Citizen-candidates, lobbies, and strategic campaigning

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March 2004

Abstract

We study a spatial model of political competition in which potential candidates need a fixed amount of money from lobbies to enter an election. We show that the set of pure strategy Nash equilibria in which lobbies finance candidates whose policies they prefer among the set of entrants coincides with the set of Nash equilibria with weakly less than two entering candidates. Fixing lobbies' preferences, if the total amount of money held by lobbies is finite, there exists some minimal distance between the two candidates' positions. This minimal distance is a bound for all such Nash equilibria and is independent of the distribution of voters' preferences.

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1 Introduction

This paper provides an analysis of the strategic incentives of lobbies and other agents who fund political campaigns. Our specific interest is in understanding the extent to which money is used to "manipulate" elections. A commonly held intuition is that third-party candidates often enter an election to "steal" votes from a candidate they do not like, thus biasing an election in their favor. We examine the extent to which this behavior is present.

To do so, we describe a multicandidate spatial model of campaign finance (see Downs, 1957). Voting is over policies. As in other models, financing affects the outcome of an election. Our innovation is that money is required for a candidate to *enter* an election. Other studies assume money buys uninformed voters, as in Baron (1994), or affects the accuracy of signals sent by candidates, as in Austen-Smith (1987). Other multicandidate models include Besley and Coate (1997, 1998, 2001), Feddersen (1992), Feddersen et al. (1990), Osborne (1995), Osborne and Slivinski (1996), and Palfrey (1984).

Our model is similar to the citizen-candidate models of Besley and Coate (1997, 1998, 2001) and Osborne and Slivinski (1996). In the citizen-candidate models, the strategic actors of the model are the citizens, who decide whether or not to enter an election. Citizens incur a cost upon entering an election. Voting over the entrants takes place at a second stage, and may be either strategic or non-strategic. A citizen is identified with some point in policy space, the point being interpreted as her ideal point. In these models, a citizen who enters an election must campaign on her ideal point. The idea is that she cannot credibly commit to implementing some other policy upon being elected. The models also assume citizens obtain some benefit to being elected, interpreted as a "spoils of office."

Our innovation is to provide a model in which agents can incur a cost to support a policy which is *not* their most preferred policy. We are interested in studying the phenomenon of "vote stealing." Osborne and Slivinski study the case in which three candidates enter an election and one of them is sure to lose. They observe that the loser enters only to steal votes from some other entrant–in other words, the loser is a "spoiler" candidate. As noted earlier, entrants in an election must campaign using their most preferred policy. Therefore, a spoiler steals votes from entrants whose policies are similar to her favorite policy. An immediate implication is that we should never observe an "extremist" who loses for sure. An extremist who steals votes from the next most extreme entrant only makes it less likely that his favorite policy (among the potentially winning policies) is selected. Taking the model at face value, this suggests that only relatively moderate candidates will ever enter an election they are certain to lose. Such a result seems to run counter to intuition, and recent real-life examples.

Key to the Osborne and Slivinski result is the ego rent. Without ego rents, a three-entrant equilibrium in Osborne and Slivinski's model does not exist. This result can be reframed in the following way. In a modified version of the Osborne and Slivinski model with no ego rents and for which citizen-candidates *are* allowed to campaign on any policy at all (incurring a cost), there cannot exist a three-entrant equilibrium in which each entrant campaigns on their ideal point. This fundamental observation foreshadows much of our analysis, and provides the basic intuition underlying our general results.

We submit that if candidates are allowed to campaign on any policy they like (subject to the implicit constraint that this policy will be implemented upon election), then a more detailed analysis of "spoiler" candidates is possible. To this end, imagine a citizen-candidate who faces costly entry to an election, but who need not campaign on her most preferred policy. We argue that this citizen-candidate should be interpreted as a "lobby," pledging money to some platform. We imagine that there is always some "agent" who can be used to campaign on this platform, and this agent is used as a candidate. In our election game, there is some fixed cost of entering the election that can be borne by several lobbies. This fixed cost can be viewed as an implicit payoff to the candidate, used in order to induce her to run for office. It can be assumed that the costs are paid to the candidate *after* she runs for office on a given platform. Implicit in this story is the idea that a potential candidate cares lexicographically about how much money she obtains, and then about the policy implemented. Lobbies can therefore use candidates as a commitment device. This is the primary distinction between lobbies as strategic actors in our model and citizen-candidates as strategic actors in the model of Osborne and Slivinski.

We study the standard single-dimensional spatial model of politics. We assume that lobbies have preferences over policies and money that are additively separable. A lobby's utility function over a policy x and money spent m can be written as $U(x, m) = u(x) - \psi(m)$, where u is concave.[1] While

[1] Our model formally allows for the case in which u is constant; but such a lobby never

most lobbies have an ideal policy, they need not pledge money to their ideal policy (or to any policy at all). We are interested in studying the conditions under which lobbies pledge money to policies they do not like. Our model is a one-shot game, in which lobbies simultaneously pledge money to different policies. There is a second (implicit) stage where the election is conducted among the entrants, this stage is entirely non-strategic (so that voters always vote for their most preferred entrant) and is therefore not part of the game.

Referring to a financed position as an "entrant," the question we then address is the following: "Which lobbies finance entrants whose policy they prefer to those of other entrants?" Lobbies need not finance any entrant at all; but among those who *do* finance an entrant, they must finance their most preferred entrant *if there are only two entrants in equilibrium*. The reasoning is straightforward; if there are two entrants and a lobby finances the entrant they do not like, then the lobby is made better off by withdrawing their funds from this entrant. Therefore, if lobbies finance entrants they do not like, there must be at least three entrants.

The main finding in this paper is that, in fact, the converse statement is true. If there are at least three entrants, then at least one lobby finances an entrant it does not like. Therefore, "truthful financing" obtains *if and only if* the number of entrants in the election is at most two.

Recall the Osborne and Slivinski result, that three entrant equilibria do not exist if there is no ego rent to holding office in their model. A straightforward corollary is that, under their preference and payoff specifications, in any three entrant equilibrium in our model, there must exist at least one lobby who does not finance their ideal point. This result is closely related to our general result stated in the previous paragraph. In addition to working with a much broader preference structure, our theorem also establishes a result for equilibria with more than three entrants (although Osborne and Slivinski also discuss the four-entrant case). But the more important contribution is the following. While Osborne and Slivinski's result shows that there exists a candidate who does not finance her *ideal point*, we show that there exists a lobby who does not finance her most preferred policy *among the set of entrants*. As Osborne and Slivinski's theorem essentially gives parametric requirements for the existence of a three-entrant equilibrium, it is not clear how to extend their theorem to establish this more general result.

plays an important role in equilibrium. Another interesting special case not usually covered in the literature is the case of single-plateaued and concave u.

The number of candidates financed in equilibrium can be observed without knowing the actual preferences of the lobbies. Therefore, we do not have to know the game that lobbies play to determine whether or not lobbies "finance truthfully." In fact, the proof of the main theorem shows that even more is true. Refer to the entrants whose positions lie furthest to the left and to the right as "extremists."[2] If there are more than two entrants, then there must be an extremist who is financed only by lobbies that prefer another entrant. In particular, an extremist entrant who loses for sure is of this nature. But it is also true that an extremist entrant who has a chance at winning but *whose position is not sufficiently extreme* is also of this nature. At most one extremist can be "sufficiently extreme," so that we can identify at least one spoiler. Again, all of these identifications can be made without knowing the underlying preferences of the lobbies.

We are also led to a study of "policy divergence." An implication of our model is that candidates' policies must be sufficiently "distinct." In fact, we show that there exists some distance such that any two candidate equilibrium features candidates whose policies are at least this distance apart. This distance depends on the preferences of lobbies, but is otherwise independent of the characteristics of the underlying electorate. This tells us that elections in which entrance is costly feature polarization of the two candidates. This type of result is common. However; we show that there is actually a uniform bound on the distance between the two positions across all two-candidate equilibria and all distributions of voters.

All of these results are, in some sense, vacuous if equilibria with more than two candidates do not exist. Thus, for any even number n, we provide a list of conditions which imply that there exists a Nash equilibrium with n entering candidates. The conditions are in no sense knife-edge.

Section 2 introduces the formal model. Section 3 provides our results. Section 4 concludes. A proof of the policy divergence result is included in an appendix.

2 The model

Let $X \subset \mathbb{R}$ be convex. The set X represents policy space. Let L be a finite set of agents. We will refer to elements of L as **lobbies**, although they may

[2] The terminology "extremist" is not meant to connote extreme in an absolute sense, only relative to the other entrants.

also be interpreted as citizens. Lobbies are the only strategic actors in our model. For all $l \in L$, let $\omega_l \geq 0$ be an endowment. Lobby l's strategy space is $S_l \equiv X \times [0, \omega_l]$, with generic element $s_l = (p_l, m_l)$. Let $S \equiv \prod_L S_l$. An element $s \in S$ will be referred to as a **strategy profile**.

The interpretation of a strategy s_l is that lobby l pledges an amount of money m_l to support a candidate whose most preferred policy is p_l. The strategy space reflects the features that no lobby can pledge more money than it has, and that a lobby can pledge money to only one policy.

Money spent on a policy can be interpreted as being given to a candidate who campaigns on that policy. Although the incentives of candidates are not explicitly modeled here, it is useful at this point to discuss the implicit incentives of the candidates. We may imagine a continuum of "potential candidates," (perhaps identical to the voters) with ideal points distributed over the policy space. One possibility is to assume that potential candidates behave naively, and simply prefer more money to less. This is along the lines of a sincere-voting assumption–potential candidates do not consider the fact that their entrance into an election may make them worse-off in the policy dimension. Alternatively, if money and policies are all that potential candidates care about, we must make the assumption that the potential candidates care about money first, and the implemented policy second. While such preferences are rightly regarded as venal, one may also imagine a "third-dimension" to preferences of potential candidates, whereby potential candidates prefer to make their platform publicly known, with the hope of gaining political exposure.[3]

To formalize ideas, let $F > 0$ be a fixed **entry fee**. For all policies $x \in X$ and all strategy profiles $s \in S$ let

$$M_x(s) \equiv \sum_{\{l : p_l = x\}} m_l$$

be the total amount of money pledged to position x under strategy profile s. If a candidate x receives at least F under strategy profile s, she becomes an "entrant" in the election. Thus, define the set of **entrants** under strategy profile s as

$$E(s) \equiv \{x \in X : M_x(s) \geq F\}.$$

Requiring some "fixed cost" to be incurred in order for a candidate to enter an election is a common modelling technique; see Besley and Coate (1997,

[3]This interpretation was suggested by an anonymous referee.

1998, 2001), Osborne and Slivinski (1996), and Weber (1998), for example.[4] However, in most previous works, the actual candidate incurs the cost. Here, the candidate may or may not be a strategic actor (depending on the interpretation of the model).

The election between the entrants occurs in a second stage, and is non-strategic. There is some distribution of voter types, where a **type** is a single-peaked binary relation over X, say R. Formally, a binary relation R over X is single-peaked if and only if there exists some $x^* \in X$ such that for all $y, z \in X$, if $x^* \leq y < z$, then yPz, and if $z < y \leq x^*$, then yPz. Let $\mathcal{R}$ be the set of all single-peaked binary relations over X. Let Σ be a σ-algebra on $\mathcal{R}$ that includes all sets of the form $\{R : xRy\}$ for all $x, y \in X$. Let μ be a countably additive, non-degenerate probability measure on Σ. No assumptions other than countable additivity are made on μ.

All voters vote for the policy which maximizes their preference (sincere voting), breaking ties by randomizing uniformly. The **vote share** of $x \in E(s)$ is defined by

$$\begin{aligned} V_x(s) \equiv\ & \mu(\{R : xPy\ \forall y \in E(s) \setminus \{x\}\}) \\ & + \mu(\{R : \exists y \in E(s) \setminus \{x\} \text{ such that } xIy, xPz\ \forall z \in E(s) \setminus \{x, y\}\})/2. \end{aligned}$$

The winner(s) of the election are the candidates with the highest vote share. If there are *no* entering candidates, the outcome of the election is some policy $x^{sq} \in X$ (call it the status quo). Formally, for a strategy profile s, let the **winners** be defined as

$$W(s) = \begin{cases} \arg\max_{x \in E(s)} V_x(s) & \text{if there exists } x \text{ such that } V_x(s) > 0 \\ x^{sq} & \text{otherwise} \end{cases}.$$

The election game results in a uniform probability measure over $W(s)$.

Lastly, we discuss the preferences of the lobbies over strategy profiles. For all $l \in L$, let $u_l : X \to \mathbb{R}$ be (weakly) concave.[5] For all $x \in X$, $u_l(x)$ represents the utility l receives when x is the policy of the winning candidate. Concavity of u_l is a strong, though common assumption in voting models with costs (Osborne and Slivinski (1996), for example, assume preferences

[4] In these papers, the cost is typically not interpreted as a monetary cost.

[5] Note that these functions need not have unique maximizers (thus, they may be single-plateaued).

are Euclidean–Feddersen (1992) considers a concave environment as well). Our results critically rely on the concavity of u_l. Throughout the paper, we will explain how our results change if we only assume that u_l is quasi-concave. Let $\psi_l : [0, \omega_l] \to \mathbb{R}_+$ be a strictly increasing cost function satisfying $\psi_l(0) = 0$. Each lobby l incurs a cost $\psi_l(m_l)$ of spending m_l.

For all $l \in L$, we define a utility function representing preferences, $U_l : S \to \mathbb{R}$, as

$$U_l(s) = \frac{\sum_{x \in W(s)} u_l(x)}{|W(s)|} - \psi_l(m_l).$$

Thus, U_l is the expected utility of a lobby l when the set of winners is $W(s)$, and when the amount of money spent by l is m_l. At this stage, it is important to discuss the interpretation of the assumption that for all $l \in L$, u_l is concave. Of course, the interpretation of this assumption is largely a function of the interpretation of the policy space X, but given that it is a convex subset of a Euclidean space, it makes sense to take averages of policies (and to speak of expectations of probability measures on X). In this sense, the concavity of u_l can be interpreted as risk-aversion over the policy dimension. For a fixed level of money donated to a candidate, a lobby prefers the expected policy of a lottery over policies to the lottery itself. Risk-aversion of lobbies is the key feature of lobby preferences driving our main result. However, not only is u_l concave, but the utility function as a whole is additively separable across policies and income. Separability implies the absence of wealth effects, so that it is without loss of generality to speak of policy preferences. The concavity of u_l also implies that lobbies are willing to pay more to be closer to a preferred policy (not necessarily at a decreasing rate–although u_l is concave, ψ_l is only assumed to be strictly increasing).

To recap, each lobby's strategy space is $S_l = X \times [0, \omega_l]$, and each agent's payoff function is given by $U_l : S \to \mathbb{R}$ as specified above. We will study the pure-strategy Nash equilibria of the **election game:** $\left(L, S, \{U_l\}_{l \in L}, \mu\right)$.

2.1 Interpretations of the model

As discussed in the introduction, the strategic agents in our model might be interpreted either as lobbies, or as citizen-candidates. The interpretation in terms of lobbies is more appropriate. If the strategic agents are interpreted as citizen-candidates, it is reasonable to suspect that voters have some information about the candidates' preferences. If a candidate is elected to office, it is also reasonable to assume (absent commitment devices) that the candidate implements her most preferred policy.

Voters should then vote according to the policy they expect to be implemented. This is why the models of both Osborne and Slivinski and Besley and Coate restrict candidates to announcing their preferred policy.

Our interpretation is that lobbies finance a candidate whose actual preferences coincide with the policy financed. In this context, it does not matter that a candidate does not have the same preferences as the lobbies that finance her. The elected candidate will be able to credibly implement the policy on which she campaigned as this will be her most preferred policy. In this sense, lobbies use the entrant as a commitment device. Again, implicit in this story is the idea that potential candidates do not behave as lobbies do. Either they lexicographically prefer money over the implemented policy, or they are happy to enter an election simply for the sake of having their viewpoint known to the voting public.

3 Results

3.1 A characterization of one-entrant equilibria

Our first theorem provides a simple set of necessary conditions that a strategy profile must satisfy if it is a Nash equilibrium of the election game with only one entrant. The conditions are also sufficient under the assumption that a lobby cannot finance a candidate on its own. The reason they are sufficient under this assumption is that in verifying that a strategy profile is a Nash equilibrium, we need not worry about those lobbies that do not finance a candidate. They cannot affect the set of entrants.

Condition *i*) states that the only candidate that receives money is the entrant, and the entrant receives just enough to enter the election.

Condition *ii*) states that anyone who finances the entrant strictly prefers the entrant's position to the status quo.

Condition *iii*) is a simple condition, placing a bound on the amount of money a lobby is willing to spend on a candidate.

Theorem 1 holds if the u_l are only required to be quasi-concave.

Theorem 1: Let s be a pure-strategy Nash equilibrium. Then if $|E(s)| = 1$, the following conditions are satisfied, where $E(s) = \{x\}$:

$i)$ $\sum_{\{l:p_l=x\}} m_l = F$, $m_l = 0$ if $p_l \neq x$,

$ii)$ $m_l > 0, p_l = x,$ implies $u_l(x) > u_l(x^{sq})$,

$iii)$ for all l, $\psi_l(m_l) \leq |u_l(x) - u_l(x^{sq})|$.

Moreover, if for all $l \in L$, $\omega_l < F$, then if a strategy profile s satisfies $|E(s)| = 1$ and the three preceding conditions, it is a Nash equilibrium with one entrant.

Proof: Let s be a pure-strategy Nash equilibrium such that $|E(s)| = 1$, where $E(s) = \{x\}$.

To verify $i)$, suppose by means of contradiction that it is false. Suppose that $\sum_{\{l:p_l=x\}} m_l > F$. (We know it cannot be less than F as x is an entrant). Let $\varepsilon > 0$ satisfy $\varepsilon < \sum_{\{l:p_l=x\}} m_l - F$. Let $l \in L$ satisfy $p_l = x$ and $m_l > 0$. Let $s'_l = (x_1, \max\{m_l - \varepsilon, 0\})$. Thus, $W(s'_l, s_{-l}) = \{x\}$, so that $U_l(s'_l, s_{-l}) > U_l(s)$, contradicting the fact that s is a Nash equilibrium. To verify the second part of $i)$, let $l \in L$ satisfy $m_l > 0$ and $p_l \neq x$. Let $s'_l = (p_l, 0)$. Then $W(s'_l, s_{-l}) = \{x\}$, so that $U_l(s'_l, s_{-l}) > U_l(s)$, contradicting the fact that s is a Nash equilibrium.

To verify $ii)$ and $iii)$, let $l \in L$ satisfy $m_l > 0$ and $p_l = x$. Let $s'_l = (x, 0)$. By $i)$, conclude $W(s'_l, s_{-l}) = \{x^{sq}\}$. As s is a Nash equilibrium, conclude

$$u_l(x) - \psi_l(m_l) \geq u_l(x^{sq}),$$

so that

$$\psi_l(m_l) \leq u_l(x) - u_l(x^{sq}).$$

As $\psi_l(m_l) > 0$, $u_l(x) > u_l(x^{sq})$. Thus,

$$\psi_l(m_l) \leq |u_l(x) - u_l(x^{sq})|.$$

If $m_l = 0$, the inequality is trivially satisfied.

It is trivial to verify that if the conditions are satisfied, nobody wishes to deviate. ■

3.2 Two-entrant equilibria

Our next theorem provides a set of necessary conditions that any Nash equilibrium of the election game with only two entrants must satisfy. As in the preceding theorem, the conditions are also sufficient if no lobby can finance a candidate on its own.

The necessary conditions we derive may be stated informally as follows.

Condition *i*) states that the entrants tie in the vote. This is not an unusual result in these types of models. It is an artifact of the fact that lobbies face no uncertainty as to the distribution of voters.

Condition *ii*) states that any entrant gets exactly enough to enter the election and no more; it also states that any lobby not financing an entrant finances no position at all. The first of these statements reflects the nature of the fact that more money does not benefit an entrant. The second statement follows as money offered to any position is not returned back to the lobby if the candidate does not enter the election.

Condition *iii*) has more content than the first two. We refer to it later as "truthful financing." It states that any lobby financing an entrant finances his most preferred entrant.

Condition *iv*) is a simple condition which effectively places an upper bound on the amount of money that lobbies are willing to spend, as a function of the entrants. Read another way, it implies that as the entrants' positions become less distinct, lobbies are willing to spend less money.

The first part of Theorem 2 holds if the u_l are only required to be quasi-concave.

The $E(s) = \{x_1, x_2\}$ is the set of entrants in the following Theorem. Conditions *ii*) and *iii*) refer to x_1; a symmetric statement holds for x_2.

Theorem 2 also presents a result which states that if each ψ_l, u_l are continuous and the aggregate endowment held by lobbies is finite, then there is a minimal degree of separation between the two entrants which is always present in a two entrant equilibrium. That is, there is some constant $k > 0$ such that, if (x_1, x_2) are the entrants, they are a distance of at least k apart. If such a theorem were not true, then although two entrants' policies could never be the same in equilibrium (indeed, this is a feature of the model), they could be "almost" the same. The last statement in the theorem holds even in the case that X is not compact (this is unusual for these types of results), mainly due to the fact that the u_l are required to be concave. If, in fact, X is compact, then each u_l need not be concave, but merely continuous.

Theorem 2: Let s be a pure-strategy Nash equilibrium. Then if $|E(s)| = 2$, the following conditions are satisfied (where $E(s) = \{x_1, x_2\}$):

i) x_1 and x_2 tie in the election *ii*) $\sum_{\{l:p_l=x_1\}} m_l = F$, $m_l = 0$ if $p_l \notin \{x_1, x_2\}$

iii) $m_l > 0, p_l = x_1,$ implies $u_l(x_1) > u_l(x_2)$

iv) for all l, $\psi_l(m_l) \le \frac{|u_l(x_1)-u_l(x_2)|}{2}$.

Moreover, if for all $l \in L$, $\omega_l < F$, if s is a strategy-profile satisfying $|E(s)| = 2$ and satisfying the preceding conditions, then s is a Nash equilibrium with two entrants.

Lastly, suppose that for all $l \in L$, ψ_l and u_l are continuous. Then there exists $k > 0$ such that for all distributions μ of voter types, if s is a pure-strategy Nash equilibrium of $(L, S, \{U_l\}, \mu)$ and $|E(s)| = 2$, where $E(s) = \{x_1, x_2\}$, then $|x_1 - x_2| \ge k$.

Proof: Let s be a pure-strategy Nash equilibrium. Suppose that $|E(s)| = 2$.

To verify *i*), suppose the statement is false. Without loss of generality, suppose $W(s) = \{x_2\}$. Let $l \in L$ satisfy $p_l = x_1, m_l > 0$. By deviating to $s'_l = (x_1, 0)$, we have $W(s'_l, s_{-l}) = \{x_2\}$, so that $U_l(s'_l, s_{-l}) > U_l(s)$, contradicting the fact that s is a Nash equilibrium.

To verify *ii*), suppose by means of contradiction that it is false. Suppose that $\sum_{\{l:p_l=x_1\}} m_l > F$. (We know it cannot be less than F as x_1 is an entrant). Let $\varepsilon > 0$ satisfy $\varepsilon < \sum_{\{l:p_l=x_1\}} m_l - F$. Let $l \in L$ satisfy $p_l = x_1$ and $m_l > 0$. Let $s'_l = (x_1, \max\{m_l - \varepsilon, 0\})$. Thus, $W(s'_l, s_{-l}) = W(s)$, so that $U_l(s'_l, s_{-l}) > U_l(s)$, contradicting the fact that s is a Nash equilibrium. Next, suppose by means of contradiction that there exists $l \in L$ such that $m_l > 0$ and $p_l \notin \{x_1, x_2\}$. Let $s'_l = (p_l, 0)$. Then $W(s'_l, s_{-l}) = W(s)$, so that $U_l(s'_l, s_{-l}) > U_l(s)$, contradicting the fact that s is a Nash equilibrium.

We verify *iii*) and *iv*). Let $l \in L$ satisfy $m_l > 0$ and $p_l = x_1$. Let $s'_l = (x_1, 0)$. By *ii*), $W(s'_l, s_{-l}) = \{x_2\}$. As s is a Nash equilibrium,

$$\frac{u_l(x_1) + u_l(x_2)}{2} - \psi_l(m_l) \ge u_l(x_2),$$

so that

$$\psi_l(m_l) \le \frac{u_l(x_1) - u_l(x_2)}{2}.$$

As $\psi_l(m_l) > 0$, conclude $u_l(x_1) > u_l(x_2)$. Thus,

$$\psi_l(m_l) \le \frac{|u_l(x_1) - u_l(x_2)|}{2}.$$

If $m_l = 0$, the inequality is trivially satisfied.

It is trivial to verify that if the conditions are satisfied, nobody wishes to deviate.

The proof of the last statement is more involved, and is relegated to an Appendix. ■

3.3 Truthful financing in equilibrium

The characterizations of one and two entrant equilibria seem to have little in common in a general sense. However, the truthful financing concept alluded to in part *iii*) of Theorem 2 is trivially satisfied for one entrant equilibria.

Let s be a strategy profile. Say that $l \in L$ finances truthfully under s if either $m_l = 0$, or $m_l > 0$ and for all $x \in E(s)$, $u_l(p_l) \geq u_l(x)$. Say a strategy profile s satisfies **truthful financing** if for all $l \in L$, l finances truthfully under s. Thus, an equilibrium satisfies truthful financing if each lobby finances their (weakly) most preferred entrant.

The preceding theorems establish that if s is a Nash equilibrium and $|E(s)| \leq 2$, then s satisfies truthful financing. Unfortunately, truthful financing does not extend to *all* pure-strategy Nash equilibria s. We provide a three-entrant example.

Example 1: Let μ be a distribution over preferences. Let s be a pure-strategy Nash equilibrium such that $|E(s)| = 3$. Suppose that $|W(s)| = 3$. Label the policies of the entrants $\{x_1, x_2, x_3\}$ (where $x_1 < x_2 < x_3$). It is clear to see that s *cannot* satisfy the truthful financing property. To see this, suppose without loss of generality that $x_2 \leq \frac{x_1+x_3}{2}$. Suppose a lobby l truthfully finances x_1, so that $u_l(x_1) \geq u_l(x_2) > u_l(x_3)$. Thus,

$$\frac{u_l(x_1) + u_l(x_2) + u_l(x_3)}{3} - \psi_l(m_l) \geq u_l(x_2).$$

The left hand side is the payoff to lobby l under strategy profile s; the right hand side is what lobby l would receive by taking back all of its money, m_l (because entrant 2 wins the election). The previous inequality implies

$$\frac{u_l(x_1) + u_l(x_3)}{2} > u_l(x_2).$$

However, by concavity of u_l, we know that

$$u_l\left(\frac{x_1 + x_3}{2}\right) \geq \frac{u_l(x_1) + u_l(x_3)}{2} > u_l(x_2).$$

Moreover, x_2 is a convex combination of $\frac{x_1+x_3}{2}$ and x_1 (as $x_1 < x_2 \leq \frac{x_1+x_3}{2}$), say, with weights λ and $1-\lambda$. That is, $x_2 = \lambda x_1 + (1-\lambda)\left(\frac{x_1+x_3}{2}\right)$. Concavity of u_l thus implies

$$u\left(\lambda x_1 + (1-\lambda)\left(\frac{x_1+x_3}{2}\right)\right) \geq \lambda u_l(x_1) + (1-\lambda) u_l\left(\frac{x_1+x_3}{2}\right) > u_l(x_2),$$

a contradiction. Therefore, the example establishes that any three winner equilibrium cannot satisfy truthful financing.

Before concluding the example, we point out that it is possible for such an equilibrium to exist (one with three entrants and three winners). Suppose, for example, that $X = [0,1]$ and μ is a nonatomic distribution over Euclidean ideal points. Suppose that this distribution has the following density function:

$$f(x) = \begin{cases} 1 \text{ for } 0 \leq x \leq 1/2 \\ 1/2 \text{ for } 1/2 < x \leq 5/6 \\ 2 \text{ for } 5/6 < x \leq 1 \end{cases}.$$

Let $x_1 = 0$, $x_2 = 2/3$, $x_3 = 1$. If $E(s) = \{x_1, x_2, x_3\}$, then $W(s) = \{x_1, x_2, x_3\}$. Suppose that $L = \{1,2,3,4,5,6\}$, and let $F = 2/9$. Suppose that for all $l \in L$, $\omega_l = 1/9$ and $\psi_l(x) = x$. For $l = 1,2,3,4$, $u_l(x) = -x$. For $l = 5,6$, $u_l(x) = -|x - 2/3|$. Fix $s_1, s_2 = (x_1, 1/9)$, $s_3, s_4 = (x_3, 1/9)$, and $s_5, s_6 = (x_2, 1/9)$. It is easy to verify that s is a Nash equilibrium, and that $E(s) = \{x_1, x_2, x_3\}$. Note the peculiar feature of this equilibrium, whereby lobbies who prefer x_1 to the remaining entrants finance both x_1 *and* x_3.

This example illustrates the idea that when there are more than two entrants in an equilibrium, it may be that some lobby finances an entrant who they do not view as their most preferred. Our main result states that this is a general principle of our model. The set of truthfully financed Nash equilibria are *characterized* as the set of Nash equilibria with weakly less than two entrants. There are *no* equilibria with more than two entrants in which some lobby does not "lie" about its' preferred entrant. In a later section, we will show that we can actually construct voting games with large numbers of entrants.

Proposition 3 of Osborne and Slivinski, which provides necessary and sufficient conditions for a strategy profile of their game to be a Nash equilibrium, foreshadows the preceding result. They establish that under the specifications of their model, if there are no ego-rents to holding office, then a three-entrant equilibrium cannot exist. Recall that they implicitly assume that all agents finance their ideal point. Our theorem contributes to their intuition by establishing that the result holds for general preferences and for general payment structures, and also holds for equilibria with more than three entrants. Moreover, Osborne and Slivinski's result can be interpreted as establishing that under their preference specifications, no three-entrant equilibrium exists in which each lobby finances *her ideal point*. Contrast this with our result, which states that there does not exist an equilibrium with three or more entrants where each lobby finances her most preferred policy *among the set of entrants*. This is a much stronger statement than the one that follows from Osborne and Slivinski, and indeed, cannot be investigated within their model. Moreover, it is not clear how to adapt their techniques to cover this case.

The intuition behind this result rests on the fact that for all $l \in L$, u_l is concave (and hence risk-averse). To understand why the theorem is true, consider some Nash equilibrium in which more than two entrants are financed. We can discuss the two entrants whose policies are the furthest to the right and left, respectively; call these entrants the "extremists." Suppose that at least one of the extremists does not get enough votes to have a chance at winning the election, say the rightmost extremist. Who would finance such an entrant? Anyone who finances the right extremist would realize that if he were not to do it, the right extremist will be forced to drop out of the election. When this extremist drops out of the election, the entrant whose position is immediately to the left will capture all of the extremist's votes. This will ensure that the entrant directly to the left wins. Thus, any lobby that finances the losing extremist prevents the entrant whose position is directly to the left of the right extremist's from winning for sure. Therefore, in equilibrium, the lobby finances the right extremist in order to give some entrant whose position is to the left of second rightmost entrant's position a chance at winning. Therefore, there is some entrant to the left that the lobby prefers to the right extremist. In fact, this result holds even if utility functions over policies are not concave, and are just quasi-concave.

The other case is one in which both extremists actually have a chance of winning. Consider an example in which there are three entrants. Suppose that all lobbies finance their most preferred entrant in equilibrium. Any lobby who finances the left extremist knows that by not doing so, the central entrant will win and the left (and right) extremists will lose. The lobby financing the left extremist therefore takes a chance that the right extremist wins in order to prevent the central entrant from winning for sure. Given that lobbies are risk-averse, this can only be the case if the central entrant's position is closer to the right extremist's position than to the left extremist's position. Of course, any lobby which finances the right extremist faces symmetric incentives, and will only finance the right extremist if the central entrant's position is closer to the left extremist's position than the right extremist's position. Obviously, the central entrant's position cannot be closer to both of the extremist's, so we obtain a contradiction. Hence, if a lobby finances an extremist whose position is not sufficiently "extreme," it does so in order to steal votes from the central entrant.

Theorem 3: A pure-strategy Nash equilibrium s satisfies truthful financing if and only if $|E(s)| \leq 2$.

Proof. First, we establish that if a pure-strategy Nash equilibrium s satisfies truthful financing, then $|E(s)| \leq 2$.

Let s be a pure-strategy Nash equilibrium that satisfies truthful financing. Suppose, by means of contradiction, that $|E(s)| > 2$. Label $E(s) = \{x_1, ..., x_m\}$, where $x_1 < x_2 < ... < x_m$ and $m > 2$. We claim that for all $x \in E(s)$, $M_x(s) = F$. To see why, suppose that there exists $x \in E(s)$ such that $M_x(s) > F$. There exists $l \in L$ such that $p_l = x$ and $m_l > 0$. Let $\varepsilon < \min\{m_l, M_x(s) - F\}$. Let $s_l' = (x, m_l - \varepsilon)$. Then $M_x(s_l', s_{-l}) > M_x(s) - \varepsilon > F$, so that $E(s_l', s_{-l}) = E(s)$. In particular, $W(s_l', s_{-l}) = W(s)$. Therefore, as ψ_l is strictly increasing, $U_l(s_l', s_{-l}) = \frac{\sum_{y \in W\left(s_l', s_{-l}\right)} u_l(y)}{\left|W\left(s_l', s_{-l}\right)\right|} - \psi_l(m_l - \varepsilon) > \frac{\sum_{y \in W(s)} u_l(y)}{|W(s)|} - \psi_l(m_l) = U_l(s)$, contradicting the fact that s is a Nash equilibrium.

Next, we claim that for all $x \in E(s)$, $V_x(s) > 0$. Suppose, by means of contradiction, that there exists $x \in E(s)$ such that $V_x(s) = 0$. Let $l \in L$ satisfy $p_l = x$ and $m_l > 0$. Let $s_l' = (x, 0)$. Then $E(s_l', s_{-l}) = E(s) \setminus \{x\}$. However, $V_x(s) = 0$, so that $W(s_l', s_{-l}) = W(s)$. Hence, $U_l(s_l', s_{-l}) = \frac{\sum_{y \in W\left(s_l', s_{-l}\right)} u_l(y)}{\left|W\left(s_l', s_{-l}\right)\right|} > \frac{\sum_{y \in W(s)} u_l(y)}{|W(s)|} - \psi_l(m_l) = U_l(s)$, contradicting the fact that s is a Nash equilibrium.

Consider the two following cases.

Case i) At most one of x_1 and x_m is an element of $W(s)$.

Suppose that $x_1 \notin W(s)$. Let $l \in L$ satisfy $p_l = x_1$ and $m_l > 0$. Let $s_l' = (x_1, 0)$. As s is a Nash equilibrium, $U_l(s) \geq U_l(s_l', s_{-l})$. As $M_{x_1}(s) = F$

and $m_l > 0$, $M_{x_1}(s'_l, s_{-l}) = F - m_l < F$, so that $x_1 \notin E(s'_l, s_{-l})$. By single-peakedness of voters' preferences, any voter who weakly prefers x_1 to x_2 must strictly prefer x_2 to every element of $E(s) \setminus \{x_1, x_2\}$. Therefore, $V_{x_2}(s'_l, s_{-l}) = V_{x_1}(s) + V_{x_2}(s) > V_{x_2}(s)$. Moreover, for all $x_k \in E(s) \setminus \{x_1, x_2\}$, $V_{x_k}(s'_l, s_{-l}) = V_{x_k}(s)$.

We claim that $W(s'_l, s_{-l}) \neq W(s)$. If the two sets were equal, then $U_l(s'_l, s_{-l}) = \frac{\sum_{y \in W(s'_l, s_{-l})} u_l(y)}{|W(s'_l, s_{-l})|} > \frac{\sum_{y \in W(s)} u_l(y)}{|W(s)|} - \psi_l(m_l) = U_l(s)$, so that s is not a Nash equilibrium.

By the preceding statement and the condition derived on the vote share, we conclude that one of the three following possibilities is true: *a*) $x_2 \in W(s)$, $W(s'_l, s_{-l}) = \{x_2\}$, *b*) $x_2 \notin W(s)$, and $W(s'_l, s_{-l}) = \{x_2\}$, or *c*) $x_2 \notin W(s)$ and $x_2 \in W(s'_l, s_{-l})$.

Case *a*): As s is a Nash equilibrium,

$$\frac{\sum_{y \in W(s)} u_l(y)}{|W(s)|} - \psi_l(m_l) = U_l(s) \geq U_l(s'_l, s_{-l}) = u_l(x_2),$$

and as $m_l > 0$,

$$\frac{\sum_{y \in W(s)} u_l(y)}{|W(s)|} > u_l(x_2).$$

Rearranging terms leads to

$$\frac{\sum_{y \in W(s) \setminus \{x_2\}} u_l(y)}{|W(s)| - 1} > u_l(x_2).$$

As $x_1 \notin W(s)$, if $y \in W(s)$, then $y > x_2$. Therefore, there exists $x_j > x_2$ such that $u_l(x_j) > u_l(x_2)$. As u_l is concave, $u_l(x_2) > u_l(x_1)$. This contradicts the truthful financing hypothesis.

Case *b*) similarly obtains:

$$\frac{\sum_{y \in W(s)} u_l(y)}{|W(s)|} > u_l(x_2).$$

Moreover, in this case, $y \in W(s)$ implies $y > x_2$. We reach the same conclusion as in case *a*).

Case *c*) similarly obtains:

$$\frac{\sum_{y \in W(s)} u_l(y)}{|W(s)|} > \frac{\sum_{y \in W(s) \cup \{x_2\}} u_l(y)}{|W(s)| + 1}.$$

In this case, rearranging terms obtains

$$\frac{\sum_{y\in W(s)} u_l(y)}{|W(s)|} > u_l(x_2).$$

Again, we reach the same conclusion.

The case in which $x_m \notin W(s)$ is symmetric. In any case, s does not satisfy truthful financing and we have obtained a contradiction.

Case $ii)$ x_1,$x_m \in W(s)$.

Let $l \in L$ satisfy $p_l = x_1$, $m_l > 0$. Let $s_l' = (x_1, 0)$. As $x_1 \in W(s)$, $V_{x_1}(s) \geq V_x(s)$ for all $x \in E(s)$. Then, as $M_{x_1}(s_l', s_{-l}) = F - m_l < F$, we obtain $E(s_l', s_{-l}) = E(s) \setminus \{x_1\}$. As all voters have single-peaked preferences, all voters who weakly prefer x_1 to the remaining elements of $E(s)$ strictly prefer x_2 to all elements of $E(s) \setminus \{x_1, x_2\}$. Therefore, $V_{x_2}(s_l', s_{-l}) = V_{x_1}(s) + V_{x_2}(s)$. As $V_{x_2}(s) > 0$, we establish that $V_{x_1}(s) + V_{x_2}(s) > V_x(s)$ for all $x \in E(s) \setminus \{x_1, x_2\}$. Hence, $W(s_l', s_{-l}) = \{x_2\}$. As s is a Nash equilibrium, calculation shows that

$$\frac{\sum_{y\in W(s)} u_l(y)}{|W(s)|} - \psi_l(m_l) = U_l(s) \geq U_l(s_l', s_{-l}) = u_l(x_2),$$

so that if $x_2 \in W(s)$,

$$\frac{\sum_{y\in W(s)\setminus\{x_2\}} u_l(y)}{|W(s)| - 1} > u_l(x_2).$$

and if $x_2 \notin W(s)$

$$\frac{\sum_{y\in W(s)} u_l(y)}{|W(s)|} > u_l(x_2).$$

As s satisfies truthful financing, $u_l(x_1) \geq \sup_{x\in E(s)} u_l(x)$. The preceding inequalities guarantee that u_l is nonconstant, and as u_l is concave, for all $x > x_2$, $u_l(x_2) > u_l(x)$.

Suppose that $x_2 \in W(s)$, so that $\frac{\sum_{y\in W(s)\setminus\{x_2\}} u_l(y)}{|W(s)|-1} > u_l(x_2)$. Then

$$\frac{\sum_{y\in W(S)\setminus\{x_2\}} u_l(y)}{|W(s)| - 1} = \frac{2}{|W(s)| - 1}\left(\frac{u_l(x_1) + u_l(x_m)}{2}\right) + \frac{|W(s)| - 3}{|W(s)| - 1}\left(\frac{\sum_{y\in W(s)\setminus\{x_1,x_m\}} u_l(y)}{|W(s)| - 3}\right).$$

As

$$\frac{\sum_{y\in W(s)\setminus\{x_1,x_m\}} u_l(y)}{|W(s)| - 3} \leq u_l(x_2),$$

we conclude that
$$\frac{u_l(x_1)+u_l(x_m)}{2} > u_l(x_2).$$

Suppose that $x_2 \notin W(s)$, so that $\frac{\sum_{y\in W(s)} u_l(y)}{|W(s)|} > u_l(x_2)$. Then

$$\frac{\sum_{y\in W(s)} u_l(y)}{|W(s)|} = \frac{2}{|W(s)|}\left(\frac{u_l(x_1)+u_l(x_m)}{2}\right)+\frac{|W(s)|-2}{|W(s)|}\left(\frac{\sum_{y\in W(s)\setminus\{x_1,x_m\}} u_l(y)}{|W(s)|-2}\right).$$

As
$$\frac{\sum_{y\in W(s)\setminus\{x_1,x_m\}} u_l(y)}{|W(s)|-2} \le u_l(x_2),$$
we conclude
$$\frac{u_l(x_1)+u_l(x_m)}{2} > u_l(x_2).$$

Thus, in either case, we conclude
$$\frac{u_l(x_1)+u_l(x_m)}{2} > u_l(x_2).$$

By the concavity of u_l, we establish that
$$u_l\left(\frac{x_1+x_m}{2}\right) \ge \frac{u_l(x_1)+u_l(x_m)}{2} > u_l(x_2).$$
As for all $x \ge x_2$, $u_l(x) \le u_l(x_2)$, we conclude that
$$\frac{x_1+x_m}{2} < x_2.$$

By considering some $l \in L$ such that $p_l = x_m$ and $m_l > 0$, we similarly obtain
$$\frac{x_1+x_m}{2} > x_{m-1}.$$
We thus conclude that
$$x_{m-1} < x_2,$$
a contradiction.

As either case leads to a contradiction, our supposition must be incorrect, so that if s satisfies truthful financing, then $|E(s)| \le 2$.

We now establish that if a pure-strategy Nash equilibrium s satisfies $|E(s)| \le 2$, then s satisfies truthful financing. Suppose that $|E(s)| \le 2$. By definition, if $|E(s)| = 1$, s satisfies truthful financing. Suppose $|E(s)| = 2$. Denote $E(s) = \{x_1, x_2\}$. We claim that $V_{x_1}(s) = V_{x_2}(s)$. Otherwise, without loss of generality, $V_{x_1}(s) > V_{x_2}(s)$, and $W(s) = \{x_1\}$. As $x_2 \in E(s)$, there exists $l \in L$ such that $p_l = x_2$ and $m_l > 0$. Further, $U_l(s) = u_l(x_1) - \psi_l(m_l)$. Let $s'_l = (x_2, 0)$. As ψ_l is strictly increasing, $U_l(s'_l, s_{-l}) = u_l(x_l) > u_l(x_1) - \psi_l(m_l) = U_l(s)$, contradicting the fact that s is a Nash equilibrium.

Therefore, $W(s) = \{x_1, x_2\}$. We claim that for all $x \in E(s)$, $M_x(s) = F$. As $x \in E(s)$, $M_x(s) \geq F$. Suppose, by means of contradiction, that $M_x(s) > F$. Let $l \in L$ satisfy $p_l = x$ and $m_l > 0$. Let $\varepsilon < \min\{m_l, M_x(s) - F\}$. Let $s'_l = (x, m_l - \varepsilon)$. Then $U_l(s'_l, s_{-l}) = \frac{u_l(x_1)+u_l(x_2)}{2} - \psi_l(m_l - \varepsilon) > \frac{u_l(x_1)+u_l(x_2)}{2} - \psi_l(m_l) = U_l(s)$, contradicting the fact that s is a Nash equilibrium.

Now, suppose that $p_l = x_1$ and $m_l > 0$. Let $s'_l = (x_1, 0)$. As s is a Nash equilibrium, $U_l(s) \geq U_l(s'_l, s_{-l})$. As $W(s) = \{x_1, x_2\}$ and $M_{x_1}(s) = M_{x_2}(s) = F$, $W(s'_l, s_{-l}) = \{x_2\}$. Therefore, $\frac{u_l(x_1)+u_l(x_2)}{2} - \psi_l(m_l) = U_l(s) \geq U_l(s'_l, s_{-l}) = u_l(x_2)$. In particular, $\frac{u_l(x_1)}{2} - \psi_l(m_l) \geq \frac{u_l(x_2)}{2}$. As $\psi_l(m_l) > 0$, $u_l(x_1) > u_l(x_2)$. A symmetric argument shows that if $p_l = x_2$ and $m_l > 0$, then $u_l(x_2) > u_l(x_1)$. Hence, s satisfies truthful financing. ■

A key feature of the theorem is that an outside observer need not observe the preferences of the lobbies in order to know whether an equilibrium satisfies truthful financing. The proof of the Theorem establishes a method of finding at least some lobbies that do not finance truthfully. Moreover, it is important to note that the theorem breaks down if the u_l functions are only quasi-concave, and not concave. Intuitively, a lobby with a quasi-concave u_l may very well prefer to face a lottery over two extremists than a centrist which is a sure-thing, no matter where the centrist's position lies relative to the two extremist's positions.

Returning to the proof of Theorem 3, we saw in the first part of step 1 that any lobby who finances an extremist who loses for sure does not finance truthfully. If both extremists have a chance of winning the election, we also saw that the extremists had to be "very" extreme in order for lobbies to be financing them truthfully. Formally, we show that if a lobby finances x_1 and also prefers x_1 to all other entrants, then $\frac{x_1+x_m}{2} < x_2$. In this case, x_1 is sufficiently "extreme" relative to the other entrants. Symmetrically, we show that if a lobby finances x_m and also prefers x_m to all other entrants, then $\frac{x_1+x_m}{2} > x_{m-1}$. In this case, x_m is sufficiently "extreme" relative to the other entrants. We thus know that if x_1 is not very extreme, so that $\frac{x_1+xm}{2} \leq x_2$, then any lobby who finances x_1 does not finance truthfully. Similarly, if x_m is not very extreme, so that $\frac{x_1+x_m}{2} \geq x_{m-1}$, then any lobby who finances x_m does not finance truthfully. The obvious interpretation here is that lobbies who finance an extremist who is not too extreme do so to steal votes from another candidate.

Thus, we have the following corollary.

Corollary 1: Let s be a Nash equilibrium of the election game. Then $|E(s)| < +\infty$. Let $E(s) = \{x_1, ..., x_m\}$, where $x_1 < x_2 < ... < x_m$.

i) Suppose that $x_1 \notin W(s)$. Let $l \in L$ satisfy $m_l > 0$ and $p_l = x_1$. Then l does not finance truthfully under s.

ii) Suppose that $x_m \notin W(s)$. Let $l \in L$ satisfy $m_l > 0$ and $p_l = x_m$. Then l does not finance truthfully under s.

iii) Suppose that $\frac{x_1+x_m}{2} \geq x_2$. Let $l \in L$ satisfy $m_l > 0$ and $p_l = x_1$. Then l does not finance truthfully under s.

iv) Suppose that $\frac{x_1+x_m}{2} \leq x_{m-1}$. Let $l \in L$ satisfy $m_l > 0$ and $p_l = x_m$. Then l does not finance truthfully under s.

In other words, we only need to observe the relative positions of the entrants to find some of the lobbies who do not fund truthfully. The result holds for any distribution of voter types and/or lobby preferences and endowments which fit into our specifications.

3.4 On the existence of Nash equilibria with large numbers of entrants

Our model has the feature that equilibria with more than two candidates are possible in many environments. The primary goal of our work is to provide qualitative conditions that allow us to observe when manipulation is occurring. Our main characterization of Nash equilibria which are being "manipulated" are in terms of the cardinality of the set of entrants. Obviously, then, it is worthwhile to understand if it is possible for there to exist equilibria which feature manipulation. Theorem 3 tells us that we only need to verify if there exist equilibria for which there are more than two entrants.

If we make the assumption that $\omega_l < F$ for all $l \in L$, then equilibria are always guaranteed to exist (namely, equilibria for which there are no entrants). Instead of trying to provide general characterizations for the n-entrant cases, we will discuss a result that gives us sufficient conditions for existence of *an* equilibrium with n entrants. When dealing with the n-agent case, there is no general structure that can be given to an equilibrium (other than that it violates truthful financing). Put simply: there are just too many possibilities for existence for us to provide a general existence theorem. In an intuitive sense; however, as long as there are enough lobbies and money, and the lobbies' preferences sufficiently diverse, it is very easy to construct equilibria with large numbers of candidates.

We show how to construct equilibria with large numbers of candidates. For large n, the equilibrium has a peculiar feature. The equilibrium will feature two candidates, each of which tie in the vote (but receive less than half of the vote). One candidate is on the right, the other on the left. We will construct our equilibrium so that these two candidates are centrally located. There will be $(n-2)$ candidates to the left and $(n-2)$ candidates to the right of the winning candidates, each of whom does not win. Not only do these entrants not win, they are financed by lobbies who do not prefer their policies among the set of entrants.

The hypotheses of the following Proposition can be weakened significantly. Let us give a brief intuition for the constructive proof. In the equilibrium constructed, there are always only two winners, and these are the centrally located entrants. All other entrants lose. Any entrant who is to the left of the central entrant is financed by lobbies who prefer entrants to the right, and any entrant who is to the right of the central entrant is financed by lobbies who prefer entrants to the left. The equilibrium is constructed so that, for example, if any losing leftist entrant exits the election, the entrant immediately to the right (who is also a leftist entrant) wins. This is why lobbies who prefer rightist entrants have the incentive to finance more and more extreme leftist entrants. Financing an extreme entrant prevents a more centrally located, yet still relatively extreme, entrant from winning. Let us remark that odd entrant examples can also be constructed (see, in particular Example 1).

Proposition 1: Suppose $X = [0, 1]$ and $n \geq 6$ and even. Suppose that $F \in \left(\frac{1}{2n}, \frac{3}{4n-2}\right)$ (for $n \geq 6$, this interval is always nonempty). Suppose that for all $l \in L$, $\omega_l < F$. Suppose that μ is a uniform distribution over Euclidean preferences. Thus, we may without loss of generality refer to the ideal points of voters as their preferences. Suppose further that all u_l are Euclidean and that for all l, $\psi_l(x) = x$. Lastly, suppose that there exist $n/2$ groups of lobbies $\{L_i\}_{i=1}^{n/2}$ such that for all $l \in \bigcup_{i=1}^{n/2} L_i$, the ideal point of l lies to the right of $1/2 + F$, for which $\sum_{l \in L_i} \omega_l \geq F$ and $n/2$ groups of lobbies $\{L'_i\}_{i=1}^{n/2}$ such that for all $l \in \bigcup_{i=1}^{n/2} L'_i$, the ideal point of l lies to the left of $1/2 - F$, for which $\sum_{l \in L'_i} \omega_l \geq F$. Then there exists a pure-strategy Nash equilibrium s for which $|E(s)| = n$.

Proof. Let $\{x_i\}_{i=1}^{n/2}$ be an increasing sequence for which $x_{n/2} = 1/2 - F$ and for which $x_i - x_{i-1}$ is strictly increasing in i. Moreover, require that $2x_1 < x_3 - x_2$, and that $(4/3)F < x_i - x_{i-1} < 2F$ for all $i = 1, ..., n/2$. (The inequalities $(4n-2)F < 3$ and $2nF > 1$ ensure that this is possible). For $i = n/2+1, ..., n$, define $x_i = 1 - x_{n-i+1}$.

For lobbies $l \in L_i$ for $i < n/2$, define $p_l = x_i$ and $m_l > 0$ so that $\sum_{l \in L_i} m_l = F$. For $l \in L_{n/2}$, define $p_l = x_{n/2+1}$ and $m_l > 0$ so that $\sum_{l \in L_{n/2}} m_l = F$. For lobbies $l \in L'_i$ for $i < n/2$ define $p_l = x_{n-i+1}$ and $m_l > 0$ so that $\sum_{l \in L'_i} m_l = F$. For $l \in L'_{n/2}$, define $p_l = x_{n/2}$ and $m_l > 0$. For all remaining lobbies, define p_l arbitrarily and define $m_l = 0$. Clearly, for this strategy profile s, $E(s) = \{x_i\}_{i=1}^{n}$, so that $|E(s)| = n$.

We claim that s is a Nash equilibrium. First, note that for $i = 2, ..., n/2 - 1$, $V_{x_i}(s) = \frac{x_i + x_{i+1}}{2} - \frac{x_i + x_{i-1}}{2} = \frac{x_{i+1} - x_{i-1}}{2}$. For $i = 1$, $V_{x_1}(s) = \frac{x_1 + x_2}{2}$. Thus, $V_{x_1}(s) < V_{x_2}(s)$. Moreover, for $i, i+1 \in \{2, ..., n/2\}$, $V_{x_{i+1}}(s) - V_{x_i}(s) = \frac{x_{i+2} - x_i}{2} - \frac{x_{i+1} - x_{i-1}}{2} = \frac{x_{i+2} - x_{i+1}}{2} - \frac{x_{i+1} - x_i}{2}$. Since $(x_{i+2} - x_{i+1}) > (x_{i+1} - x_i)$ (this is easily verified for $i = n/2 - 1$), $V_{x_{i+1}}(s) > V_{x_i}(s)$. By symmetry, we conclude that $V_{x_{n/2+1}}(s) = V_{x_{n/2}}(s) > V_{x_i}(s)$ for all $i \notin \{n/2, n/2+1\}$. Therefore, $W(s) = \left\{x_{n/2}, x_{n/2+1}\right\}$.

For all $l \notin \bigcup_{i=1}^{n/2} (L_i \cup L'_i)$, $W(s'_l, s_{-l}) = W(s)$, so that $U_l(s) \geq U_l(s'_l, s_{-l})$. Therefore, we only need to verify that the appropriate incentives are satisfied for $l \in \bigcup_{i=1}^{n/2} (L_i \cup L'_i)$. We verify for $l \in \bigcup_{i=1}^{n/2} L_i$; for $l \in \bigcup_{i=1}^{n/2} L'_i$, the analysis is symmetric. First, consider $l \in L_{n/2}$. As $m_l > 0$, by playing s'_l for which $m'_l = 0$ (an optimal deviation must be of this form), $W(s'_l, s_{-l}) \neq W(s)$. Indeed, $W(s'_l, s_{-l}) = \left\{x_{n/2}\right\}$. To see this, for all $i < n/2$, $V_{x_i}(s'_l, s_{-l}) = V_{x_i}(s)$.

For all $i > n/2+2$, $V_{x_i}(s_l', s_{-l}) = V_{x_i}(s)$. Moreover; for $i \in \{n/2, n/2+2\}$, $V_{x_i}(s_l', s_{-l}) \geq V_{x_i}(s)$. We show that $V_{x_{n/2}}(s_l', s_{-l}) > V_{x_{n/2+2}}(s_l', s_{-l})$. $V_{x_{n/2}}(s_l', s_{-l}) = \frac{x_{n/2+2} - x_{n/2-1}}{2}$ and $V_{x_{n/2+2}}(s_l', s_{-l}) = \frac{x_{n/2+3} - x_{n/2}}{2}$. Thus, we need to verify that $x_{n/2+2} - x_{n/2-1} > x_{n/2+3} - x_{n/2}$. By definition, $x_{n/2+2} = 1 - x_{n/2-1}$ and $x_{n/2+3} = 1 - x_{n/2-2}$. Therefore, the inequality reduces to $-x_{n/2-1} - x_{n/2-1} > -x_{n/2-2} - x_{n/2}$. Rewriting, we obtain $x_{n/2} - x_{n/2-1} > x_{n/2-1} - x_{n/2-2}$, which is true by hypothesis. Therefore, $W(s_l', s_{-l}) = \{x_{n/2}\}$. The ideal point for l lies to the right of $1/2 + F$, therefore, $\frac{u_l(x_{n/2}) + u_l(x_{n/2+1})}{2} - m_l \geq u_l(x_{n/2})$ (equivalently $\frac{u_l(x_{n/2+1}) - u_l(x_{n/2})}{2} \geq m_l$) is trivially verified, as $m_l < F$ and $\frac{u_l(x_{n/2+1}) - u_l(x_{n/2})}{2} = F$.

Now, consider $l \in L_i$ for $i < n/2$. Let s_l' be any strategy for which $m_l' = 0$ (an optimal deviation of l can only be of this form). We claim that $W(s_l', s_{-l}) = \{x_{i+1}\}$. To see this, for $i > 2$, as in the preceding paragraph, for all $j > i+1$, $V_{x_j}(s_l', s_{-l}) = V_{x_j}(s)$ and for all $j < i-1$, $V_{x_j}(s_l', s_{-l}) = V_{x_j}(s)$. Now, $V_{x_{i+1}}(s_l', s_{-l}) = \frac{x_{i+2} - x_{i-1}}{2}$ and $V_{x_{i-1}}(s_l', s_{-l}) = \frac{x_{i+1} - x_{i-2}}{2}$. Clearly, as in the preceding, it is the case that $V_{x_{i+1}}(s_l', s_{-l}) > V_{x_{i-1}}(s_l', s_{-l})$. We claim that $V_{x_{i+1}}(s_l', s_{-l}) > V_{x_{n/2+1}}(s_l', s_{-l})$. To see this, note that $V_{x_{n/2+1}}(s_l', s_{-l}) = \frac{x_{n/2+2} - x_{n/2}}{2}$ and $V_{x_{i+1}}(s_l', s_{-l}) = \frac{x_{i+2} - x_{i-1}}{2}$. As $x_{n/2+2} - x_{n/2} < 4F$ and $x_{i+2} - x_{i-1} > 4F$, it follows that $x_{i+2} - x_{i-1} > x_{n/2+2} - x_{n/2}$. Therefore, $W(s_l', s_{-l}) = \{x_{i+1}\}$ and hence $U_l(s) \geq U_l(s_l', s_{-l})$.

Next, we check that there is no incentive to deviate for $l \in L_1 \cup L_2$. For $l \in L_1$, it is clear that $W(s_l', s_{-l}) = \{x_2\}$; to see this, note that $V_{x_2}(s_l', s_{-l}) = \frac{x_2 + x_3}{2}$, and for all $n > 2$, $V_{x_n}(s_l', s_{-l}) = V_{x_n}(s)$. We therefore only need to verify that $x_2 + x_3 > x_{n/2+1} - x_{n/2-1}$. But again, $x_{n/2+1} = 1 - x_{n/2}$, so that this is equivalent to $x_2 + x_3 > 1 - x_{n/2} - x_{n/2-1}$. Equivalently, $x_2 + x_3 + x_{n/2-1} + x_{n/2} > 1$. To verify this inequality, as $x_2 > x_1 + (4/3)F$, $x_3 > x_1 + (8/3)F$, and $x_{n/2} = 1/2 - F$, we obtain

$$x_2 + x_3 + x_{n/2} + x_{n/2-1} > 2x_1 + (12/3)F + (1/2 - F) + x_{n/2-1}.$$

Moreover, $x_{n/2-1} > 1/2 - 3F$. Hence

$$2x_1 + (12/3)F + (1/2 - F) + x_{n/2-1} > 2x_1 + 4F + 1 - 4F > 1.$$

Therefore, we conclude that $W(s_l', s_{-l}) = \{x_2\}$, so that $U_l(s) \geq U_l(s_l', s_{-l})$.

Lastly, let us verify the case for which $l \in L_2$. Let s_l' be any strategy for which $m_l' = 0$ (an optimal deviation for l can only be of this form). We claim that $W(s_l', s_{-l}) = \{x_3\}$. To see this, for $i > 3$, as in the preceding paragraphs, $V_{x_i}(s_l', s_{-l}) = V_{x_i}(s)$. Moreover, $V_{x_1}(s_l', s_{-l}) = \frac{x_1+x_3}{2}$ and $V_{x_3}(s_l', s_{-l}) = \frac{x_4-x_1}{2}$. We want to verify that $x_4 - x_1 > x_1 + x_3$. But this is trivial; $x_4 - x_3 > x_3 - x_2 > 2x_1$ (as assumed). Therefore, the inequality holds. Lastly, we need to verify that $V_{x_3}(s_l', s_{-l}) > V_{x_{n/2+1}}(s_l', s_{-l})$, but this holds true as in the statement two paragraphs above. Therefore, $W(s_l', s_{-l}) = \{x_3\}$, and $U_l(s) \geq U_l(s_l', s_{-l})$. Therefore, s is a Nash equilibrium. ■

4 Discussion and Conclusion

Let us first discuss some extensions of the model. Firstly, we have maintained the assumption that all lobbies can finance at most one candidate. Of course, this assumption is unrealistic and empirically violated. However, versions of all of our results go through even upon dropping this assumption. If we define truthful financing in this extended model to mean that a lobby gives *some* money to a candidate whose policy is not her favorite among the set of entrants, then it is easy to see that the characterization result of Theorem 3 continues to hold.

Similarly, we have assumed that only one candidate can campaign on any given policy. In removing this assumption, there is the possibility that many losing candidates campaign on the same policy. Of course, any equilibrium in which each candidate campaigns on a *different* policy satisfies truthful financing if and only if there are weakly less than two entrants. It is trivial to see that in this extended model, if there are weakly less than two entering candidates, then the equilibrium satisfies truthful financing. The converse can also be seen to hold. Suppose that the truthful financing property holds for some Nash equilibrium. Then it is clear that no two candidates ever campaign on the same policy in this equilibrium. This is because if two candidates did campaign on the same policy, any lobby financing a candidate campaigning on that policy could increase the likelihood of the outcome of this policy simply by ceasing to finance the candidate, without changing the vote shares of any other candidate. And any lobby would want to do this if they were truthfully financing. Since no two candidates ever campaign on the same policy in such an equilibrium, all of the preceding analysis goes through unchanged and there must be weakly less than two entering candidates.

Moreover, we have said nothing about mixed strategy Nash equilibria in this paper. There are several reasons for this. Again, we wish to keep the analysis simple. However, mixed strategies complicate the analysis in a more significant way. The set of entrants in a mixed strategy equilibrium is random. This necessitates a new definition of truthful financing. There are several ways of discussing such a concept. One way is to consider only those policies that are contained in some set of entrants that has a positive probability of being realized. Another way is to define a policy to be an entrant if it is in the set of entrants with probability one. Moreover, one has to discuss the strategies of lobbies. A lobby's strategy could be said to satisfy truthful financing if it never assigns a positive probability of financing a "generalized entrant" that is not among her favorite "generalized entrants." There are certainly other natural definitions of when a lobby's strategy could be said to satisfy truthful financing as well. We do not know what happens when considering mixed strategies (depending on the definition, our results most likely will break down), but any such definition of truthful financing will certainly not be as transparent or appealing as the corresponding definition for the pure strategy case.

To conclude, our model is reminiscent of the citizen-candidate models of both Osborne and Slivinski, and Besley and Coate. Our model is closer to that of Osborne and Slivinski, who also assume a sincere-voting stage; whereas Besley and Coate assume that voters are themselves strategic actors. Both models assume that entrants must choose their favorite policy on which to campaign.

Proposition 3 of Osborne and Slivinski yields a special case of our Theorem 3 as a corollary when interpreted properly. The formal statement is as follows: Suppose s is an equilibrium of our model in an environment for which all lobbies have Euclidean preferences. Suppose that for all $l \in L$, if $m_l > 0$, then $m_l = F$. Suppose that $|E(s)| = 3$. Then there exists $l \in L$ such that $p_l \notin \arg\max_X u_l(x)$. Thus, this lobby does not finance her most preferred policy. Our result establishes more, so that $p_l \notin \arg\max_{W(s)} u_l(x)$. This result is impossible to derive from the Osborne and Slivinski result, even in the case of Euclidean preferences.

5 Appendix: Proof of policy divergence in Theorem 2

Let us recall the final statement of Theorem 2:

Suppose that for all $l \in L$, ψ_l and u_l are continuous. Then there exists $k > 0$ such that for all distributions μ of voter types, if s is a pure-strategy Nash equilibrium of $(L, S, \{U_l\}, \mu)$ and $|E(s)| = 2$, where $E(s) = \{x_1, x_2\}$, then $|x_1 - x_2| \geq k$.

First, we observe that in any two-entrant Nash equilibrium s, where $E(s) = \{x_1, x_2\}$, for all $l \in L$, $\psi_l(m_l) \leq \frac{|u_l(x_1) - u_l(x_2)|}{2}$. This statement is trivially true for all l for which $m_l = 0$. Suppose instead that $m_l > 0$. Clearly, $p_l \in \{x_1, x_2\}$. To see why, if $p_l \notin \{x_1, x_2\}$, then $E((p_l, 0), s_{-l}) = E(s)$, so that $W((p_l, 0), s_{-l}) = W(s) = \{x_1, x_2\}$. Hence, $U_l((p_l, 0), s_{-l}) = \frac{u_l(x_1) + u_l(x_2)}{2} > \frac{u_l(x_1) + u_l(x_2)}{2} - \psi_l(m_l) = U_l(s)$, contradicting the fact that s is a Nash equilibrium.

Suppose without loss of generality that $p_l = x_1$. As in the proof of Theorem 1, we can establish that $M_{x_1}(s) = F$. In particular, as s is a Nash equilibrium, and as $W(s) = \{x_1, x_2\}$, $\frac{u_l(x_1) + u_l(x_2)}{2} - \psi_l(m_l) = U_l(s) \geq U_l((p_l, 0), s_{-l}) = u_l(x_2)$. Conclude that $\psi_l(m_l) \leq \frac{u_l(x_1) - u_l(x_2)}{2} \leq \frac{|u_l(x_1) - u_l(x_2)|}{2}$. A similar statement obtains for l such that $p_l = x_2$, so that for all $l \in L$, $\psi_l(m_l) \leq \frac{|u_l(x_1) - u_l(x_2)|}{2}$.

Say $\mathbf{x} \in \mathbf{X}$ **supports** $\boldsymbol{\varepsilon} \in \mathbf{R}_+$ if there exists a vector $z \in \prod_l [0, \omega_l]$ such that

$$\sum_{\{l : u_l(x - \varepsilon) > u_l(x + \varepsilon)\}} z_l = F \text{ and } \sum_{\{l : u_l(x - \varepsilon) < u_l(x + \varepsilon)\}} z_l = F,$$

and for all $l \in L$,

$$\psi_l(z_l) \leq \frac{|u_l(x - \varepsilon) - u_l(x + \varepsilon)|}{2}.$$

Define the correspondence $g : X \to \mathbb{R}_+$ by

$$g(x) \equiv \{\varepsilon : x \text{ supports } \varepsilon\}.$$

Therefore, if there exists a Nash equilibrium s such that $E(s) = \{x - \varepsilon, x + \varepsilon\}$, then x supports ε if there exists a Nash equilibrium s such that $W(s) = \{x - \varepsilon, x + \varepsilon\}$. If we can show that there is a minimum ε which can be supported, independently of x, then the proof is finished.

We now prove a series of lemmas, leading to the proof of the theorem.

Lemma 1: The function

$$h_{-}(x,\varepsilon) \equiv \sum_{\{l:u_l(x-\varepsilon)>u_l(x+\varepsilon)\}} \min\left\{\omega_l, \psi_l^{-1}\left(\frac{u_l(x-\varepsilon)-u_l(x+\varepsilon)}{2}\right)\right\}$$

is increasing and continuous in x for fixed ε, where $\psi_l^{-1}(s) = \infty$ for any s not in the range of ψ_l.

Proof: To see this, simply note that as x increases, the set

$$\{l : u_l(x-\varepsilon) > u_l(x+\varepsilon)\}$$

increases with respect to inclusion, due to weak concavity of u_l for all l. Moreover, the quantity $u_l(x-\varepsilon) - u_l(x+\varepsilon)$ is increasing, again due to weak concavity. Therefore, as ψ_l is increasing (and hence ψ_l^{-1} increasing), we establish that $h_{-}(x,\varepsilon)$ is increasing. We see that $h_{-}(x,\varepsilon)$ is continuous in x as u_l and ψ_l^{-1} are each continuous.
■

Lemma 1 asserts that, for a fixed ε, the total money which can be raised in equilibrium for the leftmost candidate is increasing as x increases.

Lemma 2: $h_{-}(x,\varepsilon) \geq F$ if and only if there exists

$$z \in \prod_{\{l:u_l(x-\varepsilon)>u_l(x+\varepsilon)\}} [0,\omega_l]$$

such that $\sum_{\{l:u_l(x-\varepsilon)>u_l(x+\varepsilon)\}} z_l = F$ and $\psi_l(z_l) \leq \frac{u_l(x-\varepsilon)-u_l(x+\varepsilon)}{2}$.

Proof: Suppose $h_{-}(x,\varepsilon) \geq F$. Note that for each l such that $u_l(x-\varepsilon) > u_l(x+\varepsilon)$, $\min\left\{\omega_l, \psi_l^{-1}\left(\frac{u_l(x-\varepsilon)-u_l(x+\varepsilon)}{2}\right)\right\} \in [0,\omega_l]$. As

$$\sum_{\{l:u_l(x-\varepsilon)>u_l(x+\varepsilon)\}} \min\left\{\omega_l, \psi_l^{-1}\left(\frac{u_l(x-\varepsilon)-u_l(x+\varepsilon)}{2}\right)\right\} \geq F,$$

there exists some $0 < \lambda \leq 1$ such that

$$\sum_{\{l:u_l(x-\varepsilon)>u_l(x+\varepsilon)\}} \lambda \min\left\{\omega_l, \psi_l^{-1}\left(\frac{u_l(x-\varepsilon)-u_l(x+\varepsilon)}{2}\right)\right\} = F.$$

Let $z_l = \lambda \min\left\{\omega_l, \psi_l^{-1}\left(\frac{u_l(x-\varepsilon)-u_l(x+\varepsilon)}{2}\right)\right\}$. Then $z_l \in [0, \omega_l]$. Note that $z_l \leq \psi_l^{-1}\left(\frac{u_l(x-\varepsilon)-u_l(x+\varepsilon)}{2}\right)$, so that monotonicity of ψ_l implies $\psi_l(z_l) \leq \frac{u_l(x-\varepsilon)-u_l(x+\varepsilon)}{2}$.

Suppose now that the second condition is satisfied. Note that monotonicity of ψ_l implies that $z_l \leq \psi_l^{-1}\left(\frac{u_l(x-\varepsilon)-u_l(x+\varepsilon)}{2}\right)$. Moreover, $z_l \leq \omega_l$. So,

$$F = \sum_{\{l: u_l(x-\varepsilon) > u_l(x+\varepsilon)\}} z_l$$

$$\leq \sum_{\{l: u_l(x-\varepsilon) > u_l(x+\varepsilon)\}} \min\left\{\omega_l, \psi_l^{-1}\left(\frac{u_l(x-\varepsilon) - u_l(x+\varepsilon)}{2}\right)\right\},$$

which is simply $h_-(x, \varepsilon)$. Therefore, the result is proved.

■

Lemma 2 shows us that the function h_- can be used to characterize conditions under which a candidate can receive enough money in equilibrium to enter an election. This allows us to work only with the function h_-, and its counterpart for the rightmost candidate, h_+, defined below.

The function

$$h_+(x, \varepsilon) \equiv \sum_{\{l: u_l(x+\varepsilon) > u_l(x-\varepsilon)\}} \min\left\{\omega_l, \psi_l^{-1}\left(\frac{u_l(x+\varepsilon) - u_l(x-\varepsilon)}{2}\right)\right\}$$

can be shown to be decreasing and continuous, by a symmetric argument. The first lemma implies that the correspondences g_-, g_+, defined by

$$\begin{aligned} g_-(x) &\equiv \{\varepsilon : h_-(x, \varepsilon) \geq F\} \\ g_+(x) &\equiv \{\varepsilon : h_+(x, \varepsilon) \geq F\} \end{aligned}$$

are monotonic, in that $x < x'$ implies $g_-(x) \subset g_-(x')$, and $g_+(x') \subset g_+(x)$. Lastly, for all x, $g(x) = g_-(x) \cap g_+(x)$. This follows from the second lemma. Thus, g is the intersection of two monotonic correspondences.

Lemma 3: For all x, $g(x), g_+(x), g_-(x)$ are closed (possibly empty) sets not containing zero.

Proof: We show g is closed; the cases for g_-, g_+ follow similarly. Let $\{\varepsilon^n\} \subset g(x)$ be a convergent sequence with limit ε. Then, there exists a corresponding sequence $\{z^n\} \subset \prod_l [0, \omega_l]$ allowing each ε^n to be supported by x. Now, as each set $[0, \omega_l]$ is compact in $\mathbb{R}$, Tychonoff's theorem implies that the set $\prod_l [0, \omega_l]$ is compact in the product topology on $\mathbb{R}^L$. Hence, there exists a subsequence of z^n which converges to some point $z \in \prod_l [0, \omega_l]$. For this reason, we assume without loss of generality that $\{z^n\}$ converges to some $z \in \prod_l [0, \omega_l]$. Moreover, $\{z^n\}$ converges to z if and only if for all $i \in N$, $\{z_l^n\}$ converges to z_l. As ψ_l and u_l are continuous for all l, the inequality

$$\psi_l\left(z_l^n\right) \leq \frac{\left|u_l\left(x - \varepsilon^n\right) - u_l\left(x + \varepsilon^n\right)\right|}{2}$$

implies

$$\psi_l\left(z_l\right) \leq \frac{\left|u_l\left(x - \varepsilon\right) - u_l\left(x + \varepsilon\right)\right|}{2}.$$

Next, note that

$$\sum_{\{l: u_l(x-\varepsilon^n) > u_l(x+\varepsilon^n)\}} z_l^n \to \sum_{\{l: u_l(x-\varepsilon) > u_l(x+\varepsilon)\}} z_l.$$

To see this, define the sequence $\{z^{n*}\} \subset \Pi_l [0, \omega_l]$ as

$$z_l^{n*} \equiv \left\{ \begin{array}{c} z_l^n \text{ if } u_l\left(x - \varepsilon^n\right) > u_l\left(x + \varepsilon^n\right) \\ 0 \text{ otherwise} \end{array} \right. .$$

Note that

$$\sum_l z_l^{n*} = \sum_{\{l: u_l(x-\varepsilon^n) > u_l(x+\varepsilon^n)\}} z_l^n$$

and that z_l^{n*} converges to a vector z^* which is z on $\{l : u_l(x - \varepsilon) > u_l(x + \varepsilon)\}$ and zero otherwise. This follows from continuity of u; for all l such that $u_l(x - \varepsilon) > u_l(x + \varepsilon)$, there exists some m such that for all $n > m$, $u_l(x - \varepsilon^n) > u_l(x + \varepsilon^n)$. Moreover, if $u_l(x - \varepsilon) \leq u_l(x + \varepsilon)$, then if $u_l(x - \varepsilon^n) > u_l(x + \varepsilon^n)$, we know by $\psi_l(z_l^n) \leq \frac{|u_l(x-\varepsilon^n) - u_l(x+\varepsilon^n)|}{2}$ that $z_l^n \to 0$.

Now, recall that $z_l^{n*} \leq \omega_l$. As L is finite, conclude

$$\sum_l z_l^{n*} \to \sum_l z_l^*$$

so that

$$\sum_{\{l: u_l(x-\varepsilon) > u_l(x+\varepsilon)\}} z_l = F.$$

A similar argument holds for $\{l : u_l(x+\varepsilon) > u_l(x-\varepsilon)\}$. Therefore, z allows ε to be supported by x, so that $\varepsilon \in g(x)$.

It is clear that $0 \notin g(x)$. Otherwise, the z which allows 0 to be supported by x must satisfy $\psi_l(z_l) = 0$ for all l. But this would require that $z_l = 0$ for all l, so that the summation constraint is not met.

The case for g_+ and g_- follow in similar ways, using the characterization given by the second lemma.

■

Lemma 3 has shown us that for a fixed x, there is a minimal ε which can be supported in equilibrium. We now complete the proof of the theorem.

Let f be defined as:

$$f(x) \equiv \min_{\varepsilon \in g(x)} \varepsilon,$$

where we take $\min \emptyset = \infty$. Similarly, let

$$f_-(x) \equiv \min_{\varepsilon \in g_-(x)} \varepsilon$$

and

$$f_+(x) \equiv \min_{\varepsilon \in g_+(x)} \varepsilon.$$

Note that these functions are all well-defined, $f_-(x)$ is increasing, and $f_+(x)$ is decreasing. Moreover, $f(x) = \max\{f_-(x), f_+(x)\}$. Therefore, there are four possible cases.

Case i) For all $x \in X$, $f(x) = f_+(x)$. In this case, we know that $0 \leq f_-(x) \leq f(x)$. Therefore, let x^* be arbitrary. We know that $f_-(x^*) > 0$. If $x < x^*$, we have $f(x) \geq f(x^*) \geq f_-(x^*)$. If $x > x^*$, we have $f(x) \geq f_-(x) \geq f_-(x^*)$. Therefore, for all x, $f(x) \geq f_-(x^*) > 0$. In this case, set $k = 2f_-(x^*)$.

Case ii) For all $x \in X$, $f(x) = f_-(x)$. This case is symmetric to Case i).

Case iii) There exists x^* such that

$$f(x) = \begin{cases} f_+(x) & x < x^* \\ f_-(x) & x \geq x^* \end{cases}$$

Note that if $x < x^*$, then $f(x) = f_+(x) \geq f_+(x^*)$. If $x \geq x^*$, $f(x) = f_-(x) \geq f_-(x^*) \geq f_+(x^*)$. Therefore, for all x, $f(x) \geq f_+(x^*) > 0$. In this case, set $k = 2f_+(x^*)$.

Case iv) There exists x^* such that

$$f(x) = \begin{cases} f_+(x) & x \leq x^* \\ f_-(x) & x > x^* \end{cases}.$$

This case is symmetric to Case iii).

Finally, suppose (x_1, x_2) are the entrants in a Nash equilibrium. Suppose that $x_2 > x_1$, without loss of generality. Then it is clear that $\frac{x_1+x_2}{2}$ supports $\frac{x_2-x_1}{2}$; this follows by the characterization of two-entrant Nash equilibrium. Hence by the above lemmas, $\frac{x_1-x_2}{2} \in g\left(\frac{x_1+x_2}{2}\right)$, so that $\frac{x_1-x_2}{2} \geq f\left(\frac{x_1+x_2}{2}\right) \geq k/2$. We then have $x_1 - x_2 \geq k$, completing the theorem.■

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