Inequality aversion and risk aversion*

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Abstract

This note shows that for two inequality averse social welfare functions, if one is more inequality averse than the other, the household preference induced by optimally allocating aggregate bundles according to this social welfare function is more risk averse than the other. We present examples showing that this comparative static can be reversed if inequality aversion is dropped. We show that the utilitarian rule always induces the least risk averse household preference among all social welfare functions (this corresponds to the sum of certainty equivalents).

1 Introduction

The result in this note studies the interconnection between inequality aversion, on the one hand, and risk aversion, on the other. A common observation is that inequality aversion and risk aversion often go hand-in-hand. For instance, a risk averse decision maker choosing among income distributions (not knowing in which part of the distribution he will reside) is indistinguishable from an inequality averse social planner.

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In this work, we tackle the interplay between inequality aversion and risk aversion from a different perspective. A household of individuals shares risky income in order to optimize some social welfare function; the individuals may differ in terms of their risk attitudes, but hold common beliefs. The risk comes in aggregate form and must be distributed amongst the agents. We follow Samuelson (1956) and Chipman and Moore (1979) in studying household behavior under various social welfare functions. Social welfare functions can be ranked according to their propensity to reduce inequality. A simple intuition would suggest that the more inequality averse a social welfare function, the more it must reflect *all* agents' preferences. The more it reflects all individuals' preferences, in particular, the more it reflects the "weakest" or most risk-averse individual's preferences. Less inequality averse social welfare functions would not seem to have this property–maximizing aggregate utility for example would always allocate risk to the agents who benefit the most from it; these are the least risk averse agents. Using this naive intuition, we could conclude that more inequality averse households would tend to be more risk averse.

This intuition turns out to be true up to a point, and this is our main result. This simple comparative static holds for a broad family of social choice rules: namely, those which are *inequality averse* in an absolute sense: reducing inequality always benefits society. A plethora of rules exhibit this property. For this family of rules, a more inequality averse social welfare function induces a more risk averse household preference.

However, this intuition breaks down upon removing this property. It turns out the critical feature of inequality averse social welfare functions is that equal division is always a socially optimal allocation of a certain dollar. If we measure utility in dollars, equal division of a dollar always results in equal utilities. For an aggregate risky prospect to be preferred to a certain dollar, it must therefore be the case that there is an allocation of that risky prospect which is socially better than equal division of that dollar. And if this is true for a social welfare function, it is certainly also true for a less inequality averse social welfare function.

When dropping inequality aversion; however, equal division of a certain dollar need not be socially optimal. In general then a comparison of the induced preferences given two social welfare functions ranked in terms of inequality aversion need not exist. The key feature of an equal division of a dollar is that it results in an equal distribution of utilities: the property of inequality aversion is about how a list of utilities compares to equal division. Without the equal division benchmark, it turns out that anything can happen.

Finally, one can ask whether a similar comparative static holds among the family of inequality *loving* social welfare functions. A simple geometric argument shows that it does not; and in fact, it is easy to construct examples of two inequality loving social welfare functions, one of which is more inequality averse than the other, but generates a less risk averse household preference than the other. We cement this fact together with our main result by showing that the utilitarian social welfare function–the rule that adds utilities–always generates the least risk averse household preference.

Section 2 illustrates the issues involved with a simple example. Section 3 provides the main result of the paper. Section 4 discusses the case of inequality loving social welfare functions. Finally, Section 5 concludes.

2 Three social welfare functions

There are two individuals living in a household, *Alice* and *Bob*. The household faces aggregate uncertainty about a monetary payoff, which depends on the state of nature, *Rain* or *Shine* (R or S). Each individual has a quasiconcave utility function, which is homogeneous of degree one, over risky consumption. For example, Alice's utility for (x_R, x_S) is $U^A(x_R, x_S)$. Utilities are further assumed to be measured in dollars, so that for a constant consumption bundle (c, c), $U^i(c, c) = c$. The normalization of utility reflects the fact that utility is measured in dollar amounts–the utility of risky consumption is equal to its certainty equivalent. This facilitates comparison of utilities across agents (money is a natural metric).

Lastly, the utilities have a *common prior* (π_R, π_S), which is simply a vector for which for all consumption (x_R, x_S), $\pi_R x_R + \pi_S x_S \geq U^i(x_R, x_S)$. This corresponds to the standard notion of risk aversion, as utility is measured in dollar amounts.

Alice and Bob must decide how to rank aggregate consumption. They decide to allocate aggregate consumption according to a social choice rule. The social choice rule maps individual utilities into an aggregate utility: $W : \mathbb{R}^2_+ \to \mathbb{R}$. The specification of this choice rule determines the ranking over aggregate consumption. Three social choice rules, which take as input utility, are as follows:

i) The maximax rule:

$$W_{\max}\left(u^1, u^2\right) = \max\left\{u^1, u^2\right\}$$

ii) The utilitarian rule:

$$W_U\left(u^1, u^2\right) = u^1 + u^2$$

and

iii) The maximin (Rawlsian) rule

$$W_{\min}\left(u^1, u^2\right) = \min\left\{u^1, u^2\right\}.$$

The three rules differ as to their inequality aversion. The maximax rule is, in a sense, inequality loving, while the utilitarian is inequality neutral, and the maximin is inequality averse. We ask how the behavior of the rules in terms of inequality aversion affects the household's behavior in terms of risk aversion. A simple intuition is that, the more inequality averse the social welfare function, the more risk averse the induced household preference. We show that this intuition breaks down in a strong way in this example.

The household allocates risky prospects optimally, and hence the household's utility of an aggregate bundle (x_R, x_S) is given by

$$\max_{\substack{x_R^A + x_R^B = x_R \\ x_S^A + x_S^B = x_S}} W\left(U^A\left(x_R^A, x_S^A\right), U^B\left(x_R^B, x_S^B\right)\right).$$

The household utility therefore depends on the social welfare function under consideration. We can ask how these household utilities fare as to risk aversion. One way to verify this, using a classical characterization of comparative risk aversion, is to compute the at least as good as sets for a constant aggregate bundle (c, c) (these are the set of aggregate bundles that the household ranks at least as good as a sure thing paying out c). The more risk averse a household, the smaller its at least as good as set for any c.

Denote the individual at least as good as sets for (c, c) for $i = A, B$ by

$$\overline{U}^i(c, c) = \left\{(x_R, x_S) : U^i(x_R, x_S) \geq U^i(c, c) = c\right\}.$$

It is a straightforward exercise to compute the at least as good as sets for the household, given each of the social choice rules introduced above. For the maximax rule, the at least as good as set for (c, c) is

$$\overline{U}^{\max}(c, c) = \overline{U}^A(c, c) \cup \overline{U}^B(c, c).$$

For the utilitarian rule, it is

$$\overline{U}^U(c, c) = \text{co}\left(\overline{U}^A(c, c) \cup \overline{U}^B(c, c)\right).$$

For the maximin rule, it is

$$\overline{U}^{\min}(c, c) = \frac{1}{2}\overline{U}^A(c, c) + \frac{1}{2}\overline{U}^B(c, c).$$

Therefore, for this example, $\overline{U}^{\max}(c,c) \subset \overline{U}^{U}(c,c)$ and $\overline{U}^{\min}(c,c) \subset \overline{U}^{U}(c,c)$, with strict inclusions (in general). In general there is no relation between $\overline{U}^{\max}(c,c)$ and $\overline{U}^{\min}(c,c)$. These two set inclusions demonstrate that the utilitarian social welfare function leads to a less risk averse household than either the maximax or the maximin planner. This coincides with our intuition for the maximin case, but not for the maximax case.

3 The explanation

The intuition that greater inequality aversion leads to greater risk aversion turns out to be true, but only with some caveats. The comparative static does not hold for *all* pairs of social welfare functions. First, note that, in the example, while the maximax social welfare function is less inequality averse than the utilitarian social welfare function, it is not, in an absolute sense, inequality averse. This fact turns out be responsible for the breakdown in our intuition.

Let us let $N = \{1, ..., n\}$ denote a finite household of agents, and $\Omega = \{1, ..., \omega\}$ a finite set of states. Each individual has a utility function $U^i : \mathbb{R}_+^{\Omega} \to \mathbb{R}$ which is increasing and continuous.[1] We further assume, as in the example, that utility functions are normalized so that for any constant bundle $(c, c, ..., c)$, $U^i(c, ..., c) = c$. As in the example, this is a natural metric for comparing utility across agents. We lastly assume that there is a common prior $(\pi_1, ..., \pi_\omega)$ with respect to which each individual's preference is risk averse; that is

$$\sum_{s=1}^{\omega} \pi_s x_s \geq U^i(x_1, ..., x_\omega)$$

holds for all $(x_1, ..., x_\omega)$ and all $i \in N$. Of course, we have made no assumptions on the functional form of any of the U^i. They may be expected utility, but they are allowed to be

[1]We do not need to assume quasiconcavity or homogeneity of degree one of U^i. U^i is increasing when for all $(x_1, ..., x_\omega)$,$(y_1, ..., y_\omega)$, if $x_s < y_s$ for all $s \in \Omega$, then $U^i(x_1, ..., x_\omega) < U^i(y_1, ..., y_\omega)$

much more general.

A social welfare function $W : \mathbb{R}_+^N \to \mathbb{R}$ is *inequality averse* if for all $(u^1, ..., u^n)$, $W\left(\sum_{i=1}^n u^i/n, ..., \sum_{i=1}^n u^i/n\right) \geq W\left(u^1, ..., u^n\right)$. We assume all social welfare functions are monotonic and continuous. For two social welfare functions W, W', we will say that W is more inequality averse than W' if for any $(u^1, ..., u^n)$ and $(v, v, ..., v)$,

$$W\left(u^1, ..., u^n\right) \geq W\left(v, ..., v\right) \Longrightarrow W'\left(u^1, ..., u^n\right) \geq W'\left(v, ..., v\right).$$

Any deviation from equality of utility which W finds acceptable, W' finds acceptable as well.[2]

The social welfare function generates a household utility function, given by

$$U^W\left(x_1, ..., x_\omega\right) = \max_{\sum_{i=1}^n x_s^i = x_s} W\left(U^1\left(x_1^1, ..., x_\omega^1\right), ..., U^n\left(x_1^n, ..., x_\omega^n\right)\right).$$

That is, the household utility is the maximal social utility that can be achieved with an aggregate bundle.

We now say that household utility U^W is more risk averse than household utility $U^{W'}$ if for any constant aggregate bundle $(c, c, ..., c)$ and any $(x_1, ..., x_\omega)$,

$$U^W\left(x_1, ..., x_\omega\right) \geq U^W\left(c, c, ..., c\right) \Longrightarrow U^{W'}\left(x_1, ..., x_\omega\right) \geq U^{W'}\left(c, c, ..., c\right).$$

Any deviation from uncertainty which U^W finds acceptable, $U^{W'}$ also finds acceptable.[3] Note the parallel with the comparative notion of inequality aversion. This parallel is only superficial: with inequality aversion, W ranks N-vectors, whereas with risk aversion, U^W ranks Ω-vectors.

[2]The comparative notion of inequality aversion is essentially due to kolm Kolm (1969) and Atkinson (1970). It parallels the comparative notion of risk aversion pioneered by Yaari (1969), which generalizes the Arrow-Pratt notions of comparative risk aversion.

[3]As noted, this general comparative notion of risk aversion is due to Yaari (1969).

This brings us to the main result:

Theorem 1 *Suppose that W and W' are inequality averse social welfare functions. Then if W is more inequality averse than W', U^W is more risk averse than $U^{W'}$.*

The proof of the theorem is simple and is illustrated in Figure 1. By the common prior and risk aversion assumptions on individual utility functions, for any constant aggregate bundle $(c, ..., c)$, the utility possibility set is a simplex. So, fix some bundle $(x_1, ..., x_\omega)$ and a constant bundle $(c, ..., c)$ for which $U^W(x_1, ..., x_\omega) \geq U^W(c, ..., c)$. Denote the utility possibility set of $(x_1, ..., x_\omega)$ by $\mathcal{U}(x)$. Supposing that $U^W(x_1, ..., x_\omega) \geq U^W(c, ..., c)$ means there is a point on $\mathcal{U}(x)$ which W ranks as at least as high as every point on the c-simplex. But as equal division maximizes both W and W' on this c-simplex, since W is more inequality averse than W', the relevant upper contour set of W is contained in the upper contour set of W'. Therefore, there is a point on $\mathcal{U}(x)$ which W' ranks at least as high as every point on the c-simplex, so that $U^{W'}(x_1, ..., x_\omega) \geq U^{W'}(c, ..., c)$.

Proof. Let $(c, c, ..., c) \in \mathbb{R}^{\Omega}_{+}$. We claim that maximizers to the functions

$$W\left(U^1\left(x_1^1, ..., x_\omega^1\right), ..., U^n\left(x_1^n, ..., x_\omega^n\right)\right)$$

and

$$W'\left(U^1\left(x_1^1, ..., x_\omega^1\right), ..., U^n\left(x_1^n, ..., x_\omega^n\right)\right)$$

subject to $\sum_{i=1}^{n} x_s^i = c$ both occur at $x_s^i = \frac{c}{n}$ for all $i \in N$, $s \in \Omega$. To see this, note that by risk aversion, the allocation $\{(x_1^1, ..., x_\omega^1), ..., (x_1^n, ..., x_\omega^n)\}$ of $(c, c, ..., c)$ is weakly Pareto dominated by $\{(\sum_{s=1}^{\omega} \pi_s x_s^1, ..., \sum_{s=1}^{\omega} \pi_s x_s^1), ..., (\sum_{s=1}^{\omega} \pi_s x_s^n, ..., \sum_{s=1}^{\omega} \pi_s x_s^n)\}$, which is a feasible constant allocation of $(c, ..., c)$.

As for any constant allocations $\{(d^1, ..., d^1), ..., (d^n, ..., d^n)\}$ of c, $U^i(d^i, ..., d^i) = d^i$, as

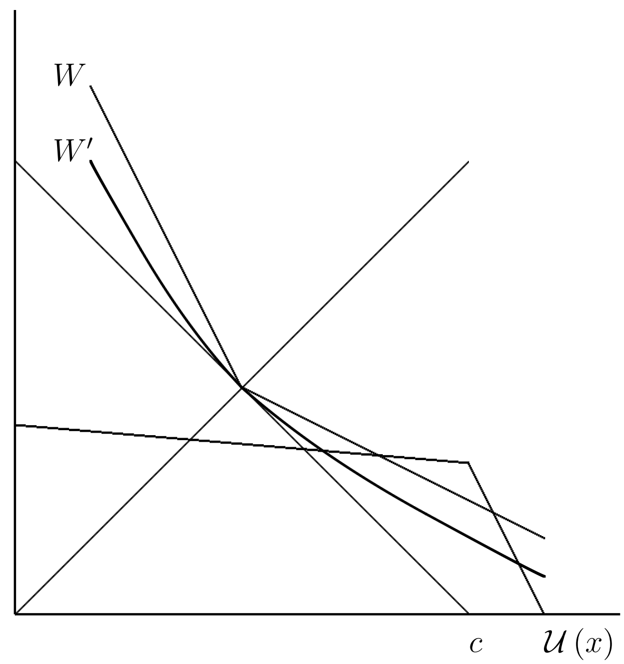

Figure 1: Proof of main result

both W and W' are monotonic, it follows that they have maximizers on

$$\left\{ \left(d^1, ..., d^n\right) : \sum_{i=1}^{n} d^i = c \right\}.$$

As they are inequality averse, $\left(\frac{c}{n}, ..., \frac{c}{n}\right)$ maximizes each of them; these utilities are given by the allocation $\left\{\left(\frac{c}{n}, ..., \frac{c}{n}\right), ..., \left(\frac{c}{n}, ..., \frac{c}{n}\right)\right\}$. Now, suppose that $U_W(x_1, ..., x_\omega) \geq U_W(c, ..., c)$. Let $\{(\overline{x}_1^1, ..., \overline{x}_\omega^1), ..., (\overline{x}_1^n, ..., \overline{x}_\omega^n)\}$ solve

$$\max_{\sum_{i=1}^{n} x_s^i = x_s} W\left(U^1\left(x_1^1, ..., x_\omega^1\right), ..., U^n\left(x_1^n, ..., x_\omega^n\right)\right).$$

Then

$$W\left(U^1\left(\overline{x}_1^1, ..., \overline{x}_\omega^1\right), ..., U^n\left(\overline{x}_1^n, ..., \overline{x}_\omega^n\right)\right) \geq W\left(\frac{c}{n}, ..., \frac{c}{n}\right).$$

Consequently, as W is more inequality averse than W',

$$W'\left(U^1\left(\overline{x}_1^1,...,\overline{x}_\omega^1\right),...,U^n\left(\overline{x}_1^n,...,\overline{x}_\omega^n\right)\right) \geq W'\left(\frac{c}{n},...,\frac{c}{n}\right).$$

Therefore,

$$\begin{aligned} & \max_{\sum_{i=1}^n x_s^i = x_s} W'\left(U^1\left(x_1^1,...,x_\omega^1\right),...,U^n\left(x_1^n,...,x_\omega^n\right)\right) \\ \geq\ & W'\left(\frac{c}{n},...,\frac{c}{n}\right) \\ =\ & \max_{\sum_{i=1}^n x_s^i = c} W'\left(U^1\left(x_1^1,...,x_\omega^1\right),...,U^n\left(x_1^n,...,x_\omega^n\right)\right). \end{aligned}$$

Consequently $U^{W'}(x_1,...,x_\omega) \geq U^{W'}(c,...,c)$. ■

4 On inequality loving social welfare functions

In general, no comparative static result of the type in Theorem 1 can be established for inequality loving social welfare functions. However, it is of interest to note that a reverse comparative static can often obtain.

Figure 2 displays the upper contour sets for two inequality loving social welfare functions, W and W'. W is the more inequality averse of the two. The figure depicts W' as being maximized on the simplex. However, from the diagram, it is clear that when W is maximized on the simplex, the upper contour set of W will not be contained in the upper contour set of W', and hence we could not conclude that U^W is more risk averse than $U^{W'}$. That the reverse comparative static often obtains can be demonstrated by considering the social welfare functions $W_p(u^1,...,u^n) = \sum_{i=1}^n (u^i)^p$ for $p \geq 1$. As p decreases, the social welfare function becomes more inequality averse, but the induced household preference becomes less risk averse.

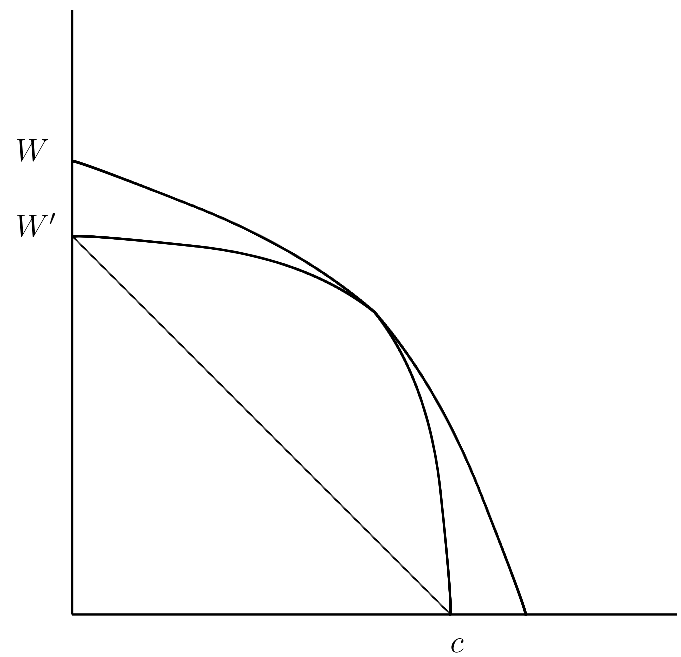

Figure 2: Inequality loving social welfare functions

There is one general statement about risk aversion for inequality loving social welfare functions. In fact, the following result is very general and applies to all social welfare functions. The *utilitarian social welfare function* is defined by $W_U(u^1, ..., u^n) = \sum_{i=1}^n u^i$. The utilitarian social welfare function always induces the least risk averse household preference among social welfare functions satisfying the Pareto property. In a sense, the further a household deviates from inequality neutrality *in either direction*, the more risk averse the household becomes.

Theorem 2 *For any social welfare function W, U^{W_U} is less risk averse than U^W.*

Proof. Let W be a social welfare function; recall we always assume W to be monotonic and continuous. Suppose that $U^W(x_1, ..., x_\omega) \geq U^W(c, ..., c)$. As in the proof of Theorem 1, the Pareto efficient allocations of $(c, ..., c)$ are of the form $\{(d^1, ..., d^1), ..., (d^n, ..., d^n)\}$, where $\sum_{i=1}^n d^i = c$. Consequently, there exists $\{(\overline{x}_1^1, ..., \overline{x}_\omega^1), ..., (\overline{x}_1^n, ..., \overline{x}_\omega^n)\}$ for which for all $s \in \Omega$, $\sum_{i=1}^{n} \overline{x}_s^i = x_s$ such that $W\left(U^1\left(\overline{x}_1^1, ..., \overline{x}_\omega^1\right), ..., U^n\left(\overline{x}_1^n, ..., \overline{x}_\omega^n\right)\right) \geq W\left(d^1, ..., d^n\right)$ whenever $\sum_{i=1}^{n} d^i = c$. As W is monotonic, $\sum_{i=1}^{n} U^i\left(\overline{x}_1^i, ..., \overline{x}_\omega^i\right) \geq c$. But as for any allocation $\left\{\left(x_1^1, ..., x_\omega^1\right), ..., \left(x_1^n, ..., x_\omega^n\right)\right\}$ of $(c, ..., c)$, $W_U\left(U^1\left(x_1^1, ..., x_\omega^1\right), ..., U^n\left(x_1^n, ..., x_\omega^n\right)\right) \leq c$, and $W_U\left(U^1\left(\overline{x}_1^1, ..., \overline{x}_\omega^1\right), ..., U^n\left(\overline{x}_1^n, ..., \overline{x}_\omega^n\right)\right) = \sum_{i=1}^{n} U^i\left(\overline{x}_1^i, ..., \overline{x}_\omega^i\right)$, we conclude that $U^{W_U}\left(x_1, ..., x_\omega\right) \geq U^{W_U}\left(c, ..., c\right)$, from which we establish the claim.

■

We note that, although Theorem 2 is stated in cardinal terms, the comparative notion of risk aversion is ordinal. One way to read the theorem in an ordinal framework states that the social welfare function which can be represented by the *sum of individual certainty equivalents* always induces the least risk averse household among all numerically representable social welfare functions satisfying the Pareto property.

5 Conclusion

The results of this note indicate that oftentimes, a household whose allocation decisions are motivated by equity will exhibit more risk averse behavior than one whose are not. This illuminates a basic tradeoff between household equity and aggregate risk attitudes.

While we have stated the result on the domain of inequality averse social welfare functions, the proof makes clear that the critical property of these social welfare functions is their recommendation to equitably divide any dollar. Any pair of social welfare functions which recommend an equal division of a certain dollar will satisfy the comparative static discussed here.

Finally, we note that a partial converse of our result is available. Suppose we are given two *quasiconcave* inequality averse social welfare functions, W and W'. Suppose it is known that for *every* list of utility functions with a common prior, U^W is more risk averse than

$U^{W'}$. Then it follows that W is more inequality averse than W'.[4]

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[4]This can be verified by a simple duality argument. Consider any upper contour set of W, passing through the ray of equal coordinates at $(c/n, ..., c/n)$, for example. Consdider any positive direction $p \in \mathbb{R}^{\Omega}_{++}$. Now consider the tangent hyperplane in direction p to the upper contour set of W. This hyperplane is the boundary of a half space which is the utility possibilities set for some profile of utilities which are homogeneous of degree one (such a profile is easy to construct) and some aggregate bundle x which returns positive payoff only in one state. The household utitily which is so constructed will rank x as indifferent to c according to U^W. As a consequence of the comparative risk aversion hypothesis, x will be at least as good as c according to $U^{W'}$. Therefore, the tangent to the upper contour set of W' in direction p will be further back than the corresponding tangent hyperplane of W. So long as each upper contour set is convex, this verifies that the upper contour set of W' contains that of W, and hence W is more inequality averse than that of W'. Unlike Theorem 1, this argument *does* rely critically on the fact that each of W and W' are quasiconcave.