Note on Symmetric Utility

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Abstract

This note studies necessary and sufficient conditions for consumer demand data to be generated by a symmetric utility function. We find that a dataset of prices and consumption decisions can be rationalized by a symmetric utility function if and only if the symmetrized dataset satisfies the generalized axiom of revealed preference.

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1 Introduction

In this paper, we study revealed preference conditions that are necessary and sufficient for a finite dataset of prices and consumption choices to be rationalized by a symmetric and locally nonsatiated utility. In particular, a dataset is rationalized by a symmetric utility function if and only if a *symmetrized* data set satisfies the generalized axiom of revealed preference. To the best of our knowledge, the conditions for a symmetric and locally nonsatiated utility function are unknown.

Models of consumer demand are foundational in economics. Afriat (1967) provides necessary and sufficient conditions for a finite dataset of prices and consumption choices to be consistent with an individual maximizing a locally nonsatiated preference ordering. Varian (1982, 1983) coins the term generalized axiom of revealed preference (GARP) and studies restrictions on utility such as homotheticity and separability. For a review of other works on revealed preference, see Chambers and Echenique (2016).

The study of a symmetric utility function is a natural feature of preference to study. For example, when performing a revealed preference test on consumer demand data, a researcher may examine consumption choices for a subset of all offered goods. A researcher does not know *a priori* whether an individual treats each good symmetrically. Thus, a test of symmetric utility determines whether the goods under study are treated symmetrically.

Studying symmetric utility may be of interest outside of the standard consumer problem. For example, many behavioral models make use of symmetric attention costs. Fudenberg et al. (2015) studies stochastic choice data generated by an additively separable and symmetric cost function. Matejka and McKay (2014) studies rational inattention assuming the Shannon entropy cost function.[1] The approach in this paper studies the implications of symmetry without additive separable costs. This result can be used to characterize the properties of a symmetric cost function in the stochastic choice model of Allen and Rehbeck (2016).

The remainder of the note proceeds as follows: Section 2 presents the dataset of interest and presents basic definitions. Section 3 presents the main result and an example dataset that refutes symmetric utility.

[1]The Shannon entropy for a finite probability distribution $q \in \{z \in \mathbb{R}^M \mid \sum_{m=1}^M z_m = 1 \ , \ z_m \geqslant 0 \ \forall m \in \{1, \ldots, M\}\}$ is given by $c(q) = -\sum_{m=1}^M q_m \log(q_m)$.

2 Definitions

A *consumer choice dataset* is a finite collection of indexed pairs: $\{(x^k, p^k)\}_{k=1}^K$, where each $x^k \in \mathbb{R}^n_+$ is a consumption bundle and $p^k \in \mathbb{R}^n_{++}$ is a vector of prices. A dataset is *weakly rationalizable* by a locally nonsatiated preference if there is a complete, transitive, and locally nonsatiated relation $\succeq$ such that for all k, if $p^k \cdot x^k \geqslant p^k \cdot x$, then $x^k \succeq x$. A dataset is weakly rationalizable by a utility function u if for all k, if $p^k \cdot x^k \geqslant p^k \cdot x$, then $u(x^k) \geqslant u(x)$.

A utility function is *symmetric* if for all $x \in \mathbb{R}^n_+$ and all permutations $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, $u(x_1, \ldots, x_n) = u(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. A preference is symmetric if for all $x \in \mathbb{R}^n_+$ and all permutations $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, $(x_1, \ldots, x_n) \sim (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. We say a dataset is *symmetrically rationalizable* if it is weakly rationalizable by a symmetric utility function.

Given any dataset, we can define its *revealed preference pair* $\langle \succeq^R, \succ^R \rangle$ by $x^k \succeq^R x$ if $p^k \cdot x^k \geqslant p^k \cdot x$ and $x^k \succ^R x$ if $p^k \cdot x^k > p^k \cdot x$. We say the revealed preference pair is *acyclic* if there is no sequence $\{k_1, \ldots, k_m\}$ of indices for which $x_{k_1} \succeq^R \ldots \succeq^R x_{k_m} \succ^R x_{k_1}$. We note that a revealed preference pair is acyclic if and only if it satisfies GARP.

Let Σ be the set of permutations $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. For a dataset $D = \{(x^k, p^k)\}_{k=1}^K$, denote by $D_\sigma = \{(x^k_{\sigma(1)}, \ldots, x^k_{\sigma(n)}), (p^k_{\sigma(1)}, \ldots, p^k_{\sigma(n)})\}_{k=1}^K$. Let $x^k_\sigma = (x^k_{\sigma(1)}, \ldots, x^k_{\sigma(n)})$ and $p^k_\sigma = (p^k_{\sigma(1)}, \ldots, p^k_{\sigma(n)})$. We can concatenate the datasets D_σ across all $\sigma \in \Sigma$. The resulting dataset D_Σ is called the *symmetrized* data set.

The proof of the main result uses a version of Afriat's Theorem (see for example Chambers and Echenique (2016)).

Lemma 1 (Afriat's Theorem)**.** *Let $D = \{(x^k, p^k)\}_{k=1}^K$ be a finite consumption dataset. The revealed preference generated by a consumption dataset satisfies acyclicity if and only if there exist numbers U^k and $\lambda^k > 0$ such that*

$$U^k \leqslant U^\ell + \lambda^\ell p^\ell \cdot (x^k - x^\ell)$$

for every pair $(x^k, p^k), (x^\ell, p^\ell) \in D$.

3 Main Result

We now present the main result.

Proposition 1. *The following are equivalent for the dataset $D = \{(x^k, p^k)\}_{k=1}^K$.*

(i) *D is weakly rationalizable by a concave, continuous, increasing,*[2] *and symmetric utility function.*

(ii) *D is weakly rationalizable by a symmetric and locally nonsatiated preference.*

(iii) *The revealed preference pair generated by* D_Σ *is acyclic.*

(iv) *There exist* $U^{(\sigma,k)} \in \mathbb{R}$ *and* $\lambda^{(\sigma,k)} > 0$ *such that*

$$U^{(\sigma,k)} \leqslant U^{(\eta,\ell)} + \lambda^{(\eta,\ell)} p^\ell_\eta \cdot (x^k_\sigma - x^\ell_\eta)$$

for every pair $(x^k_\sigma, p^k_\sigma), (x^\ell_\eta, p^\ell_\eta) \in D_\Sigma$ *where* $\sigma, \eta \in \Sigma$.

(v) *There exist* $U^k \in \mathbb{R}$ *and* $\lambda^k > 0$ *such that*

$$U^k \leqslant U^\ell + \lambda^\ell p^\ell_\eta \cdot (x^k_\sigma - x^\ell_\eta)$$

for every pair $(x^k_\sigma, p^k_\sigma), (x^\ell_\eta, p^\ell_\eta) \in D_\Sigma$ *where* $\sigma, \eta \in \Sigma$.

Proof. The proof is a straightforward application of Afriat's Theorem. Observe that the first condition in the statement obviously implies the second, the second condition obviously implies the third. Afriat's Theorem gives the equivalence between the third and fourth conditions. We show that the fourth condition implies the first.

The fourth condition implies there is a continuous, increasing, concave utility function

$$u(x) = \min\left\{U^{(\sigma,k)} + \lambda^{(\sigma,k)} p^k_\sigma \cdot (x - x^k_\sigma) \mid k = 1, \ldots, K\ ,\ \sigma \in \Sigma\right\}$$

which weakly rationalizes D_Σ. For each $\sigma \in \Sigma$ define $u_\sigma(x) = u(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Observe that by construction, each u_σ also weakly rationalizes D_Σ. Finally,

$$u^*(x) = \sum_{\sigma \in \Sigma} u_\sigma(x)$$

is continuous, increasing, concave, and symmetric. It also clearly rationalizes D_Σ, and hence D.

Finally, we show that the fourth and fifth condition are equivalent; clearly, the fifth implies the fourth. On the other hand, given the numbers $U^{(\sigma,k)}$ and $\lambda^{(\sigma,k)}$ as specified in the fourth condition, define $U^k = \sum_{\eta \in \Sigma} U^{(\eta,k)}$ and $\lambda^k = \sum_{\eta \in \Sigma} \lambda^{(\eta,k)}$.

We want to show that for all k, ℓ, $U^k \leqslant U^\ell + \lambda^\ell p^\ell_\eta \cdot (x^k_\sigma - x^\ell_\eta)$.

[2] A utility function is increasing if $x_i \geqslant y_i$ for all i and $x \neq y$, then $x > y$.

We have $U^{(\sigma,k)} \leqslant U^{(\eta,\ell)} + \lambda^{(\eta,\ell)} p^\ell_\eta \cdot (x^k_\sigma - x^\ell_\eta)$. Further, we know that

$$p^\ell_\eta \cdot (x^k_\sigma - x^\ell_\eta) = p^\ell_\tau \cdot (x^k_{\tau\circ\eta^{-1}\circ\sigma} - x^\ell_\tau)$$

so

$$U^{(\tau\circ\eta^{-1}\circ\sigma,k)} \leqslant U^{(\tau,\ell)} + \lambda^{(\tau,\ell)} p^\ell_\eta \cdot (x^\ell_\sigma - x^l_\eta).$$

Now sum these inequalities across τ to get:

$$U^k \leqslant U^\ell + \lambda^\ell p^\ell_\eta \cdot (x^k_\sigma - x^l_\eta).$$

□

The set of permutations of consumption bundles and prices is typically large. The following corollary gives a simple way to find a refutation of symmetric rationalizability. In particular, if one finds a transposition of two elements that satisfies the below inequality, then the dataset cannot be rationalized by a symmetric utility function.

Corollary 1. *The dataset D is not rationalized by a symmetric utility function if there exists $k \in \{1, \ldots, K\}$ and a pair $\sigma, \eta \in \Sigma$ such that $p^k_\eta \cdot x^k_\eta > p^k_\eta \cdot x^k_\sigma$.*

Proof. It is immediate that this rules out a solution to (v) in Proposition 1. □

Corollary 1 implies that a dataset with a single observation can violate symmetric rationalizability. For example, the dataset $x = (1,0)$ and $p = (2,1)$ does not have a symmetric rationalization by Corollary 1. There is only one permutation of the indices, so we denote $p_\sigma = (1,2)$ and $x_\sigma = (0,1)$. The dataset satisfies the condition of Corollary 1 since $p \cdot x = (2,1) \cdot (1,0) > (2,1) \cdot (0,1) = p \cdot x_\sigma$. Thus, there is no symmetric utility function that rationalizes the data. We also see the dataset violates acyclicity since $p_\sigma \cdot x_\sigma = (1,2) \cdot (0,1) > (1,2) \cdot (1,0) = p_\sigma \cdot x$.

Finally, we remark that tests of the joint hypotheses of symmetry and other properties (for example, quasi-linearity (Brown and Calsamiglia (2007)) or homotheticity (Varian (1983))) have straightforward generalizations of the idea here.

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