^{i,j,k}$ can be chosen to be of the form

$$u_{\{i,j,k\}}(x_{\{i,j,k\}}) = c^{\{i,j,k\}}(x_i^2 + x_j^2 + x_k^2) + v^{\{i,j,k\}} \cdot (x_i, x_j, x_k).$$

Then for any $\{i, j, k\} \subseteq \{1, \ldots, n\}$ of cardinality 3 we have two additive representations on $\mathbf{R}^{\{i,j,k\}}$: $w_i(x_i) + w_j(x_j) + w_k(x_j)$ and $\sum_{h \in \{i,j,k\}} c^{\{i,j,k\}}(x_h^2) + v_h^{\{i,j,k\}} x_h$. By Theorem 2 in Chapter 6.2.4 of Krantz et al. (1971), there exists $\alpha^{\{i,j,k\}}$ and $\beta^{\{i,j,k\}} > 0$ with

$$\beta^{\{i,j,k\}} w_h(x_h) + \alpha^{\{i,j,k\}} = c^{\{i,j,k\}}(x_h^2) + v_h^{\{i,j,k\}} x_h$$

for all x_h. This is true for all x_h iff there is β, c_h and v_h with $\beta^{\{i,j,k\}} = \beta > 0$, $c^{\{i,j,k\}} = c$, $v_h^{\{i,j,k\}} = v_h$ and $\alpha^{\{i,j,k\}} = 0$.[4] Hence, $\beta w_h(x_h) = cx_h^2 + v_h x_h$.

[4]Normalize $\beta^{\{i,j,k\}} = 1$. Then, for $k \neq l$ we have $w_i(x_i) + \alpha^{\{i,j,k\}} = c^{\{i,j,k\}}(x_i^2) + v_i^{\{i,j,k\}} x_i$ and $w_i(x_i) + \alpha^{\{i,j,l\}} = c^{\{i,j,l\}}(x_i^2) + v_i^{\{i,j,l\}} x_i$. Hence, $\alpha^{\{i,j,k\}} - \alpha^{\{i,j,l\}} = (c^{\{i,j,k\}} -$

□

7. Proof of Theorem 4

We prove sufficiency. So let U be as in the statement of the theorem. Let

$$f(x) = \frac{1}{2}\left[U(z+x) - U(z)\right] + \frac{1}{2}\left[U(z-x) - U(z)\right]$$

and define $g(x) = U(x) - f(x)$. The following lemmas show sufficiency.

Lemma 16.

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

Proof.

$$\begin{aligned}
A = f(x-y) - f(x) - f(y) =& \frac{1}{2}U(q+(x-y)) + \frac{1}{2}U(q-(x-y)) - U(q) \\
&- \frac{1}{2}U(q'+x) - \frac{1}{2}U(q'-x) + U(q') \\
&- \frac{1}{2}U(q''+y) - \frac{1}{2}U(q''-y) + U(q'') \\
=& \frac{1}{2}U(z+(x-y)) + \frac{1}{2}U(z-(x-y)) - U(z) \\
&- \frac{1}{2}U(z+(x+y)) - \frac{1}{2}U(z-(x-y)) + U(z+y) \\
&- \frac{1}{2}U(z-(x-y)) - \frac{1}{2}U(z-(x+y)) + U(z-x)
\end{aligned}$$

Where the first equality is by definition of f, with arbitrary $q, q', q'' \in \mathbf{R}^n$. The second uses $q = z$, $q' = z + y$ and $q'' = z - x$.

$c^{\{i,j,l\}}) + (v_i^{\{i,j,k\}} - v_i^{\{i,j,l\}})x_i$. This can only hold for all $x_i \in \mathbf{R}$ if $\alpha^{\{i,j,k\}} - \alpha^{\{i,j,l\}} = c^{\{i,j,k\}} - c^{\{i,j,l\}} = v_i^{\{i,j,k\}} - v_i^{\{i,j,l\}} = 0$.

Then we have that

$$A = -f(x+y) - 2U(z) + U(z+y) + U(z-x)$$
$$+\frac{1}{2}\left[U(z+(x-y)) + U(z-(x-y)) - U(z-(x-y)) - U(z-(x-y))\right]$$
$$= -f(x+y) - 2U(z) + U(z+y) + U(z-x)$$
$$+\frac{1}{2}\left[U(z+(x-y)) - U(z-(x-y))\right]$$
$$= -f(x+y) + f(x) + f(y) - \frac{1}{2}\left[U(z+x) + U(z-x) + U(z+y) + U(z-y)\right]$$
$$+U(z+y) + U(z-x) + \frac{1}{2}\left[U(z+(x-y)) - U(z-(x-y))\right]$$
$$= -f(x+y) + f(x) + f(y) + \frac{1}{2}\left[U(z+y) + U(z-x) - U(z+x) - U(z-y)\right]$$
$$+\frac{1}{2}\left[U(z+(x-y)) - U(z-(x-y))\right]$$

Let $y' = -y$. Then by Axiom 2 we can set z such that

$$U(z-y')+U(z-x)-U(z+x)-U(z+y')+U(z+(x+y'))-U(z-(x+y')) = 0.$$

Thus

$$f(x-y) - f(x) - f(y) = A = -f(x+y) + f(x) + f(y).$$

□

The function f is continuous and uniquely identified from U. Then Lemma 16 and Proposition 4 of Chapter 11 of Aczél and Dhombres (1989) implies that there is a unique function $S : \mathbf{R}^{2n} \to \mathbf{R}$ such that S is symmetric, bi-linear, and $f(x) = S(x,x)$.

Lemma 17.

$$g(x+y) = g(x) + g(y)$$

Proof. We have $g(x+y)-g(x)-g(y)=U(x+y)-U(x)-U(y)-(f(x+y)-f(x)-f(y))$. Hence, for any choice of z,z',z'':

$$\begin{aligned}g(x+y)-g(x)-g(y)=&U(x+y)-U(x)-U(y)+\frac{1}{2}\left[U(z+x-y)-U(z)\right]\\&+\frac{1}{2}\left[U(z-x+y)-U(z)\right]\\&-\frac{1}{2}\left[U(z'+x)-U(z')\right]-\frac{1}{2}\left[U(z'-x)-U(z')\right]\\&-\frac{1}{2}\left[U(z''+y)-U(z'')\right]-\frac{1}{2}\left[U(z''-y)-U(z'')\right]\end{aligned}$$

In particular, for $z'=y$ and $z''=x$, and using that $U(0)=0$, we obtain that

$$\begin{aligned}g(x+y)-g(x)-g(y)=&\frac{1}{2}\left[U(z+x-y)-U(z)\right]+\frac{1}{2}\left[U(z-x+y)-U(z)\right]\\&-\frac{1}{2}\left[U(y-x)-U(0)\right]-\frac{1}{2}\left[U(x-y)-U(0)\right]=0,\end{aligned}$$

by the axiom. □

Note that g is continuous because U is continuous. Then 17 implies that g is a linear function by Corollary 2 of Chapter 4 of Aczél and Dhombres (1989).

For necessity: Status-quo independence is a simple calculation. Eventual Linearity is established by the following calculation.

$$
\begin{aligned}
U(w+(x+y))-U(w-(x+y)) &= g(w+(x+y))-g(w-(x+y)) \\
&+ S(w+(x+y),w+(x+y)) \\
&- S(w-(x+y),w-(x+y)) \\
&= 2g(x)+2g(y)+2S(w,x+y)+S(x+y,x+y) \\
&+ 2S(w,x+y)-S(x+y,x+y) \\
&= 2g(x)+2g(y)+4S(w,x)+4S(w,y)+S(x,x) \\
&+ 2S(x,y)+S(y,y) \\
&= [g(w)+g(x)+S(w,w)+2S(w,x)+S(x,x)] \\
&- [g(w)+g(-x)+S(w,w)+2S(w,-x)+S(-x,-x)] \\
&+ [g(y)+g(w)+S(w,w)+2S(w,y)+S(y,y)] \\
&- [g(-y)+g(w)+S(w,w)-2S(w,-y)-S(-y,-y)] \\
&= U(w+x)-U(w-x)+U(w+y)-U(w-y)
\end{aligned}
$$

8. Proof of Proposition 7

We first establish the result for $w=0$, so suppose that $\|x\|=\|y\|$, where $x \succeq y$.

We first establish the result for positive integer β. The proof proceeds by induction. Let $a \in \mathbf{R}^n$ for which $a \perp x$ and $a \perp y$, further $\|a\|=\|x\|=\|y\|$. Such a exists because $n \geq 3$.

By OIOA, it follows that $2a+x \succeq 2a+y$. Further, $(x-a) \perp (x+a)$ and $(y-a) \perp (y+a)$. Since $a+(x-a) \succeq a+(y-a)$ and $a+(x+a) \succeq a+(y+a)$, OIOA implies that $a+(x-a)+(x+a) \succeq a+(y-a)+(y+a)$, or $a+2x \succeq a+2y$. By OIOA, if $2y \succ 2x$, we would have $a+2y \succ a+2x$, a contradiction. So, in fact $2x \succeq 2y$.

Suppose now that $x \succeq y$, and that we have shown $kx \succeq ky$ for $k \in \mathbf{N}$. We claim that $(k+1)x \succeq (k+1)y$. By $(k+1)a \perp kx$, $(k+1)a \perp ky$ and

OIOA, $(k+1)a + kx \succeq (k+1)a + ky$ (or $a + (kx + ka) \succeq a + (ky + ka)$). Moreover, $a + (x - a) \succeq a + (y - a)$. Observe that $(kx + ka) \perp (x - a)$ and $(ky + ka) \perp (y - a)$. Consequently, by OIOA, $a + (x - a) + k(x + a) \succeq a + (y - a) + k(y + a)$, or $ka + (k+1)x \succeq ka + (k+1)y$. Again it must follow that $(k+1)x \succeq (k+1)y$.

By induction, $kx \succeq ky$ for all $k \in \mathbf{N}$ with $k > 0$. Note that the same argument shows that if $x \succ y$ then $kx \succ ky$.

Now let $q > 0$ be a rational number, $q = k/l$ with $k, l \in \mathbf{N}$. Then it must hold that $qx \succeq qy$, as $qy \succ qx$ would imply that $lqy = ky \succ kx = lqx$ by the first step and the fact that $\|qx\| = \|qy\|$.

Finally, by continuity of $\succeq$ we obtain that $\beta x \succeq \beta y$ for all real $\beta > 0$. This proves the result for $w = 0$.

To see that the result holds for arbitrary w, it is enough to observe that the ranking $x \succeq_w y$ iff $(x + w) \succeq (y + w)$ satisfies OIOA and apply the previous argument.

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