Proper scoring rules for general decision models

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Abstract

If a decision maker whose behavior conforms to the max-min expected utility model (Gilboa and Schmeidler [15]) is faced with a scoring rule for a subjective expected utility decision maker, she will always announce a probability belonging to her set of priors; moreover, for any prior in the set, there is a scoring rule inducing the agent to announce that prior. We also show that on the domain of Choquet expected utility preferences (Schmeidler [27]) with risk neutral lottery evaluation and totally monotone capacities, proper scoring rules do not exist. This implies the non-existence of proper scoring rules for any larger class of preferences (CEU with convex capacities, multiple priors). Keywords: experimental procedures, scoring rule, subjective expected utility, implementation, multiple priors, Choquet expected utility, probability elicitation. JEL classification: D81, C49

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1. Introduction

A typical experimental economics procedure for eliciting subjective beliefs of agents is the "scoring rule."[1] A scoring rule is a menu of actions. A decision maker is asked to choose an element from this menu; optimizing behavior of the decision maker reveals her probability measure.

Consider an environment with an exogenous set of states of the world. In classical models of subjective expected utility, decision makers make choices between *acts*, or state-contingent outcomes. Savage [24] establishes that a decision maker whose behavior conforms with several intuitive axioms acts as if she behaves in an expected utility fashion. For such a decision maker, there exists a unique probability measure over the states of the world, and a utility function over money. The decision maker always makes choices over acts in order to maximize her expected utility. In theory, this unique probability measure can be recovered by observing all possible choices between pairs of acts. While Savage's theory requires an infinite set of states of the world, other theories, most notably that of Anscombe and Aumann [2], do not.

Suppose that the decision maker in question is risk-neutral, so that her utility index over monetary payoffs is affine (we will see that this is without loss of generality). It is not necessary to observe all possible choices between all pairs of acts in order to elicit the unique probability measure representing beliefs. It is enough to offer such a decision maker *one* choice over a *menu* of acts, an insight originally due to Brier [4]. Optimizing behavior of the decision maker reveals her probability measure. Such a menu of acts is referred to as a *scoring rule*. A scoring rule is *proper* if the unique optimizing choice is to reveal her probability measure. The theory of scoring rules can easily be viewed as a subset of the implementation literature (surveyed, for example, by Jackson [18])–a scoring rule is a single-agent mechanism whereby it is always a strictly dominant strategy for a decision maker to reveal her true preference.

The theory of scoring rules meshes well with the "as if" approach of classical decision theory. A decision maker is only required to make a choice from among a menu of *acts*; there need not be any mention of the word "probability" by whoever offers this menu to the decision maker.

Brier provides the first example of a proper scoring rule and McCarthy [19]

[1] When we use the term beliefs, or the belief of an event E, we mean the maximal monetary amount an agent would pay for a bet which returns one monetary unit if E obtains and zero otherwise.

(see also de Finetti [8] and Savage [25]) fully characterizes the proper scoring rules. The first to interpret scoring rules as incentive devices is Good [16]. Scoring rules are commonly used in the experimental economics literature to elicit probabilities, starting with the work of McKelvey and Page [20], as noted by Camerer [5] (p. 592-593) (who also discusses the general use of scoring rules in experimental economics). The typical protocol is as follows. Subjects are assumed to form probabilistic beliefs over some binary outcome. The experimenter offers the subject a menu of acts conforming to some scoring rule, typically the *quadratic* scoring rule. Subjects are informed that it is in their best interest to choose that act corresponding to their belief. Subjects then choose accordingly. This procedure is used throughout experimental economics; two prominent examples are McKelvey and Page [20] and Nyarko and Schotter [22].

The preceding protocol operates under the presumption that a given decision maker's behavior conforms to the subjective expected utility axioms. Of course, the subjective expected utility paradigm is not universal. Ellsberg [10] demonstrates this. The behavior of many decision makers does *not* conform to either the Savage or Anscombe and Aumann axioms. Informally speaking, there may be uncertainty about probabilities of certain events, referred to in the literature as "ambiguity."[2] Given that the experimental protocol *presumes* that subjects form probabilistic beliefs, it is important to understand how they behave if they do not. This brings us to the first main result of our note. A well-known model that accommodates Ellsberg-type behavior is the max-min expected utility model, axiomatized by Gilboa and Schmeidler [15]. This decision maker can be viewed as possessing a set of priors. She evaluates the utility of an act by taking the minimal expected utility of the act across all priors in her set. We uncover which probabilities such a subject might announce.

Our first main result shows that she will choose an act corresponding to some probability measure in her set of priors. We also conclude that for every such probability, there exists a proper scoring rule for which she chooses the act corresponding to the probability. This result allows us to conclude that decision makers facing "ambiguity" systematically overstate their beliefs when facing a traditional proper scoring rule. This result should serve as a caution to experimentalists to ensure that their subjects indeed possess probabilistic beliefs.

Ultimately, one would like to design a scoring rule allowing a decision maker to express beliefs reflecting subjectively ambiguous situations. We investigate

[2] General theories of ambiguity are found in the works of Epstein [11], Epstein and Zhang [12], and Ghirardato and Marinacci [13].

the possibility of this approach in a model which is much less general than max-min expected utility. A well-known model due to Schmeidler [27] features non-probabilistic beliefs. A decision maker whose behavior conforms to the Choquet expected utility model has a unique, possibly non-additive, set function representing beliefs. Our second result is that there exists no analogue of a proper scoring rule for this model. This demonstrates the impossibility of recovering beliefs by observing the choice from a *single menu*. When consequences are suitably interpreted, using a single menu is without loss of generality. A decision maker who is not subjective expected utility will "look ahead," reducing any sequence or collection of choices into a single decision by considering only the final outcome as one large act. Otherwise, this decision maker is susceptible to Dutch books (the idea is originally due to Ramsey [23] and de Finetti [7]; see also Savage [24] and Schick [26]).

The result is demonstrated on the smallest well-known extension of the subjective expected utility paradigm, giving a broad-ranging impossibility result for non-expected utility models. It establishes that the existence of proper scoring rules for subjective expected utility models is knife-edge.

Section 2 introduces the model. Section 3 discusses our primary results. Section 4 concludes. All proofs are in an Appendix.

2. The model

Let Ω be a finite set of states of the world. An **act** is a function $x:\Omega\rightarrow\mathbb{R}$. The set of acts is denoted $\mathcal{F}$. A **capacity** is a function $\nu:2^{\Omega}\rightarrow\mathbb{R}$ which is monotonic (i.e. for all $E,F\subset\Omega$, if $E\subset F$, then $\nu(E)\leq\nu(F)$), and is normalized so that $\nu(\varnothing)=0$, and $\nu(\Omega)=1$. A capacity is **totally monotone** if for all $\{E_1,...,E_n\}\subset 2^{\Omega}$,

$$\nu\left(\bigcup_{j=1}^{n}E_j\right)\geq\sum_{J\subset\{1,...,n\}}(-1)^{|J|+1}\,\nu\left(\bigcap_{j\in J}E_j\right).$$

Denote the set of totally monotone capacities on Ω by $\mathcal{TM}(\Omega)$ and denote the set of probability measures on Ω by $\Delta(\Omega)$.

All probability measures are totally monotone capacities and all totally monotone capacities are convex. A capacity is a probability measure if and only if the corresponding α assigns positive value only to singletons.

A risk-neutral Choquet expected utility maximizer evaluates acts $x : \Omega \to \mathbb{R}$ through the use of the Choquet integral [3]:

$$E_\nu [x] = \int_\Omega x(\omega) \, d\nu(\omega).$$

The Choquet integral is defined as

$$E_\nu [x] = \int_0^\infty \nu(\{\omega : f(\omega) \geq t\}) \, dt + \int_{-\infty}^0 [\nu(\{\omega : f(\omega) \geq t\}) - 1] \, dt.$$

Preferences conforming to the Choquet expected utility model were introduced and axiomatized by Schmeidler [27].[4]

Say a preference ordering is a **max-min expected utility** ordering if there exists a nonempty, closed, convex set $P \subset \Delta(\Omega)$ such that the decision maker evaluates acts x according to $\min_{p \in P} E_p [x]$. Max-min expected utility preferences were first axiomatized by Gilboa and Schmeidler [15].

Let $\mathcal{C}$ be some set of capacities. A **scoring rule on $\mathcal{C}$** is a function $f : \mathcal{C} \to \mathcal{F}$ for which for all $\nu, \nu' \in \mathcal{C}$,

$$E_\nu [f(\nu)] \geq E_\nu [f(\nu')].$$

Obviously, scoring rules exist; simply fix $x \in \mathcal{F}$, and let $f(\nu) \equiv x$ for all $\nu \in \mathcal{C}$. A scoring rule is **proper**[5] if for all $\nu, \nu' \in \mathcal{C}$ for which $\nu \neq \nu'$, $E_\nu [f(\nu)] > E_\nu [f(\nu')]$. A decision maker facing a proper scoring rule acts in her best interest (in an ex-ante sense) by choosing the act corresponding to her capacity.

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[3] Note that we require our decision-maker to be risk-neutral. This is also a feature of the pioneering work of Brier [4], McCarthy [19], and Savage [25]. However, the key feature of risk-neutrality is that the decision maker has a von Neumann-Morgenstern utility index which is linear. If one accepts the theory of Anscombe and Aumann [2], then decision makers need not be risk-neutral, and one can use lotteries as payoffs. Instead of monetary compensation, state-contingent payoffs would be in the probability of winning some alternative. This idea is first introduced in Allen [1]. The only difference is the requirement that probabilities must lie in the unit interval $[0, 1]$, whereas monetary payoffs could be potentially unbounded. However, any monetary proper scoring rule will also necessarily be bounded, at least in the subjective expected utility model.

[4] Other references include Nakamura [21] and Chew and Karni [9].

[5] Sometimes, such a scoring rule is referred to as **strictly proper**.

3. Results

The following result is due to McCarthy [19], Theorem 1.[6] In the theorem, g is a function of beliefs which returns the expected payoff to a decision maker when announcing optimally. The primary requirement is that this function is strictly convex.

Theorem 1 (McCarthy): The function $f : \Delta(\Omega) \to \mathcal{F}$ is a proper scoring rule on $\Delta(\Omega)$ if and only if there exists some strictly convex function $g : \Delta(\Omega) \to \mathbb{R}$ for which for all $\nu \in \Delta(\Omega)$, the function $h_\nu : \Delta(\Omega) \to \mathbb{R}$ defined by $h_\nu(\nu') \equiv E_{\nu'}[f(\nu)]$ is an element of the subdifferential of g at ν.[7]

The question addressed here is the following. Let f be a scoring rule on $\Delta(\Omega)$, and let $P \subset \Delta(\Omega)$ be closed and convex. For which p^* is it true that

$$\min_{p \in P} E_p[f(p^*)] \geq \min_{p \in P} E_p[f(p')]$$

for all $p' \in \Delta(\Omega)$? When a max-min expected utility decision maker is faced with a scoring rule on $\Delta(\Omega)$, which probability measure will she reveal? The question is important for experimental economics. For the max-min expected utility model, decision makers systematically overstate their "beliefs."

If the scoring rule f is not continuous, a solution to the optimization problem need not exist, as simple examples verify. To this end, we are concerned primarily with *continuous* scoring rules on $\Delta(\Omega)$; those are the scoring rules for which the corresponding function from Theorem 1 $g : \Delta(\Omega) \to \mathbb{R}$ is everywhere differentiable.

Theorem 2: Let f be a continuous proper scoring rule on $\Delta(\Omega)$ and let P be convex and compact. Then $\arg\max_{p^* \in \Delta(\Omega)} \min_{p \in P} E_p[f(p^*)]$ exists, is a singleton, and is equal to $\arg\min_{p^* \in P} \max_{p \in \Delta(\Omega)} E_{p^*}[f(p)]$. In particular, the unique solution to the optimization problem is an element of P. Furthermore, for all $p \in P$, there exists a continuous probabilistic scoring rule whose corresponding solution is p.

[6] An infinite-dimensional extension is provided by Hendrickson and Buehler [17]. While McCarthy's original motivation has little to do with belief elicitation, the formal results found there are frequently interpreted as statements on the possibility of belief elicitation through scoring rules (for such interpretations, see Savage [24] and Hendrickson and Buehler [17]).

[7] Formally speaking, $g \geq h_\nu$, and $g(\nu) = h_\nu(\nu)$.

We will show that there does not exist a proper scoring rule on $\mathcal{TM}(\Omega)$. The set $\mathcal{TM}(\Omega)$ can be identified with a set which is of exponentially higher dimension than $\Delta(\Omega)$. It is more difficult to use acts in $\mathcal{F}$ to distinguish between decision makers with differing capacities.

Theorem 3: If $|\Omega| > 1$, there does not exist a proper scoring rule on the domain $\mathcal{TM}(\Omega)$.

Corollary 1: For all $\mathcal{C}$ for which $\mathcal{TM}(\Omega) \subset \mathcal{C}$, there does not exist a proper scoring rule on $\mathcal{C}$.

The preceding corollary applies to the case of those individuals who evaluate acts with respect to convex capacities and to the general model of biseparable preferences of Ghirardato and Marinacci [14].

We may define a scoring rule in which decision makers announce sets of priors. Denote by $\mathcal{K}(\Delta(\Omega))$ the nonempty, compact, convex subsets of $\Delta(\Omega)$.

Corollary 2: There does not exist a function $f : \mathcal{K}(\Delta(\Omega)) \to \mathcal{F}$ for which for all $P, P' \in \mathcal{K}(\Delta(\Omega))$ with $P \neq P'$

$$\min_{p \in P} E_p[f(P)] > \min_{p \in P} E_p[f(P')].$$

4. Conclusion

A natural question is exactly what *can* be elicited from a decision maker facing a single menu of acts. What Theorem 2 (coupled with Theorem 1) establishes is that for a max-min expected utility decision maker with prior set P, we may elicit $\arg\min_{p \in P} g(p)$ for any strictly convex and smooth function $g : \Delta(\Omega) \to \mathbb{R}$. This is done by constructing a scoring rule on $\Delta(\Omega)$ which, for each $p \in \Delta(\Omega)$, returns an element of the subdifferential of g.

In fact, it is not difficult to see that for any convex (not necessarily strictly) and smooth function $g : \Delta(\Omega) \to \mathbb{R}$, one may construct a function $f : \Delta(\Omega) \to \mathbb{R}$ such that the probability maximizing $\min_{p' \in P} E_{p'}[f(p)]$ over $\Delta(\Omega)$ also minimizes g over P; again f is an element of the subdifferential. Hence we can, for example, elicit for a given vector λ the value of the support function of P in direction λ.

This observation demonstrates that, if independent decisions can be observed, and if it is known that a decision maker is Choquet expected utility with a convex capacity, one can elicit the capacity in at most $2^{\Omega} - 2$ steps. The fact that capacities can be elicited in a finite number of steps by asking a decision maker to reveal suitable "marginal willingnesses to pay" is known (see Chambers and Melkonyan [6]). The procedure we describe gives a choice-based procedure for such elicitation, instead of asking decision makers to report differentials of their utility function.

5. Appendix: Proofs

Proof of Theorem 1 (sketch): The existence of a strictly convex function $g : \Delta(\Omega) \to \mathbb{R}$ for which for all $\nu \in \Delta(\Omega)$, h_ν is an element of the subdifferential of g for all ν is equivalent to the function $g : \Delta(\Omega) \to \mathbb{R}$ being defined $g(\nu) = \sup_{\nu'} h_{\nu'}(\nu)$ and uniquely maximized (by strict convexity) for ν at $\nu' = \nu$ (as h_ν is an element of the subdifferential of g at ν). This is equivalent to $g(\nu) = h_\nu(\nu)$ and $g(\nu) > h_{\nu'}(\nu)$ for all $\nu' \neq \nu$. This latter condition is equivalent to $E_\nu[f(\nu)] > E_\nu[f(\nu')]$ for all pairs ν, ν' for which $\nu' \neq \nu$. This is equivalent to f being a proper scoring rule. ■

Proof of Theorem 2. As f is continuous, existence follows as $E_p[f(p^*)]$ is upper semicontinuous (as an infimum of continuous functions) as a function of p^*, and as $\Delta(\Omega)$ is compact. The remainder of the proof is an application of the minimax theorem. Define X to be the convex hull of $\{f(p)\}_{p\in\Delta(\Omega)}$. As f is continuous in p and as $\Delta(\Omega)$ is compact, the set $\{f(p)\}_{p\in\Delta(\Omega)}$ is compact, thus X is compact (as Ω is finite). Consider the function $G : P \times X \to \mathbb{R}$ defined by

$$G(p, x) = E_p[x].$$

This function is clearly bilinear. Moreover, by the Sion minimax Theorem (Berge [3], p. 210), there exist $p^* \in P$ and $x^* \in X$ such that for all $(p, x) \in P \times X$,

$$E_{p^*}[x] \leq E_{p^*}[x^*] \leq E_p[x^*].$$

We may therefore conclude

$$\min_{p\in P} \max_{x\in X} E_p[x] = \max_{x\in X} \min_{p\in P} E_p[x].$$

and is achieved at $p = p^*$, $x = x^*$. Moreover, for a given p, the unique maximizer of $E_p[x]$ over x is $f(p)$ (as f is a proper scoring rule and by definition of X), so that $x^* = f(p^*)$. Hence, we may conclude

$$\min_{p\in P} \max_{p'\in\Delta(\Omega)} E_p[f(p')] = \max_{p'\in\Delta(\Omega)} \min_{p\in P} E_p[f(p')]$$

and is achieved at $p = p^*$, $p' = p^*$. We claim that p^* is the unique element of $\arg\min_{p \in P} \max_{p' \in \Delta(\Omega)} E_p [f (p')]$; this follows trivially by the strict convexity of $g (p) = \max_{p' \in \Delta(\Omega)} E_p [f (p')]$ and the fact that P is convex and compact. Moreover, p^* is also the unique element of $\arg\max_{p' \in \Delta(\Omega)} \min_{p \in P} E_p [f (p')]$. To see this, let $p' \in \Delta (\Omega)$, $p' \neq p^*$. Then

$$\begin{aligned} & \min_{p \in P} E_p [f (p')] \\ \leq \; & E_{p^*} [f (p')] \\ < \; & E_{p^*} [f (p^*)] \\ = \; & \max_{p'' \in \Delta(\Omega)} \min_{p \in P} E_p [f (p'')] . \end{aligned}$$

Here, the first inequality follows as $p^* \in P$ and the second follows as f is a proper scoring rule on $\Delta (\Omega)$. Hence, $\max_{p' \in \Delta(\Omega)} \min_{p \in P} E_p [f (p')]$ is achieved (uniquely) at $p^* \in P$, which is the unique minimizer of the strictly convex function $g (p) = \max_{p' \in \Delta(\Omega)} E_p [f (p')]$ over P.

To see that for all $p^* \in P$, there exists a continuous probabilistic scoring rule for which $\arg\max_{p \in \Delta(\Omega)} \min_{p^* \in P} E_{p^*} [f (p)] = \{p^*\}$, simply let g be a strictly convex and smooth function $g : \Delta (\Omega) \to \mathbb{R}$ whose global minimum is achieved at p^* (for example, let $g (p) \equiv \|p - p^*\|^2$, where $\|\cdot\|$ denotes the Euclidean norm). Let $f : \Delta (\Omega) \to \mathbb{R}$ be the continuous probabilistic scoring rule which consists of the subdifferentials of g. Then $g (p') = \max_{p \in \Delta(\Omega)} E_{p'} [f (p)]$, from which we utilize our preceding result. ■

Proof of Theorem 3. Suppose, by means of contradiction, that there exists a proper scoring rule f on the domain $\mathcal{TM} (\Omega)$. For all $E \subset \Omega$, $E \neq \varnothing$, define $\nu_E (F) = \begin{cases} 1 \text{ if } E \subset F \\ 0 \text{ otherwise} \end{cases}$. A classical representation theorem, due to Dempster, Shafer, and Shapley (see Shapley [28], for example), states that ν is a totally monotone capacity if and only if for all $E \subset \Omega$, there exists $\alpha (\nu, E) \geq 0$ for which $\sum_{F \neq \varnothing} \alpha (\nu, F) = 1$ such that $\nu = \sum_F \alpha (\nu, F) \nu_F$. Moreover, it is also well-known that, in this case, the Choquet integral becomes

$$E_\nu [x] = \sum_{E \neq \varnothing} \min_{\omega \in E} \{x (\omega)\} \alpha (\nu, E) .$$

Without loss of generality, we may work with the Dempster, Shafer, and Shapley representation of totally monotone capacities; therefore, assume that f maps from $\Delta\left(2^{\Omega}\backslash\varnothing\right)$ into $\mathcal{F}$. If $\alpha,\beta\in\Delta\left(2^{\Omega}\backslash\varnothing\right)$ and $\alpha\neq\beta$, by properness of f,

$$\sum_{E\in 2^{\Omega}\backslash\varnothing}\min_{\omega\in E}\left\{f\left(\alpha\right)\left(\omega\right)\right\}\alpha\left(E\right)>\sum_{E\in 2^{\Omega}\backslash\varnothing}\min_{\omega\in E}\left\{f\left(\beta\right)\left(\omega\right)\right\}\alpha\left(E\right).$$

As the payoff from telling the truth is $U\left(\alpha\right)=\sum_{E\in 2^{\Omega}\backslash\varnothing}\min_{\omega\in E}\left\{f\left(\alpha\right)\left(\omega\right)\right\}\alpha\left(E\right)$, the function

$$U\left(\alpha\right)=\sup_{\beta\in\Delta\left(2^{\Omega}\backslash\varnothing\right)}\sum_{E\in 2^{\Omega}\backslash\varnothing}\min_{\omega\in E}\left\{f\left(\beta\right)\left(\omega\right)\right\}\alpha\left(E\right)$$

is a strictly convex supremum of linear functionals. Note that there is no problem in requiring U to be convex; the impossibility will result in requiring it to be *strictly* convex.

We will show that it is impossible for U to be strictly convex. Let $W\left(\Omega\right)$ be the set of weak orders on Ω.[8] We will say an act $x:\Omega\rightarrow\mathbb{R}$ **is monotonic with respect to** $\preceq\in W\left(\Omega\right)$ if $x\left(\omega\right)\geq x\left(\omega'\right)\Leftrightarrow\omega\succeq\omega'$. Note that $W\left(\Omega\right)$ is a finite set.

For all acts $x\in\mathcal{F}$, there exists an order $\preceq$ with respect to which x is monotonic. Note that if x is monotonic with respect to $\preceq$, then

$$\sum_{E\in 2^{\Omega}\backslash\varnothing}\min_{\omega\in E}\left\{x\left(\omega\right)\right\}\alpha\left(E\right)=\sum_{E\in 2^{\Omega}\backslash\varnothing}x\left(\arg\min_{\omega\in E}\preceq\right)\alpha\left(E\right).$$

Denote by $x\left(\preceq\right)$ the set of acts that are monotonic with respect to $\preceq$. For all $\preceq$ for which $x\left(\preceq\right)\cap f\left(\mathcal{TM}\left(\Omega\right)\right)$ is nonempty, define $U^{\preceq}\left(\alpha\right)=\sup_{x\in x\left(\preceq\right)\cap f\left(\mathcal{TM}\left(\Omega\right)\right)}\sum_{E\in 2^{\Omega}\backslash\varnothing}x\left(\arg\min_{\omega\in E}\preceq\right)\alpha\left(E\right)$. Note that each such $U^{\preceq}$ is convex and subdifferentiable on the boundary of $\Delta\left(2^{\Omega}\backslash\varnothing\right)$. Here, the definitions are required because our proof requires us to partition the range of the scoring rule into the ordinal score-orderings of states.

We claim that for all V open in $\Delta\left(2^{\Omega}\backslash\varnothing\right)$ and convex, there exists $\alpha,\beta\in V$, $\alpha\neq\beta$, and $\lambda\in\left(0,1\right)$ such that $U^{\preceq}\left(\alpha\right)=U^{\preceq}\left(\beta\right)=U^{\preceq}\left(\lambda\alpha+\left(1-\lambda\right)\beta\right)$, so that $U^{\preceq}$ is not strictly convex. Let $\omega^{*}\in\arg\min_{\omega\in\Omega}\preceq$, and consider any $E\neq\left\{\omega^{*}\right\}$ which contains ω^{*}. Then in particular, $\omega^{*}\in\arg\min_{\omega\in E}\preceq^{*}$ and $\omega^{*}\in\arg\min_{\omega\in\left\{\omega^{*}\right\}}\preceq$. As V is open, there exists $\alpha\in V$ for which $\alpha\left(\left\{\omega^{*}\right\}\right)>0$. Let $\varepsilon<\alpha\left(\left\{\omega^{*}\right\}\right)$ be small enough so that

$$\beta\left(F\right)=\left\{\begin{array}{c}\alpha\left(F\right)-\varepsilon\text{ if }F=\left\{\omega^{*}\right\}\\ \alpha\left(F\right)+\varepsilon\text{ if }F=E\\ \alpha\left(F\right)\text{ otherwise}\end{array}\right\}$$

[8] An order $\preceq$ is a weak order if it is complete and transitive.

is contained in V (as V is open, such an ε exists). As for all $F \notin \{\{\omega^*\}, E\}$, $\beta(F) = \alpha(F)$, it is clear that

$$\begin{aligned} & \sup_{x \in x(\preceq) \cap f(\mathcal{TM}(\Omega))} \sum_{E \in 2^\Omega \setminus \varnothing} x\left(\arg\min_{\omega \in E} \preceq\right) \alpha(E) \\ = & \sup_{x \in x(\preceq) \cap f(\mathcal{TM}(\Omega))} \sum_{E \in 2^\Omega \setminus \varnothing} x\left(\arg\min_{\omega \in E} \preceq\right) \beta(E). \end{aligned}$$

Moreover, it is also clear that for any $\lambda \in (0,1)$,

$$\begin{aligned} & \sup_{x \in x(\preceq) \cap f(\mathcal{TM}(\Omega))} \sum_{E \in 2^\Omega \setminus \varnothing} x\left(\arg\min_{\omega \in E} \preceq\right) \alpha(E) \\ = & \sup_{x \in x(\preceq) \cap f(\mathcal{TM}(\Omega))} \sum_{E \in 2^\Omega \setminus \varnothing} x\left(\arg\min_{\omega \in E} \preceq\right) (\lambda\alpha + (1-\lambda)\beta)(E). \end{aligned}$$

Therefore, $U^{\preceq}$ is not strictly convex on *any* open neighborhood V.

Clearly, $U = \sup_{\preceq} U^{\preceq}$. We claim that there exists some open V and some $\preceq$ for which $U|_V = U^{\preceq}|_V$. The theorem will then be complete, as U is not strictly convex on V. Enumerate the functions $\{U^{\succeq}\}$ as $\{U^1, ..., U^K\}$. Clearly, $\Delta(2^\Omega \setminus \varnothing)$ is relatively open. Set $V^1 = \Delta(2^\Omega \setminus \varnothing)$. For all $i = 2, ..., K$, set $V^i = \{\alpha \in V^{i-1} : \text{there exists } j \text{ such that } U^j(\alpha) > U^{i-1}(\alpha)\}$. Set $V^{K+1} = \varnothing$. Clearly, V^i is open for all i, and for all $i = 1, ...K$, $V^i \subset V^{i-1}$. Let i^* satisfy $V^{i^*} \neq \varnothing$ and $V^{i^*+1} = \varnothing$ (clearly such an i^* must exist). As $V^{i^*+1} = \varnothing$, it must be that $U^{i^*}(\alpha) \geq U^i(\alpha)$ for all i on V^{i^*}. We therefore establish that there exists some open set V for which there exists $\preceq^*$ for which $U = U^{\preceq^*}$. This is an immediate contradiction to the strict convexity of U. Therefore, there exists no proper scoring rule on $\mathcal{TM}(\Omega)$. ■

Proof of Corollary 2. For all $\nu \in \mathcal{TM}(\Omega)$, let $C(\nu) = \{p \in \Delta(\Omega) : p(E) \geq \nu(E) \text{ for all } E \subset \Omega\}$. A classic result of Schmeidler (for example, see Schmeidler [27], p. 582-583) states that for all $E \subset \Omega$, $\nu(E) = \min\{p(E) : p \in C(\nu)\}$, and moreover, $E_\nu[x] = \min_{p \in C(\nu)} E_p[x]$ for all $x \in \mathcal{F}$. The corollary now follows trivially. ■

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