TESTABLE IMPLICATIONS OF TRANSLATION INVARIANCE AND HOMOTHETICITY: VARIATIONAL, MAXMIN, CARA AND CRRA PREFERENCES

CHRISTOPHER P. CHAMBERS, FEDERICO ECHENIQUE, AND KOTA SAITO

ABSTRACT. We provide revealed preference axioms that characterize models of translation invariant preferences. In particular, we characterize the models of variational, maxmin, CARA and CRRA utilities. In each case we present a revealed preference axiom that is satisfied by a dataset if and only if the dataset is consistent from the corresponding utility representation. Our results complement traditional exercises in decision theory that take preferences as primitive.

1. INTRODUCTION

We work out the testable implications of models with translation invariant preferences. Given a finite dataset on purchases of state-contingent assets, we give a revealed preference axiom that describes the datasets that are consistent with different models of translation invariant preferences.

These models include risk neutral variational preferences (Maccheroni et al., 2006), risk neutral maxmin preferences (Gilboa and Schmeidler, 1989), and subjective expected utility preferences with constant absolute risk aversion: so-called CARA preferences. Analogously to the

Chambers is affiliated with the Department of Economics, University of California, San Diego. Echenique and Saito are affiliated with the Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena CA 91125. Emails: cpchambers@ucsd.edu (Chambers), fede@caltech.edu (Echenique), ksaito@hss.caltech.edu (Saito). We are very grateful to Simon Grant and Massimo Marinacci, who posed questions to us that led to some of the results in this paper.

CARA case, we also work out the testable implications of subjective expected utility preferences with constant relative risk aversion, the CRRA preferences (these form the "homothetic" class alluded to in the title). The models have been used by economists for different purposes. Variational and maxmin preferences are the most commonly-used models of ambiguity aversion. They are also used to capture model robustness (Hansen and Sargent, 2008). CARA and CRRA preferences are very common in applied work in macroeconomics and finance, among other fields.

Our contribution is to start from finite data on state-contingent consumption purchases, such as one would observe from a market experiment on choice under uncertainty (Hey and Pace, 2014; Ahn et al., 2014; Bayer et al., 2012; Bossaerts et al., 2010). We describe the datasets that are rationalizable as consistent with a preference relation that satisfies translation invariance. When we say that we describe the datasets that are rationalizable, we mean that we provide a property, a "revealed preference axiom," that the data satisfies if and only if it is consistent with the theory in question.

The models we study have well known axiomatizations when one takes preferences as primitive, but not when one takes consumption data as given. The axiomatization of variational preferences is due to Maccheroni et al. (2006) (see also Grant and Polak (2013) for a variation on their arguments). The axiomatization of maxmin is due to Gilboa and Schmeidler (1989). These papers are often thought to provide the behavioral counterpart of certain theories of choice: the preference relation captures an agent's behavior, and the theorems in these papers describe the behaviors that are consistent with the theory. Our focus is on behavior in the market, not on preferences. The primitive is a finite list of purchases of state-contingent payments, each one made at a different price vector.

In contrast with most papers on ambiguity, we do not work in the Anscombe-Aumann framework. For this reason, we must restrict attention to risk-neutral variational and maxmin preferences. The Anscombe-Aumann framework has the advantage that it (essentially) allows the utility over outcome to be observable. In a similar vein, our results extend beyond the risk neutral case by adding a utility function as an "observation" to our datasets.

It would of course be desirable to obtain results without the assumption of risk neutrality; but these are likely difficult to come by. One exception is the case of maxmin utility with two states: we give a characterization of the data sets that are rationalizable with risk neutral (concave utility over money) maxmin in Section 6. The two-state case is of course restrictive, but probably of interest for experiments on ambiguity: some of the most basic experiments illustrating ambiguity aversion involve two states.

The closest papers to ours are Varian (1988), Bayer et al. (2012) and Polisson and Quah (2013). Our results on CARA and CRRA are close to Varian (1988). The main difference is that Varian considers the case of objective probabilities, not subjective. Bayer et al. (2012) and Polisson and Quah (2013) looks at the testable implications of models of ambiguity aversion for the same kinds of data that we assume in this paper. They give a characterization in terms of the solution of a system of inequalities. Our contribution is different because we give a revealed preference axiom that has to be satisfied for the data to be rationalizable.

The papers by (Kubler et al., 2014) and (Echenique and Saito, 2013) are also related. Kubler et al. solves the same problem as we do here, but for the case of expected utility theory with known (objective) probabilities over states. Echenique and Saito solve the problem for subjective expected utility.

2. Definitions.

Let S be a finite set of states of the world. An *act* is a function from S into $\mathbf{R}$. So $\mathbf{R}^S$ is the set of acts. An act can be interpreted as a state-contingent monetary payment. Define $\|x\|_1 = \sum_s x_s$.

A *preference relation* on $\mathbf{R}^S$ is a binary relation $\succeq$ that is complete and transitive. Given a preference relation $\succeq$, we denote by $\succ$ the strict part of $\succeq$. A function $u : \mathbf{R}^S \to \mathbf{R}$ defines a preference relation $\succeq$ by

$x \succeq y$ if and only if $u(x) \geq u(y)$. We say that u *represents* $\succeq$, or that it is a utility function for $\succeq$.

A preference relation $\succeq$ on $\mathbf{R}^S$ is *locally nonsatiated* if for every x and every $\varepsilon > 0$ there is y such that $\|x - y\| < \varepsilon$ and $y \succ x$.

3. Preferences, utilities, and data.

A *data set* D is a finite collection $\{(p^k, x^k)\}_{k=1}^K$, where each $p^k \in \mathbf{R}^S_{++}$ is a vector of strictly positive (Arrow-Debreu) prices, and each $x^k \in \mathbf{R}^S$ is an act. The interpretation of a dataset is that each pair (p^k, x^k) consists of an act x^k chosen from the budget $\{x \in \mathbf{R}^S : p^k \cdot x \leq p^k \cdot x^k\}$ of affordable acts.[1]

A data set $\{(p^k, x^k)\}_{k=1}^K$ is *rationalizable* by a preference relation $\succeq$ if $x^k \succeq x$ whenever $p^k \cdot x^k \geq p^k \cdot x$. So a data set is rationalizable by a preference relation when the choices in the dataset would have been optimal for that preference relation.

A data set $\{(p^k, x^k)\}_{k=1}^K$ is *rationalizable* by a utility function u if it is rationalizable by the preference relation represented by u. So a data set is rationalizable by a utility function when the choices in the dataset would have maximized that utility function in the relevant budget set.

A preference relation $\succeq$ is *translation invariant* if for all $x, y \in \mathbf{R}^S$ and all $c \in \mathbf{R}$, we have $x \succeq y$ if and only if $x + (c, \dots, c) \succeq y + (c, \dots, c)$.

A preference relation $\succeq$ is *homothetic* if for all $x, y \in \mathbf{R}^S$ and all $\alpha > 0$, we have $x \succeq y$ if and only if $\alpha x \succeq \alpha y$.

A preference $\succeq$ is a *(risk-neutral) variational preference* if there is a convex and continuous function c such that the utility function

$$\inf_{\pi \in \Delta(S)} \pi \cdot x + c(\pi)$$

represents $\succeq$. If a data set is rationalizable by a variational preference relation, we will say that the dataset set is *(risk-neutral) variational-rationalizable.*

[1] Arrow-Debreu prices make sense in a setting of complete markets and absence of arbitrage. Arrow-Debreu prices can then be recovered from asset prices. We also imagine experimental data from markets in which Arrow-Debreu securities are traded (Hey and Pace, 2014; Ahn et al., 2014; Bayer et al., 2012).

A special case of variational preference is maxmin: A preference relation is *(risk-neutral) maxmin* if there is a closed and convex set $\Pi \subseteq \Delta(S)$ such that the utility function

$$\inf_{\pi \in \Pi} \pi \cdot x$$

represents $\succeq$. If a data set is rationalizable by a risk neutral maxmin preference relation, we will say that the dataset set is *(risk-neutral) maxmin-rationalizable*.

A utility $u : \mathbf{R}^S \to \mathbf{R}$ is *constant absolute risk aversion (CARA)* if there is $a > 0$ and $\pi \in \Delta(S)$ for which

$$u(x) = \sum_{s \in S} \pi_s \left(-\exp(-ax) \right).$$

Note that CARA is a special case of subjective expected utility.

A utility $u : \mathbf{R}^S \to \mathbf{R}$ is *constant relative risk aversion (CRRA)* if there is $a \in (0,1)$ and $\pi \in \Delta(S)$ for which

$$u(x) = \sum_{s \in S} \pi_s \left(\frac{x^{1-a}}{1-a} \right).$$

If a data set is rationalizable by a CARA (CRRA) utility, we will say that the dataset set is *CARA (CRRA) rationalizable*.

4. Variational preferences

We present the results on variational and maxmin rationalizability as Theorems 1 and 3. In each case, the model in question assumes a linear utility index: so the model captures ambiguity aversion but risk neutrality. These results beg the question of the empirical content of risk aversion together with ambiguity aversion. In Section 6 we present a result on maxmin utility with risk aversion. It is restricted to environments with two states.

1. **Theorem.** *The following statements are equivalent:*

(1) *Dataset D is rationalizable by a locally nonsatiated, translation invariant preference.*

(2) *Dataset D is rationalizable by a continuous, strictly increasing, concave utility function satisfying the property $u(x+(c,\dots,c)) = u(x)+c$.*
(3) *Dataset D is variational-rationalizable.*
(4) *For every $l = 1,\dots,M$, and every sequence $\{k_l\} \subseteq \{1,\dots,K\}$, we have $\sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}) \geq 0$ (here addition is modulo M, as usual).*

2. *Remark.* The preceding result can be generalized. Suppose we were interested in the testable implications of preferences which are β-translation invariant, for some $\beta \geq 0$, $\beta \neq 0$. That is, we want to know whether for all x, y, we have $x \succeq y$ if and only if $x+\beta \succeq y+\beta$. Define the seminorm $\|x\|_1^\beta = \sum_i |\beta_i x_i|$. Then it is an easy exercise to verify that the testable implications of β-translation invariance are given by equation (4), replacing $\|\cdot\|_1$ with $\|\cdot\|_1^\beta$. Hence, the test given here should be compared with the one given by Brown and Calsamiglia (2007), and other tests for risk preferences.

We now turn out attention to maxmin preferences. Note that the equivalence between (2) and (3) in Theorem 3 is well known, but here we prove it through an application of Theorem 1.

We say that a function $u : \mathbf{R}^S \to \mathbf{R}$ is linearly homogeneous if for all $x \in \mathbf{R}^S$ and all $\alpha > 0$, we have $u(\alpha x) = \alpha u(x)$.

3. **Theorem.** *The following statements are equivalent:*

(1) *Dataset D is rationalizable by a locally nonsatiated, homothetic and translation invariant preference.*
(2) *Dataset D is rationalizable by a continuous, strictly increasing, linearly homogeneous and concave utility function satisfying the property that $u(x+(c,\dots,c)) = u(x)+c$.*
(3) *Dataset D is maxmin-rationalizable.*
(4) *For every k and l,*

$$\frac{p^k}{\|p^k\|_1} \cdot x^k \leq \frac{p^l}{\|p^l\|_1} \cdot x^k.$$

5. CARA and CRRA

The previous section considers translation invariance and homotheticity as general properties of preferences in choice under uncertainty. Here we focus on the case of subjective expected utility. So we consider models in which the agent has a single prior over states, and maximizes expected utility. The prior is unknown though, and must be inferred from her choices. In the subjective expected utility case, translation invariance gives rise to CARA preferences, and homotheticity to CRRA.

4. **Theorem.** *A dataset D is CARA rationalizable if and only if there is $\alpha^* > 0$ such that (1) holds; and CRRA rationalizable if and only if there is $\alpha^* \in (0,1)$ such that (2) holds.*

$$\alpha^*(x^k_t - x^k_s + x^{k'}_s - x^{k'}_t) = \ln(\frac{p^k_s}{p^k_t}\frac{p^{k'}_t}{p^{k'}_s}) \tag{1}$$

$$\alpha^* \ln(\frac{x^k_t}{x^k_s}\frac{x^{k'}_s}{x^{k'}_t}) = \ln(\frac{p^k_s}{p^k_t}\frac{p^{k'}_t}{p^{k'}_s}) \tag{2}$$

The conditions in Theorem 4 may look like existential conditions: essentially Afriat inequalities. Afriat inequalities are indeed the source of equations (1) and (2), as evidenced by the proof of Theorem 4, but note that the statements are equivalent to non-existential statements. Equation (1) says that when $(x^k_t - x^k_s + x^{k'}_s - x^{k'}_t) \neq 0$,

$$\frac{\ln(\frac{p^k_s}{p^k_t}\frac{p^{k'}_t}{p^{k'}_s})}{(x^k_t - x^k_s + x^{k'}_s - x^{k'}_t)}$$

is independent of k, t, k' and s; and that when $(x^k_t - x^k_s + x^{k'}_s - x^{k'}_t) = 0$ then $\ln(\frac{p^k_s}{p^k_t}\frac{p^{k'}_t}{p^{k'}_s}) = 0$. Similarly for Equation (2).

It is worth pointing out that, except in the case when for all observations, all prices are equal, and consumption of all goods are equal, equation (1) can have only one solution. Hence, risk preferences are uniquely identified. The next corollary also shows that beliefs are identified.

When $\pi \in \Delta(S)$ and $a > 0$, let $U_a = \sum_{s\in S} \pi_s \left(-\exp(-ax)\right)$ denote the associated subjective expected CARA utility.

5. **Corollary.** *$\alpha^* > 0$ solves (1) if and only if there is $\pi \in \Delta(S)$ such that π and U_{α^*} CARA rationalizes D. Further, for any such $\alpha^* > 0$, there is a unique $\pi^* \in \Delta(S)$ such that if π' and U_{α^*} CARA rationalizes D, then $\pi' = \pi$. Similarly for (2) and CRRA rationalizability.*

6. Risk averse max-min with two states

The prior result is about risk neutral maxmin. Here we turn to maxmin with risk aversion. A preference relation is *maxmin* if there is a closed and convex set $\Pi \subseteq \Delta(S)$ and a concave utility $u : \mathbf{R}^S \to \mathbf{R}$ such that the utility function

$$\inf_{\pi \in \Pi} \sum_{s=1,2} \pi_s u(x_s)$$

represents $\succeq$. If a data set is rationalizable by a maxmin preference relation, we will say that the dataset set is *maxmin-rationalizable.*

Assume a dataset $\{(p^k, x^k)\}_{k=1}^K$ in which $x^k_s \neq x^{k'}_{s'}$ when $(k, s) \neq (k', s')$.

Let K_1 be the set of all k such that $x^k_1 < x^k_2$, and K_2 be the set of all k such that $x^k_1 > x^k_2$. Suppose that $K = K_1 \cup K_2$.

Given a sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^n$, consider the following notation: Let $I_{l,s} = \{i : k_i \in K_l \text{ and } s_i = s\}$ $I'_{l,s} = \{i : k'_i \in K'_l \text{ and } s'_i = s\}$, for $l = 1, 2$ and $s = 1, 2$.

Strong Axiom of Revealed Maxmin Utility (SARMU): *For any sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^n$ in which*

1. $x^{k_i}_{s_i} \geq x^{k'_i}_{s'_i}$ *for all i;*
2. *each k appears as k_i (on the left of the pair) the same number of times it appears as k'_i (on the right);*
3. $|I_{1,1}| - |I'_{1,1}| = |I'_{2,1}| - |I_{2,1}| \leq 0$

The product of prices satisfies that

$$\prod_{i=1}^n \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} \leq 1.$$

One can alternatively define the axiom with $|I_{2,2}| - |I'_{2,2}| = |I'_{1,2}| - |I_{1,2}| \leq 0$ in condition (3).

6. **Theorem.** *A dataset is maxmin rationalizable if and only if it satisfies SARMU.*

6.1. **Discussion.** Echenique and Saito (2013) show that the following axiom characterizes rationalizability by subjective expected utility.

Strong Axiom of Revealed Subjective Expected Utility (SARSEU): *For any sequence of pairs* $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ *in which*

(1) $x^{k_i}_{s_i} \geq x^{k'_i}_{s'_i}$ *for all* i*;*
(2) *each* k *appears as* k_i *(on the left of the pair) the same number of times it appears as* k'_i *(on the right);*
(3) $|I_{1,1}| + |I_{2,1}| = |I'_{1,1}| + |I'_{2,1}|$

The product of prices satisfies that

$$\prod_{i=1}^{n} \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} \leq 1.$$

Note that condition (3) of SARSEU is equivalent to $|I_{2,2}| + |I_{1,2}| = |I'_{2,2}| + |I'_{1,2}|$ because

$$|I_{1,1}| + |I_{2,1}| + |I_{2,2}| + |I_{1,2}| = n = |I'_{1,1}| + |I'_{2,1}| + |I'_{2,2}| + |I'_{1,2}|.$$

Inspection of SARSEU and SARMU yields the following

7. **Proposition.** *If a dataset satisfies SARSEU then it satisfies SARMU.*

For a dataset to be maxmin rationalizable, but inconsistent with subjective expected utility, it needs to contain a sequence in the conditions of SARSEU in which $|I_{1,1}| + |I_{2,1}| = |I'_{1,1}| + |I'_{2,1}|$, but where $|I_{1,1}| - |I'_{1,1}| > 0$.

As we have emphasized, the result in Theorem 6 is for two states. There are two simplifications afforded by the assumption of two states, and the two are crucial in obtaining the theorem. The first is that with two states there are only two extreme priors to any set of priors. With the assumption that u is monotonic, one can know which of the two extremes is relevant to evaluate any given act. The second simplification is a bit harder to see, but it comes from the fact that one can normalize the probability of one state to be one and only keep track of the probability of the other state. Then the property of beign an extreme prior carries over to the probability of the state that is left "free."[2]

7. Proofs

7.1. **Proof of Theorem 1.** That (3) $\Longrightarrow$(1) is obvious. We shall first prove that (1) $\Longrightarrow$(4)

Suppose, towards a contradiction, D is a dataset satisfying (1) but not (4). Then we have a cycle $\sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}) < 0$. Let us without loss assume the sequence is $x^1, \dots, x^M$ so as to avoid cumbersome notation. Let $Z = \sum_{l=1}^{M} \frac{p^l}{\|p^l\|_1} \cdot (x^{l+1} - x^l) < 0$.

Define a new sequence $(y^1, \dots, y^M)$ inductively. Let $y^1 = x^1$, and let $y^k = x^k + (c^k, \dots, c^k)$ where c^k is chosen so that $\frac{p^k}{\|p^k\|_1} \cdot (y^{k+1} - y^k) = \frac{Z}{M}$. Specifically, $c^1 = 0$ and

$$c^{k+1} = c^k + \frac{Z}{M} - \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k)$$

for $k = 1, \dots, M-1$. Let $q^k = \frac{p^k}{\|p^k\|_1}$ and consider the dataset (q^k, y^k), $k = 1, \dots M$.

Observe that

$$\sum_{k=1}^{M-1} q^k \cdot (y^{k+1} - y^k) + q^M \cdot (y^1 - y^M) = \sum_{k=1}^{M} \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k) = Z,$$

and that $q^k \cdot (y^{k+1} - y^k) = Z/M$ for $k = 1, \dots, M-1$. Therefore, $q^M \cdot (y^1 - y^M) = Z/M$

The original dataset is rationalizable by some locally non-satiated and translation invariant preference $\succeq$. It is easy to see that the same preference rationalizes the dataset (q^k, y^k). Indeed, if $q^k \cdot y^k \geq q^k \cdot y$ then $p^k \cdot x^k \geq p^k \cdot (y - (c^k, \dots, c^k))$, by definition of y^k and q^k. So $x^k \succeq (y - (c^k, \dots, c^k))$, and thus $y^k \succeq y$ by translation invariance of $\succeq$.

Observe that

$$\sum_{k=1}^{M-1} q^k \cdot (y^{k+1} - y^k) + q^M \cdot (y^1 - y^M) = \sum_{k=1}^{M-1} \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k) + c^M = Z,$$

[2]This can be seen in the proof of Lemma 8 when we go from $\bar{\pi} \geq \underline{\pi}$ to $\bar{\mu}_1 \geq \underline{\mu}_1$.

and that $q^k \cdot (y^{k+1} - y^k) = Z/M$ for $k = 1, \ldots, M-1$. Therefore, $q^M \cdot (y^1 - y^M) = Z/M$. In particular, $q^k \cdot (y^{k+1} - y^k) = Z/M < 0$ for $k = 1, \ldots, M \pmod M$. Thus $y^k \succ y^{k+1}$ as (q^k, y^k) is rationalizable by $\succeq$ and $\succeq$ is locally nonsatiated. This contradicts the transitivity of $\succeq$.

Now we show that (4) $\Longrightarrow$(2). Let $x \in \mathbf{R}^S$. Let Σ_x be the set of all subsequences $\{k_l\}_{l=1}^M \subseteq \{1, \ldots, K\}$ for which $k_1 = 1$ and define $x^{k_{M+1}} = x$. By (4), if $\{k_l\}_{l=1}^M \in \Sigma_x$ has a cycle (meaning that $k_l = k_{l'}$ for $l, l' \in \{1, \ldots, M\}$ with $l \neq l'$), then there is a shorter sequence $\{k_j\}_{j=1}^{M'} \in \Sigma_x$ with

$$\sum_{j=1}^{M'} \frac{p^{k_j}}{\|p^{k_j}\|_1} \cdot (x^{k_{j+1}} - x^{k_j}) \leq \sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}).$$

Therefore, $u(x) = \inf\{\sum_{l=1}^M \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}) : \{k_l\}_{l=1}^M \in \Sigma_x\}$ is well defined, as the infimum can be taken over a finite set.

That $u : \mathbf{R}^S \to \mathbf{R}$ defined in this fashion is concave, strictly increasing and continuous is immediate. To see that it rationalizes the data, suppose that $p^k \cdot x^l \leq p^k \cdot x^k$. Then $\frac{p^k}{\|p^k\|_1} \cdot x^l \leq \frac{p^k}{\|p^k\|_1} \cdot x^k$. It is clear then by definition that $u(x^l) \leq u(x^k) + \frac{p^k}{\|p^k\|_1} \cdot (x^l - x^k) \leq u(x^k)$.

Finally, to show that $u(x) + (c, \ldots, c)) = u(x) + c$, note that for any p^k, we have $\frac{p^k}{\|p^k\|_1} \cdot (x + (c, \ldots, c)) = c + \frac{p^k}{\|p^k\|_1} \cdot x$. The result then follows by construction.

We end the proof by showing that (2) $\Longrightarrow$(3) Let $u : \mathbf{R}^S \to \mathbf{R}$ be as in the statement of (2). Define the concave conjugate of u by

$$\begin{aligned} f(\pi) &= \inf\{\pi \cdot x - u(x) : x \in \mathbf{R}^S\} \\ &= \inf\{\pi \cdot x + c\pi \cdot \mathbf{1} - u(x) - c : x \in \mathbf{R}^S, c \in \mathbf{R}\} \\ &= \inf\{\pi \cdot x - c(1 - \pi \cdot \mathbf{1}) - u(x) : x \in \mathbf{R}^S, c \in \mathbf{R}\}, \end{aligned}$$

where the second inequality uses that $u(x + (c, \ldots, c)) = u(x) + c$. Now note that $f(\pi) = -\infty$ if $(1 - \pi \cdot \mathbf{1}) \neq 0$. Note also that the monotonicity of u implies that $f(\pi) = -\infty$ if there is s such that $\pi_S < 0$. Hence the domain of f is a subset of $\Delta(S)$.

Now since u is continuous, we have that $u(x) = \inf_{\pi \in \Delta(S)} \pi \cdot x - f(p)$. Since u rationalizes the dataset, the dataset is variational rationalizable.

7.2. **Proof of Theorem 3.** It is obvious that (3) $\Longrightarrow$(2) and that (2) $\Longrightarrow$(1). Hence, to show the theorem, it suffices to show that (4) implies (3) and that (1) implies (4).

For a dataset D, let $\pi^k = \frac{p^k}{\|p^k\|_1}$. It is easy to see that (4) $\Longrightarrow$(3). Let Π be the convex hull of $\{\pi^k : k = 1, \dots, K\}$. Then it is immediate that $U(x) = \min_{\pi \in \Pi} \pi \cdot x$ rationalizes D.

We prove that (1) $\Longrightarrow$(4). Suppose that D satisfies (1) but not (4). Then there is k and l for which $\pi^l \cdot x^k < \pi^k \cdot x^k$. Let $\succeq$ be a preference relation as stated in (1). Homotheticity implies that $\succeq$ rationalizes the data $\{(x^j, \pi^j) : j = 1, \dots, K\} \cup \{(\theta x^l, \pi^l)\}$ for any scalar $\theta > 0$. Now, $\pi^l \cdot x^k < \pi^k \cdot x^k$ implies that

$$x^k \cdot (\pi^l - \pi^k) + \theta x^l \cdot (\pi^k - \pi^l) < 0$$

for $\theta > 0$ small enough. But this is a violation of (4) in Theorem 1. A contradiction because $\succeq$ is translation invariant and locally nonsatiated.

7.3. **Proof of Theorem 4.** The idea in the proof is to solve the first-order conditions for the unknown terms. Consider first the case of CARA. Let $\pi \in \Delta(S)$ and $\alpha > 0$ rationalize D. Then we know that x^k maximizes $\sum_s \pi_s - \exp(-\alpha x_s)$ subject to $p^k \cdot x \leq p^k \cdot x^k$. By considering the Lagrangean and the first order conditions, we may conclude that for every $s, t \in S$ and every $k \in \{1, \dots, K\}$, we have

$$\frac{\pi_s \exp(-\alpha x^k_s)}{p^k_s} = \frac{\pi_t \exp(-\alpha x^k_t)}{p^k_t}.$$

Conclude that $\frac{p^k_s \pi_t}{p^k_t \pi_s} = \exp(-\alpha(x^k_s - x^k_t))$. By taking logs, the system becomes:

$$\ln(\pi_s) - \ln(\pi_t) + \alpha(x^k_t - x^k_s) = \ln(p_s) - \ln(p_t). \quad (3)$$

In the case of CRAA, the existence of a rationalizing π and parameter α imply a first-order condition of the form

$$\ln(\pi_s) - \ln(\pi_t) + \alpha \ln(x_t^k / x_s^k) = \ln(p_s) - \ln(p_t). \tag{4}$$

We can denote $\ln(\pi_s)$ by z_s in Equations (3) and (4). Thus we obtain that D is rationalizable if and only if there exist $z_s \in \mathbf{R}$ and $\alpha > 0$ such that the following equation is solved for all s, t, k with $s \neq t$:

$$z_s - z_t + \alpha(y_t^k - y_s^k) = \ln(p_s^k) - \ln(p_t^k),$$

where $y_t^k = x_t^k$ for CARA rationalizability, and $y_t^k = \ln x_t^k$ for CRRA rationalizability.

Now the necessity of the axioms is obvious. Let $k \neq k'$, then

$$\alpha(y_t^k - y_s^k) - \ln(p_s^k / p_t^k) = z_s - z_t = \alpha(y_t^{k'} - y_s^{k'}) - \ln(p_s^{k'} / p_t^{k'})$$

for any s and t. Thus

$$\alpha(y_t^k - y_s^k - y_t^{k'} + y_s^{k'}) = \ln(\frac{p_s^k}{p_t^k} \frac{p_t^{k'}}{p_s^{k'}}).$$

So (1) is satisfied for the case of CARA rationalizability, and (2) is satisfied for the case of CRRA rationalizability.

To prove sufficiency, let

$$d^p(s, t, k) = \log(p_s^k / p_t^k)$$
$$d^x(s, t, k) = y_s^k - y_t^k.$$

Let α^* be such that for all k, k', s, s' and t,

$$\alpha^*(y_t^k - y_s^k - y_t^{k'} + y_s^{k'}) = \ln(\frac{p_s^k}{p_t^k} \frac{p_t^{k'}}{p_s^{k'}}).$$

Then in particular, for all k, k', s, s' and t,

$$d^p(s, t, k) + \alpha^* d^x(s, t, k) + d^p(t, s, k') + \alpha^* d^x(t, s, k') = 0. \tag{5}$$

Note also that

$$\begin{aligned} &d^p(s, t, k) + d^p(t, s', k) + d^p(s', s, k) \\ &\qquad + \alpha^*(d^x(s, t, k) + d^x(t, s', k) + d^x(s', s, k)) = 0. \end{aligned} \tag{6}$$

Fix $s_0 \in S$ and let $z_{s_0} \in \mathbf{R}$ be arbitrary. For any $s \in S$, define z_s by

$$z_s = z_{s_0} + \alpha^* d^x(s_0, s, k) + d^p(s, s_0, k),$$

for some k. In fact, by Equation (5) this definition is independent of k because $d^p(s, s_0, k) + \alpha^* d^x(s, s_0, k) = d^p(s, s_0, k') + \alpha^* d^x(s, s_0, k')$.

Given this definition, note that

$$\begin{aligned} z_s - z_t &= \alpha^*(d^x(s_0, s, k) - d^x(s_0, t, k)) + d^p(s, s_0, k) - d^p(t, s_0, k) \\ &= \alpha^*(d^x(s_0, s, k) - d^x(s_0, t, k)) + d^p(s, s_0, k) - d^p(t, s_0, k) \\ &+ d^p(s, t, k) + d^p(t, s_0, k) + d^p(s_0, s, k) \\ &+ \alpha^*(d^x(s, t, k) + d^x(t, s_0, k) + d^x(s_0, s, k)) \\ &= d^p(s, t, k) + \alpha^* d^x(s, t, k). \end{aligned}$$

Where the second equality uses Equation (6).

Hence, with the constructed $(z_t)_{t \in S}$ we have

$$z_s - z_t + \alpha^*(y_t^k - y_s^k) = \log(p_s^k / p_t^k),$$

for all s, t, and k. The first-order conditions for rationalizability are therefore satisfied.

7.4. **Proof of Theorem 6.**

8. **Lemma.** *A dataset D is rationalizable if and only if there are v_s^k, λ^k, $s = 1, 2$, $k = 1, \ldots, K$, and $\bar{\pi}, \underline{\pi} \geq 0$ with $\bar{\pi} \geq \underline{\pi}$, such that:*

$$\begin{aligned} \pi v_1^k &= \lambda^k p_1^k \\ v_2^k &= \lambda^k p_2^k, \end{aligned}$$

for all $k = 1, \ldots, K$, where $\pi = \bar{\pi}$ when $x_1^k < x_2^k$ and $\pi = \underline{\pi}$ when $x_1^k > x_2^k$. The numbers also satisfy that $v_s^k \leq v_{s'}^{k'}$ when $x_s^k > x_{s'}^{k'}$.

Proof. To prove sufficiency, let v_s^k, λ^k, $s = 1, 2$, $k = 1, \ldots, K$, and $\bar{\pi}, \underline{\pi} \geq 0$ with $\bar{\pi} \geq \underline{\pi}$ be as in the statement of the lemma. Define $\bar{\mu}, \underline{\mu} \in \Delta(S)$ as follows. Let $\bar{\mu}_1 = \bar{\pi}/(1 + \bar{\pi})$ $\bar{\mu}_2 = 1/(1 + \bar{\pi})$ and $\underline{\mu}_1 = \underline{\pi}/(1 + \underline{\pi})$ $\underline{\mu}_2 = 1/(1 + \underline{\pi})$ Note that $\bar{\mu}_1 \geq \underline{\mu}_1$ and $\bar{\mu}_2 \leq \underline{\mu}_2$, as $\bar{\pi} \geq \underline{\pi}$. Define $\theta^k = \lambda^k/(1 + \bar{\pi})$ if $x_1^k < x_2^k$ and $\theta^k = \lambda^k/(1 + \underline{\pi})$ if

$x_1^k > x_2^k$. Then we have that $\mu_s v_1^k = \theta^k p_2^k$, with $\mu_s = \bar{\mu}_s$ when $x_1^k < x_2^k$; and $\mu_s = \underline{\mu}_s$ when $x_1^k > x_2^k$.

Given the numbers v_s^k it is now routine to define a correspondence ρ such that if $x \leq x'$, $y \in \rho(x)$ and $y' \in \rho(x')$ then $y \geq y' > 0$, and with $\rho(x_s^k) \ni v_s^k$. This gives a concave and increasing function u with $\partial u(c) = \rho(x)$. So $\frac{\theta^k p_s^k}{\mu_s} \in \partial u(x_s^k)$ for all (x, s), and hence the first order conditions are satisfied for maxmin rationalization.

We omit the proof of necessity. □

Let A be a matrix with $2K + 2 + K + 1$ columns, and $2K$ rows. The first $2K$ columns are labeled with a different pair (k, s). The next 2 columns are labeled $\underline{\pi}$ and $\underline{\pi}$. The next K columns are labeled with a $k \in \{1, \ldots, K\}$. Finally the last column is labeled p.

For each (k, s) with $k \in K_1$, A has a row with all zero entries with the following exception. It has a 1 in the column labeled (k, s), among the first group of $2K$ columns. It has a 1 in the column labeled k. In the column labeled p it has $-\log(p_s^k)$. Finally, if $s = 1$ then it has a 1 in the column labeled $\bar{\pi}$. For each (k, s) with $k \in K_2$, A has a row defined as above. The only difference is that when $s = 1$ then it has a 1 in the column labeled $\underline{\pi}$ instead of having one in $\bar{\pi}$.

Let B be a matrix with the same number of columns as A, and one row for each pair $(x_s^k, x_{s'}^{k'})$ with $x_s^k > x_{s'}^{k'}$. The columns of B are labeled like those of A. The row for $x_s^k > x_{s'}^{k'}$ has all zeroes except for a 1 in column (k', s') and a -1 in column (k, s). Finally, B has one more row. This row as a 1 in the column for $\bar{\pi}$ and a -1 in the column for $\underline{\pi}$, and it is labeled $s = 1$ for future reference.

Let E be a matrix with the same number of columns as A, labeled as above, and a single row. The row has all zeroes except for a 1 in column p.

By Lemma 8, there is no rationalizing maxmin preference if and only if there is no solution to the system of inequalities $A \cdot x = 0$, $B \cdot x \geq 0$ and $E \cdot x > 0$.

Suppose that all $\log(p_s^k)$ are rational numbers. We shall use the following version of the Theorem of the Alternative, which can be found as Theorem 1.6.1 in (Stoer and Witzgall, 1970).

9. **Lemma.** *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E belong to a commutative ordered field $\mathbf{F}$. Exactly one of the following alternatives is true.*

(1) *There is $u \in \mathbf{F}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, $E \cdot u \gg 0$.*
(2) *There is $\theta \in \mathbf{F}^r$, $\eta \in \mathbf{F}^l$, and $\pi \in \mathbf{F}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

Then the non-existence of a solution to the system $A \cdot x = 0$, $B \cdot x \geq 0$ and $E \cdot x > 0$ is equivalent to the existence of integer vectors η, θ and γ such that $\theta \geq 0$, $\gamma > 0$, and $\eta \cdot A + \theta \cdot B + \gamma E = 0$.

For a matrix D with $2K + 2 + K + 1$ columns, let D_1 denote the submatrix corresponding to the first $2K$ columns, D_2 correspond to the next 2, D_3 to the next K, and D_4 to the last column. Note that, by construction of A, B and E, $\eta \cdot A + \theta \cdot B + \gamma E = 0$ implies that $\eta \cdot A_1 + \theta \cdot B_1 = 0$, $\eta \cdot A_2 + \theta \cdot B_2 = 0$, $\eta \cdot A_3 = 0$, $\eta \cdot A_4 + \gamma = 0$. In fact, we can without loss assume that η, θ and γ take values of -1, 0 or 1. (This assumption is without loss because we can replace each row of matrices A, B and E with as many copies as indicated by the corresponding vector η, θ or γ.)

From the existence of such vectors it follows that we can obtain a sequence $(x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^n$ with $x_{s_i}^{k_i} > x_{s'_i}^{k'_i}$. The source of each pair $(x_{s_i}^{k_i}, x_{s'_i}^{k'_i})$ is that the column (k_i, s_i) of A is multiplied by $\eta_{(k_i,s_i)} > 0$ and the column (k'_i, s'_i) of A is multiplied by $\eta_{(k'_i,s'_i)} < 0$. The vector η must then have $\eta_{(k_i,s_i)} > 0$ and $\eta_{(k'_i,s'_i)} > 0$, with a -1 in the first column and a 1 in the second.

We shall prove that the sequence $(x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^n$ satisfies the properties stated in the axiom.

Firstly, $\eta \cdot A_3 = 0$ means that for each k, the number of is for which $k = k_i$ equals the number of is for which $k = k'_i$.

Secondly, $\eta \cdot A_2 + \theta \cdot B_2 = 0$ implies that:

$$\sum_{k \in K_1} \eta_{(k,1)} + \theta_{s=1} = 0$$
$$\sum_{k \in K_2} \eta_{(k,1)} - \theta_{s=1} = 0.$$

Note that $\sum_{k \in K_1} \eta_{(k,1)} = |\{i : k_i \in K_1, s = 1\}| - |\{i : k_i' \in K_1, s = 1\}|$, and similarly for $\sum_{k \in K_2} \eta_{(k,1)}$. Hence,

$$\begin{aligned} &|\{i : k_i \in K_1 \text{ and } s_i = 1\}| - |\{i : k_i' \in K_1 \text{ and } s_i = 1\}| \\ &\quad = |\{i : k_i' \in K_2 \text{ and } s_i = 1\}| - |\{i : k_i \in K_2 \text{ and } s_i = 1\}| \leq 0, \end{aligned}$$

as

$$\sum_{k \in K_1} \eta_{(k,1)} = - \sum_{k \in K_2} \eta_{(k,1)} = -\theta_{s=1} \leq 0.$$

Therefore the sequence $(x^{k_i}_{s_i}, x^{k_i'}_{s_i'})_{i=1}^n$ satisfies the second property stated in the axiom.

Finally,

$$\sum_{i=1}^n \log(p^{k_i'}_{s_i'}/p^{k_i}_{s_i}) = \sum_{(k,s))} \eta_{(k,s)} = -\gamma < 0,$$

as $\eta \cdot A + \gamma = 0$. Hence

$$\prod_{i=1}^n \frac{p^{k_i}_{s_i}}{p^{k_i'}_{s_i'}} > 1.$$

The above proof assumes that the log of prices is rational. The proof of the theorem follows along the same lines as Echenique and Saito (2013). Specifically, we have shown the following

10. **Lemma.** *If $\{(x^k, p^k)\}$ is a dataset satisfying SARMU, in which $\log p^k \in \mathbf{Q}$ for all k, then the dataset is maxmin rationalizable.*

One can then prove the following

11. **Lemma.** *If $\{(x^k, p^k)\}$ is a dataset that satisfies SARMU, and $\varepsilon > 0$ then there is a collection of prices $\{q^k)\}$ such that $\log q^k \in \mathbf{Q}_+$, $\|p^k - q^k\| < \varepsilon$, and the dataset $\{(x^k, q^k)\}$ satisfies SARMU.*

The proof of Lemma 11 is exactly as in (Echenique and Saito, 2013).

Lemma 10 establishes the result in datasets in which the log of prices is rational. Consider an arbitrary data set $\{(x^k, p^k)\}$, with prices that may not be rational.

Suppose towards a contradiction that the dataset satisfies SARMU, but that it is not maxmin rational. Specifically then, by Lemma 8, suppose that there is no solution to the system $A \cdot x = 0$, $B \cdot x \geq 0$ and $E \cdot x > 0$. Then by Lemma 9 there are real vectors η, θ and γ such that $\theta \geq 0$, $\gamma > 0$, and $\eta \cdot A + \theta \cdot B + \gamma E = 0$.

Let $(q^k)_{k=1}^K$ be vectors of prices such that the dataset $(x^k, q^k)_{k=1}^K$ satisfies SARMU and $\log q_s^k \in \mathbf{Q}$ for all k and s. (Such $(q^k)_{k=1}^K$ exists by Lemma 11.) Furthermore, the prices q^k can be chosen arbitrarily close to p^k. Construct matrices A', B', and E' from this dataset in the same way as A, B, and E above. Note that only the prices are different in $\{(x^k, q^k)\}$ compared to $\{(x^k, p^k)\}$. So $E' = E$, $B' = B$ and $A'_i = A_i$ for $i = 1, 2, 3$. Since only prices q^k are different in this dataset, only A'_4 may be different from A_4.

By Lemma 11, we can choose prices q^k such that $|\theta \cdot A'_4 - \theta \cdot A_4| < \gamma/2$. We have shown that $\theta \cdot A_4 = -\gamma$, so the choice of prices q^k guarantees that $\theta \cdot A'_4 < 0$. Let $\gamma' = -\theta \cdot A'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \gamma' E_i = 0$ for $i = 1, 2, 3$. And $B_4 = 0$ so

$$\theta \cdot A'_4 + \eta \cdot B'_4 + \gamma' E_4 = \theta \cdot A'_4 + \gamma' = 0.$$

We also have that $\eta \geq 0$ and $\gamma' > 0$. Therefore θ, η, and γ' exhibit a solution to the dual system for dataset $\{(x^k, q^k)\}$, a contradiction with Lemma 10.

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