Resource allocation with partial responsibilities for initial endowments

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Abstract

This paper studies a resource allocation problem in which each individual is responsible but in general only partially for his initial endowment. We consider pure-exchange economies with initial endowments but we do not assume the individual rationality axiom, taking that the society consists of citizens who cannot opt out from it. We characterize a class of allocation rules which are parametrized by income redistribution codes. In particular, we characterize a one-parameter family of income redistribution codes, in which one extreme corresponds to the case that everybody is 100% responsible for his initial endowment and the other extreme corresponds to the case that nobody is responsible for his endowment at all.

1 Introduction

This paper studies a resource allocation problem in which each individual is responsible but in general only partially for his initial endowment.

Existing normative and axiomatic studies on resource allocation assume either that everybody is 100% responsible for his initial endowment or that any such notion of individual responsibility or entitlement is irrelevant. The former line of thought typically assumes

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that nobody should get worse off than his initial endowment, which is the individual rationality axiom, and characterizes the Walras rule combined with other axioms (see for example Hurwicz [2], Gevers [1], Thomson [6], Nagahisa [4], Nagahisa and Suh [5]). The latter line assumes that only social aggregate of endowments should matter and imposes certain equity axioms, which typically characterizes the Walras rule starting from equal division (see for example Nagahisa and Suh [5], Thomson [7], Thomson and Zhou [8]).

We view that how much individuals are responsible for and entitled to their initial endowments is a *quantitative* question, and that we should be able to characterize a *class* of allocation rules which allows flexibility of the degrees of responsibility to impose, but is sufficiently fine in order to help such quantitative social decision.

We consider pure-exchange economies with initial endowments but we do not assume the individual rationality axiom, taking that the society consists of citizens who cannot opt out from it and the individual rationality axiom is taken to be a distributive criterion rather than a voluntary participation condition. This is nothing but the idea behind income redistribution.

Dropping the individual rationality axiom leaves it unspecified how much individuals are responsible for their initial endowments. In our argument it is rather *calibrated* as a part of pinning down the solution. It is not an obvious question if such calibration works, as it is not obvious how much one's endowment is valuable for the society, because in general individuals' preferences are diverse and complementarity and substitution between goods are complex. Therefore we focus on the cases in which we can unambiguously define how much an individual contributes to the society. In particular, we focus on the cases that all individuals have an identical and linear preference and hence any allocation is efficient and only the distributive properties of allocation are the issue. We impose an axiom that welfare level for each individual should be pinned down uniquely in such cases of purely redistribution problems.

Together with other axioms which are standard, we characterize the class of allocation rules which are parametrized by income distribution codes. In particular, we characterize a one-parameter family of income redistribution codes, in which one extreme corresponds to the case that each individual is 100% responsible for his initial endowment and the other extreme corresponds to the case that everybody is not responsible for this endowment at all.

2 Model and axioms

We consider pure exchange economies in which there are n individuals and l goods. Assume $n \geq 3$. The consumption space for each individual is $\mathbb{R}^l_+$, where consumptions are taken to be column vectors. Each individual generically denoted by $i = 1, \cdots, n$ has initial endowment $e_i \in \mathbb{R}^l_{++}$ which is taken to be variable. Let

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \in \mathbb{R}^{nl}_{++}$$

denote an entire profile of initial endowment vectors.

Let $\mathcal{R}$ denote the set of preferences over $\mathbb{R}^l_+$ which are complete, transitive, continuous, convex and strongly monotone.

Let $\mathcal{D} \subset \mathcal{R}^n$ be the domain of preferences to be considered. Given any profile $R = (R_1, \cdots, R_n) \in \mathcal{D}$, the object R_i denotes individual i's weak preference relation, P_i denotes the corresponding strict preference relation, and I_i denotes the corresponding indifference relation.

We assume that $\mathcal{D}$ contains the subdomain of identical linear preferences, denoted $\mathcal{D}_{IL}$, in which any $R = (R_1, \cdots, R_n) \in \mathcal{D}_{IL}$ satisfies

$$x_i R_i y_i \iff px_i = py_i$$

for all $i = 1, \cdots, n$ with respect to some common row vector $p \in \Delta^\circ$, where $\Delta = \left\{p \in \mathbb{R}^l_+ : \sum_{k=1}^l p_k = 1\right\}$ and $\Delta^\circ = \Delta \cap \mathbb{R}^l_{++}$.

The subdomain $\mathcal{D}_{IL}$ plays an important role in our argument. It is the domain in which any allocation is efficient and only distributive properties of allocations are the issue, and we propose some axioms for such class of *pure redistribution problems.*

We now define the correspondence of feasible allocations and the social choice correspondence.

Definition 1 The *feasibility correspondence* $F : \mathbb{R}^{nl}_{++} \twoheadrightarrow \mathbb{R}^{nl}_+$ is defined by

$$F(e) = \left\{ x \in \mathbb{R}^{nl}_+ : \sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i \right\}$$

for each $e \in \mathbb{R}^{nl}_{++}$.

Definition 2 *Social choice correspondence* is a correspondence $\varphi : \mathbb{R}^{nl}_{++} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{nl}_{+}$ such that $\varphi(e, R) \subset F(e)$ for all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$.

We consider the following axioms on the social choice correspondence. First one states that the recommended allocations should not be Pareto-dominated.

Efficiency: For all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$ and $x \in \varphi(e, R)$ there is no $x' \in F(e)$ such that $x'_i R_i x_i$ for all $i = 1, \cdots, n$ and $x'_i P_i x_i$ for at least one i.

Definition 3 The *Pareto correspondence* is a correspondence $P : \mathbb{R}^{nl}_{++} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{nl}_{++}$ such that $P(e, R)$ is the set of efficient allocations for each $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$.

Second axioms states that if there is an allocation which is Pareto-indifferent to the allocation already recommended then it should not be excluded.

Non-Discrimination: For all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$, $x \in \varphi(e, R)$ and $x' \in F(e)$, if $x'_i I_i x_i$ for all $i = 1, \cdots, n$, then $x' \in \varphi(e, R)$.

Dropping the individual rationality axiom leaves it unspecified how much individuals are responsible for their initial endowments. We argue that it is rather *calibrated* as a part of pinning down the solution. It is not an obvious question if such calibration works, as it is not obvious how much one's endowment is valuable for the society, because in general individuals' preferences are diverse and complementarity and substitution between goods are complex. Therefore we focus on the cases that all individuals have an identical and linear preference and hence any allocation is efficient and only the distributive properties of allocation are the issue. The axiom below states that welfare level for each individual should be pinned down uniquely in such pure redistribution problems.

Welfare Uniqueness under Identical Linear Preferences: For all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$, and $x, x' \in \varphi(e, R)$, $x'_i I_i x_i$ for all $i = 1, \cdots, n$.

This axiom is weaker than the standard individual rationality axiom, which states that nobody should get worse off than his initial endowment.

Individual Rationality: For all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$, and $x \in \varphi(e, R)$, $x_i R_i e_i$ for all $i = 1, \cdots, n$.

Lemma 1 Individual Rationality implies Welfare Uniqueness under Identical Linear Preferences.

Proof. For all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$, pick any $x \in \varphi(e, R)$. IR requires $x_i R_i e_i$ for all $i = 1, \cdots, n$. If there is some i with $x_i P_i e_i$, under the identical linear preference and feasibility constraint there is j with $e_j P_j x_j$, which violates IR. Therefore $x_i I_i e_i$ for all $i = 1, \cdots, n$, which implies welfare uniqueness. ■

Next two axioms are about informational efficiency and implementability which are standard in the literature. The Maskin monotonicity axiom is known to be necessary for implementability in Nash equilibria and also sufficient for it in the current setting, because the no veto power condition is vacuously met under $n \geq 3$ (Maskin [3]).

Maskin Monotonicity: For all $(e, R), (e, R') \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$ and $x \in \varphi(e, R)$, if

$$x_i R_i y_i \implies x_i R'_i y_i$$

for all $i = 1, \cdots, n$ and $y_i \in \left\{ z_i \in \mathbb{R}^l_+ : \exists z_{-i} \in \mathbb{R}^{(n-1)l}_+, \ (z_i, z_{-i}) \in F(e) \right\}$, then $x \in \varphi(e, R')$.

The following Gevers monotonicity axiom is weaker than Maskin Monotonicity (Gevers [1]).

Gevers Monotonicity: For all $(e, R), (e, R') \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$ and $x \in \varphi(e, R)$, if

$$x_i R_i y_i \implies x_i R'_i y_i$$

for all $i = 1, \cdots, n$ and $y_i \in \mathbb{R}^l_+$, then $x \in \varphi(e, R')$.

The above axioms characterize a class of market-based mechanisms which are indexed by income distribution codes.

Definition 4 An *income distribution code* is a function $t : \Delta^\circ \times \mathbb{R}^{nl}_{++} \to \mathbb{R}^n_+$ such that

$$\sum_{i=1}^{n} t_i(p, e) = p \sum_{i=1}^{n} e_i$$

hold for all $(p, e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$.

Here is the definition of the Walras rule with income distribution code.

Definition 5 *Walras rule with income distribution code* t is a correspondence $W_t : \mathbb{R}^{nl}_{++} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{nl}_{++}$ such that for all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$ an allocation $x \in \mathbb{R}^{nl}_{+}$ belongs to $W_t(e, R)$ if and only if $x \in F(e)$ and there exists $p \in \Delta^{\circ}$ such that for all $i = 1, \cdots, n$ it holds

$$px_i \leq t_i(p, e)$$

and it holds

$$px'_i \leq t_i(p, e) \implies x_i R_i x'_i$$

for all $x'_i \in \mathbb{R}^{l}_{+}$.

Likewise we can define the constrained version of the above, in which possible individual deviation is limited to socially feasible consumptions.

Definition 6 *Constrained Walras rule with income distribution code* t is a correspondence $CW_t : \mathbb{R}^{nl}_{++} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{nl}_{++}$ such that for all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$ an allocation $x \in \mathbb{R}^{nl}_{+}$ belongs to $CW_t(e, R)$ if and only if $x \in F(e)$ and there exists $p \in \Delta^{\circ}$ such that for all $i = 1, \cdots, n$ it holds

$$px_i \leq t_i(p, e)$$

and it holds

$$px'_i \leq t_i(p, e) \implies x_i R_i x'_i$$

for all $x'_i \in \{z_i \in \mathbb{R}^{l}_{+} : \exists z_{-i} \in \mathbb{R}^{(n-1)l}_{+},\ (z_i, z_{-i}) \in F(e)\}$.

The following lemma is straightforward.

Lemma 2 $W_t \subset CW_t \subset P$ for any income distribution code t.

The lemma below characterizes the Walras rule with income distribution code in the domain of identical linear preferences.

Lemma 3 If a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences and Non-Discrimination, then there is an income distribution code t such that $\varphi(e, R) = W_t(e, R)$ for all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$.

Proof. For $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$ with the identical linear preference being described by normal vector p, by Welfare Uniqueness under Identical Linear Preferences there exists a unique vector $t(p, e) \in \mathbb{R}^n$ such that $px_i = t_i(p, e)$ for all $x \in \varphi(e, R)$ and $i = 1, \cdots, n$, and by Efficiency $\sum_{i=1}^n t_i(p, e) = p \sum_{i=1}^n e_i$ is met, and thus $\varphi(e, R) \subset W_t(e, R)$. By Non-Discrimination we have $\varphi(e, R) \supset W_t(e, R)$. ■

Because $W_t(e, R) = CW_t(e, R)$ for all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$, we also have

Lemma 4 If a social choice correspondence φ satisfies Welfare Uniqueness under Identical Linear Preferences and Non-Discrimination, then there is an income distribution code t such that $\varphi(e, R) = CW_t(e, R)$ for all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$.

Here is our first main result.

Theorem 1 Assume $\mathcal{D} = \mathcal{R}^n$. If a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences, Non-Discrimination and Gevers Monotonicity then there is an income distribution code t such that $\varphi(e, R) \supset W_t(e, R)$ for all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$ and equality holds on $\mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$.

Proof. For any $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$, pick any $x \in W_t(e, R)$. Let p be a corresponding equilibrium price and let $R^* \in \mathcal{D}_{IL}$ be the profile of identical linear preferences represented by $u(z) = pz$ for $z \in \mathbb{R}^l_+$.

Then $x \in W_t(e, R^*)$ and by the previous lemma we have $x \in \varphi_t(e, R^*)$. By Monotonicity we obtain $x \in \varphi_t(e, R)$. ■

Replacing Gevers Monotonicity by Maskin Monotonicity characterizes the constrained Walras rule with income distribution code.

Theorem 2 Assume $\mathcal{D} = \mathcal{R}^n$. If a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences, Non-Discrimination and Maskin Monotonicity then there is an income distribution code t such that $\varphi(e, R) \supset CW_t(e, R)$ for all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$ and equality holds on $\mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$.

Proof. For any $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$, pick any $x \in CW_t(e, R)$. Let p be a corresponding equilibrium price and let $R^* \in \mathcal{D}_{IL}$ be the profile of identical linear preferences represented by $u(z) = pz$ for $z \in \mathbb{R}^l_+$.

Then $x \in CW_t(e, R^*)$ and by the previous lemma we have $x \in \varphi_t(e, R^*)$. By Maskin Monotonicity we obtain $x \in \varphi_t(e, R)$. ■

3 Domain of smooth or identical linear preferences

Let $\mathcal{S} \subset \mathcal{R}$ be the set of preferences which are differentiable on $\mathbb{R}^l_{++}$ and satisfies the boundary condition, i.e., $\{z' \in \mathbb{R}^l_+ : zIz'\} \subset \mathbb{R}^l_{++}$ for all $R \in \mathcal{S}$ and for all $z \in \mathbb{R}^l_{++}$.

For each individual i, given $R_i \in \mathcal{S}$ and its differentiable representation u_i, the marginal rate of substitution of good k for m at $x_i \in \mathbb{R}^l_{++}$ for individual i is given by

$$MRS_{km}(x_i, R_i) = \frac{\frac{\partial u_i(x_i)}{\partial x_{im}}}{\frac{\partial u_i(x_i)}{\partial x_{ik}}}$$

Here we consider the domain of smooth or identical linear preferences $\mathcal{D} = \mathcal{S} \cup \mathcal{D}_{IL}$, where $\mathcal{D}_{IL}$ is viewed as consisting or limit points of points in $\mathcal{S}$.

The following local independence axiom (Nagahisa [4]) is an informational efficiency condition which states that only marginal rates of substitution should matter.

Local Independence: For all $(e, R), (e, R') \in \mathbb{R}^{nl}_{++} \times (\mathcal{S}^n \cup \mathcal{D}_{IL})$ and $x \in \mathbb{R}^{nl}_{++}$, if $MRS_{km}(x_i, R_i) = MRS_{km}(x_i, R'_i)$ for all $i = 1, \cdots, n$ and $k, m = 1, \cdots, l$, then $x \in \varphi(e, R)$ if and only if $x \in \varphi(e, R')$.

Lemma 5 Under Efficiency, Local Independence implies Gevers Monotonicity on $\mathcal{D} = \mathcal{S}^n \cup \mathcal{D}_{IL}$.

Proof. Pick any $(e, R), (e, R') \in \mathbb{R}^{nl}_{++} \times (\mathcal{S}^n \cup \mathcal{D}_{IL})$ and $x \in \varphi(e, R)$. By Efficiency x must be an interior allocation. Suppose it holds that

$$x_i R_i y_i \implies x_i R'_i y_i$$

for all $i = 1, \cdots, n$ and $y_i \in \mathbb{R}^l_+$. Then, by the nature of smooth preferences it must be that $MRS_{km}(x_i, R_i) = MRS_{km}(x_i, R'_i)$ for all $i = 1, \cdots, n$ and $k, m = 1, \cdots, l$, otherwise the indifference surfaces cross. By Local Independence we have $x \in \varphi(e, R')$. ■

Together with other axioms the local independence axiom implies that the social choice correspondence must be a selection of the Walrasian correspondence with some income distribution code.

Theorem 3 Assume $\mathcal{D} = \mathcal{S}^n \cup \mathcal{D}_{IL}$. If a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences and Local Independence, then there is an income distribution code t such that $\varphi(e, R) \subset W_t(e, R)$ for all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}$ and equality holds on $\mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$.

Proof. For any $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{S}^n$, pick any $x \in \varphi(e, R)$. By Efficiency and the smoothness of preference x must be an interior allocation and there is a unique supporting vector p. Let $R^* \in \mathcal{D}_{IL}$ be the profile of identical linear preferences represented by $u(z) = pz$ for $z \in \mathbb{R}^l_+$.

By Local Independence we have $x \in \varphi(e, R^*)$. Since $\varphi(e, R^*) = W_t(e, R^*)$ as $R^* \in \mathcal{D}_{IL}$, we have $x \in W_t(e, R^*)$. By the property of W_t, we obtain $x \in W_t(e, R)$. ■

Because Local Independence implies Gevers Monotonicity on the domain of smooth or identical linear preferences we have a full characterization of the Walras rule with income distribution code.

Theorem 4 Assume $\mathcal{D} = \mathcal{S}^n \cup \mathcal{D}_{IL}$. Then a social choice correspondence φ satisfies Efficiency, Welfare Uniqueness under Identical Linear Preferences, Non-Discrimination and Local Independence if and only if there is an income distribution code t such that $\varphi = W_t$.

Proof. It is straightforward to see that W_t satisfies all the axioms. So we prove sufficiency of the axioms.

Because Local Independence implies Gevers Monotonicity, we can apply the same argument as in Theorem 1 and obtain $\varphi \supset W_t$.

On the other hand, $\varphi \subset W_t$ follows from Theorem 3. ■

4 Comparative properties

Now we investigate comparative properties of the allocation rule. One natural requirement will be that as one contributes more resources to the society he should be rewarded more. However, because of diversity of preference it is not obvious what we mean by contributing more. Moreover, it is not obvious even when all individuals have identical preferences, since having more resources may change how goods are substituted with each other. We view, however, that making more contribution is unambiguously defined when all individuals have identical and linear preferences, because having more more valuable resources can never change the way how goods are substituted with each other.

Restricted Endowment Monotonicity: For all $R \in \mathcal{D}_{IL}$ and $e, e' \in \mathbb{R}^{nl}_{++}$, if $e'_i R_i e_i$ for all $i = 1, \cdots, n$ then for all $x \in \varphi(e, R)$ and $x' \in \varphi(e', R)$ it holds $x'_i R_i x_i$ for all $i = 1, \cdots, n$.

Restricted Endowment Monotonicity characterizes the class of income distribution codes in which only the lists of individual incomes should matter.

Lemma 6 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$. Then φ satisfies Restricted Endowment Monotonicity additionally if and only if there exists a function $\tau : \Delta^\circ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that for each $i = 1, \cdots, n$, $\tau_i(p, w)$ is nondecreasing in w and $\sum_{i=1}^n \tau_i(p, w) = \sum_{i=1}^n w_i$ for all $(p, w) \in \Delta^\circ \times \mathbb{R}^n_+$, and it holds

$$t(p, e) = \tau(p, (pe_1, \cdots, pe_n)).$$

for all $(p, e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$.

Next axiom states that when all individuals have identical and linear preferences allocations are linear in initial endowment vectors. Consider that there are two different resource allocation problems with different initial endowment vectors and that the society is deciding allocation after consolidating the two problems. Then, when all individuals have identical and linear preferences the order of such consolidation should not matter.

Restricted Endowment Linearity: For all $R \in \mathcal{D}_{IL}$ and $e, e' \in \mathbb{R}^{nl}_{++}$, for all $\alpha, \beta \geq 0$, for all $x \in \varphi(e, R)$ and $x' \in \varphi(e', R)$, it holds $\alpha x + \beta x' \in \varphi(\alpha e + \beta e', R)$.

Restricted Endowment Linearity characterizes the class of income distribution codes which are linear in initial endowment vectors.

Lemma 7 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$. Then φ satisfies Restricted Endowment Linearity additionally if and only if there exists a function $T : \Delta^\circ \to \mathbb{R}^{n \times nl}_+$, which maps each $p \in \Delta^\circ$ into an $n \times nl$ matrix $T(p)$, such that it holds

$$t(p, e) = T(p)e$$

and

$$\sum_{i=1}^n T_{i,(j,k)}(p) = p_m, \quad j = 1, \cdots, n, \; k = 1, \cdots, l$$

for all $(p, e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$.

To explain the next axiom, let us introduce a mixture operation over the domain of identical linear preferences. Given $R, R' \in \mathcal{D}_{IL}$ and $\alpha \in [0, 1]$, with $R = (R_1, \cdots, R_n)$

satisfying

$$x_i R_i y_i \iff p x_i \geq p y_i$$

for all i with some common $p \in \Delta^\circ$, and $R' = (R'_1, \cdots, R'_n)$ satisfying

$$x_i R'_i y_i \iff p' x_i \geq p' y_i$$

for all i with some common $p' \in \Delta^\circ$, let $\alpha R \oplus (1-\alpha)R' \equiv (\alpha R_1 \oplus (1-\alpha)R'_1, \cdots, \alpha R_n \oplus (1-\alpha)R'_n) \in \mathcal{D}_{IL}$ be the profile of identical linear preferences given by

$$x_i(\alpha R_i \oplus (1-\alpha)R'_i)y_i \iff (\alpha p + (1-\alpha)p')x_i \geq (\alpha p + (1-\alpha)p')y_i$$

for all i.

To illustrate, imagine that two societies merge so that household i in one society forms a joint household with its counterpart household i in the latter, where the proportion of merging is the same across household identities. As all the households in both societies have linear preferences merging has no complementarity effect. As all the households in each society before merging have identical preferences it is natural that the merging with common proportion does not change welfare weights put over the households. Hence the allocation for the society consisting of the merged households is simply the corresponding mixture of the allocations given in the original societies before merging. The is what is said by the axiom below.

Restricted Preference Linearity: For all $R, R' \in \mathcal{D}_{IL}$ and $e \in \mathbb{R}^{nl}_{++}$, for all $\alpha \in [0,1]$, for all $x \in \varphi(e, R)$ and $x' \in \varphi(e, R')$, it holds $\alpha x + (1-\alpha)x' \in \varphi(e, \alpha R \oplus (1-\alpha)R')$.

To motivate further, imagine a situation that there are l states of the world and the society is deciding how to allocate state-contingent consumptions, in which with probability α the society consists of risk neutral households all with an identical belief p and with probability $1-\alpha$ the society consists of risk neutral households all with an identical belief p'. Then, from an ex-ante viewpoint the society is viewed as consisting of risk neutral households with an identical belief $\alpha p + (1-\alpha)p'$. Then the axioms states that the household should receive the ex-ante expected values of welfare.

Restricted Preference Linearity characterizes the class of income distribution codes which are linear in price vectors.

Lemma 8 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$. Then φ satisfies Restricted Preference Linearity additionally if and only if there exists a function $F : \mathbb{R}^{nl}_{+} \to \mathbb{R}^{nl}_{+}$ such that it holds

$$t_i(p, e) = \sum_{k=1}^{l} p_k F_{ik}(e)$$

for each $i = 1, \cdots, n$ and

$$\sum_{i=1}^{n} F_{ik}(e) = \sum_{i=1}^{n} e_{ik} \quad k = 1, \cdots, l$$

for all $(p, e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$.

The last axiom is a weaker version of the anonymity axiom, which will not need explanation.

Restricted Anonymity: For all $(e, R) \in \mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$ and any permutation $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$, $x \in \varphi(e, R)$ if and only if $x^\pi \in \varphi(e^\pi, R^\pi)$, where $z^\pi_i = z_{\pi^{-1}(i)}$ for each $i = 1, \cdots, n$ for any object z with n-entries.

The following lemma is immediate.

Lemma 9 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$. Then φ satisfies Anonymity additionally if and only if for all $(p, e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$ and any permutation $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$ it holds

$$t(p, e^\pi) = (t(p, e))^\pi$$

for all $(p, e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$.

The above four additional axioms characterize a one-parameter family of income distribution codes.

Theorem 5 Suppose $\varphi = W_t$ holds for some income distribution code t on $\mathbb{R}^{nl}_{++} \times \mathcal{D}_{IL}$. Then φ satisfies, Restricted Endowment Monotonicity, Restricted Endowment Linearity, Restricted Preference Linearity and Restricted Anonymity additionally if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$t_i(p, e) = \lambda p e_i + (1 - \lambda) p \overline{e}_{-i}$$

for all $(p, e) \in \Delta^\circ \times \mathbb{R}^{nl}_{++}$ and $i = 1, \cdots, n$, where $\overline{e}_{-i} = \frac{1}{n-1} \sum_{j \neq i} e_j$.

Proof. From Restricted Endowment Linearity and Restricted Preference Linearity, and Restricted Endowment Monotonicity, there is an $n \times n$ matrix T such that

$$t(p, e) = T \begin{pmatrix} pe_1 \\ \vdots \\ pe_n \end{pmatrix}$$

By Restricted Anonymity, we have $T_{ii} = T_{jj}$, $T_{ij} = T_{ji}$ and $T_{ik} = T_{il}$ for all i, j and $k, l \neq i$. Then T takes the form

$$T = \lambda I + (1 - \lambda)E,$$

where I is the $n \times n$ identity matrix and E is the $n \times n$ matrix with all diagonal entries being 0 and all off-diagonal entries being $\frac{1}{n-1}$. ■

Note that in the above class: (i) when $\lambda = 1$ it is the case of no redistribution, where $t_i(p, e) = pe_i$ for each i; (ii) when $\lambda = \frac{1}{n}$ it is the case of equal division of social income, where $t_i(p, e) = p\bar{e}$ for each i with $\bar{e} = \frac{1}{n}\sum_{j=1}^{n} e_j$, and (iii) when $\lambda = 0$ it is the case of receiving the others' average income, where $t_i(p, e) = p\bar{e}_{-i}$.

5 Conclusion

In this paper we have axiomatically studied the problem of resource allocation in which each individual is taken to be responsible but in general only partially for his initial endowment. In our argument how much individuals are responsible is rather calibrated as a part of pinning down the solution.

We have characterized a class of allocation rules which lie between the Walras rule taking initial endowments as they are and the Walras rule starting from equal division, which are parametrized by income redistribution codes.

Also, we have characterized a one-parameter family of income redistribution codes, in which one extreme corresponds to the case that each individual is 100% responsible for his initial endowment and the other extreme corresponds to the case that everybody is not responsible for this endowment at all.

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