Money metric utilitarianism

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Abstract

We discuss a method of ranking allocations in economic environments which applies when we do not know the names or preferences of individual agents. We require that two allocations can be ranked with the knowledge only of their aggregate bundles and community indifference sets–a condition we refer to as aggregate independence. We also postulate a basic Pareto and continuity property, and a property stating that when two disjoint economies and allocations are put together, the ranking in the large economy should be consistent with the rankings in the two smaller economies (reinforcement). We show that a ranking method satisfies these axioms if and only if there is a probability measure over the strictly positive prices for which the rule ranks allocations on the basis of the random-price money-metric utilitarian rule. This is a rule which computes the money-metric utility for each agent at each price, sums these, and then takes an expectation according to the probability measure.

1 Introduction

The question addressed in this paper is on the evaluation of social welfare in large economies in which information about individual characteristics is not available. Data in a large economy come in the form of aggregate statistics. Data about the preferences or consumption of all individuals in society are prohibitively costly or impossible to obtain. Indeed, the US Census, for example, typically provides only an estimate of the *number* of people residing

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in the US, let alone their names, preferences, or consumption. The social choice literature has generally ignored this issue. Social choice is largely concerned with distributional considerations–considerations which reliance on aggregate data rule out forthright.[1,2]

In contrast, economic indicators such as the GNP are easy to compute with aggregate data. However, preferences only enter into such indicators through price. Any welfare indicator depending on prices and consumption alone cannot respect the Pareto property (on this, see Samuelson [34] p. 146-156).[3] The Pareto property is a minimal property that a measure of welfare should satisfy. Consequently, to make basic welfare judgments, we need to know more than just aggregate consumption and prices in equilibrium.

In this work, we take an intermediate stance. Our aim is to show that some basic statements about welfare can be made, even if the preferences of individuals are unknown. Instead, we suppose that data about *community* preferences and aggregate consumption can be attained. The names, preferences, and consumption of individuals are not available. Our main contribution is a characterization of a well-known class of social welfare functionals based on this informational parsimony requirement. A description of this class follows.

Many measures of well-being of a society are stated in monetary terms in order to facilitate cost benefit analysis. One such measure works as follows. It depends on some initial efficient allocation, which is supported by prices p. The social welfare of another allocation is the aggregate amount of money at prices p needed to bring everybody to their individual level of welfare. This is sometimes referred to as the "aggregate equivalent variation." If this amount of money is positive, then a change to the new allocation is deemed beneficial. Like the GNP,

[1] Strictly speaking, data about the income distribution is readily available. However, this data does not allow us to identify the preferences of individuals with given incomes. From a Bergson-Samuelson welfare perspective then, it is of no use. Moreover, even if preferences could be backed out of the income distribution, welfare would necessarily depend non-trivially on price except in very special circumstances (on this, see Roberts [32]).

[2] Another possible reason that social choice theory assumes all individual preferences are known is that its' most celebrated result–Arrow's general possibility theorem [2]–shows that paradoxical results obtain even when we know binary comparisons of all individuals.

[3] Other problems can result as well; see Gorman [20] or Chipman and Moore [7], for example.

aggregate equivalent variation depends on an initial allocation, and leads to transitivity (as well as symmetry) violations. Further, given an equilibrium (p, x) in an exchange economy, for any other efficient allocation y with supporting prices $q \neq p$, the aggregate equivalent variation in moving to (q, y) at prices p is strictly negative. The rule is thus rendered practically useless in such an environment as no change will ever be recommended from an initially efficient allocation.[4]

Hence, in order to use money to evaluate social welfare meaningfully, prices must be exogenous.[5] For fixed prices p, define the social welfare of allocation x to be the amount of money society would need to distribute to make every agent at least as well off as at x. This involves finding, for each agent in society, the "money-metric" utility (the term was coined by Samuelson [36]), and summing these across agents. We call such a rule "money-metric utilitarian."[6] This welfare function has a long history in applied welfare economics.

Fixing prices may be inappropriate–Donaldson [12] claims that the choice of prices can have non-trivial ethical consequences. For example, relatively high prices of "luxury goods" assign higher importance to the wealthy. To circumvent this issue, we could allow prices to be random. Instead of finding the amount of money needed to make society as well off as a given allocation, we instead calculate the *expected* amount of money needed to make society as well off as a given allocation. We call such a social welfare function *random-price money-metric utilitarian*. Unfortunately, even this added generality will typically rule out most notions of distributive justice.

Our main result shows that a rule can be computed from aggregate data alone and satisfies basic efficiency properties if and only if it is random-price money-metric utilitarian.

It is worth emphasizing that we axiomatize this family of social welfare orderings as a

[4]For more on this, see Boadway [6].

[5]Axiomatizations of Hicks' variation concepts which rely on price as a primitive can be found in Ebert [13].

[6]It has also been called an "aggregate generalized variation," in the sense of Chipman and Moore [9] or an "aggregate equivalent income."

function of *preferences alone*. Neither prices nor utility is a primitive of our model, distinguishing this work from works which take utility as a primitive–such as D'Aspremont and Gevers [3] or Maskin [28]. We study a variable population model to provide *rankings* of allocations.[7]

Our main axiom is called *aggregate independence*. To understand how the axiom works, we first discuss a stronger axiom, called *strong aggregate independence*. Given an allocation x and a list of preferences, consider the set of all allocations y which weakly Pareto dominate x. Each such allocation y induces an aggregate allocation (summing the allocations across agents). The set of all such aggregate allocations is called the *Scitovsky upper contour set* [38]. The Scitovsky upper contour set is necessarily convex and closed, and indeed resembles an upper contour set for an individual agent. The "indifference curve" which is on the boundary of the Scitovsky upper contour set is often referred to as a "community indifference curve." However, one must be careful with such terminology. In general, the Scitovsky upper contour sets and corresponding community indifference curves do not behave as individuals' indifference curves do, unless the restrictive properties of Gorman [19, 21] are satisfied. For example, community indifference curves often cross.

Now, strong aggregate independence says that a ranking of two allocations should depend only on their Scitovsky upper contour sets. The ranking should be independent of the names of the agents in society, their individual preferences, and their individual consumption. To formalize this, imagine we have two different societies of agents, and are ranking allocations x and y in one society versus z and w in another. If the Scitovsky upper contour set of x is equal to the Scitovsky upper contour set of z, and the Scitovsky upper contour set of y is equal to the Scitovsky upper contour set of w, then the ranking of x and y should be the same as the ranking of z and w. Now, aggregate independence is a weaker axiom which allows

[7] Arrow's work lays the foundations for ranking economic allocations [2]. More recently, a line of investigation initiated by Fleurbaey and Maniquet [14] studies the theory of equitable rankings in economic environments.

the ranking of two allocations to depend on the aggregate allocations under consideration in addition to their Scitovsky upper contour sets. Asking that less information be used in ranking allocations will typically result in impossibility; on this, see Fleurbaey et al [17].

The remaining axioms are standard. We require that a ranking of allocations should respect the Pareto principle. Second, we require an analogue of Young's [41] reinforcement principle. Two disjoint economies when considered as a whole should rank allocations in a manner consistent with the ranking of the allocations of the original economies. Finally, in the hope of being able to measure welfare in monetary units, we require a continuity axiom, ensuring a real-valued representation. However, money plays no role whatsoever in our analysis.

Our result states that a social welfare ordering satisfies these properties *if and only if* there exists a probability measure over the set of strictly positive prices for which social rankings are made on the basis of the random price money-metric utilitarian rule.

Any random money-metric utilitarian rule typically lacks basic equity properties. This point has been made by Blackorby and Donaldson [5], Hammond [22], and McKenzie [29]. Our result illustrates exactly why this is the case. Any rule satisfying aggregate independence necessarily ignores distributional issues, as it relies only on aggregate data. To some extent, this is unavoidable for a rule that can only consider aggregate data. One benefit of the axiomatic approach is in illuminating such tradeoffs (informational parsimony versus equity, in this case).

Section 2 describes the model. Section 3 is devoted to the discussion of our primary axioms. Section 4 states the main theorem, and Section 5 provides a proof. Section 6 concludes.

2 The model

Let $X = \mathbb{R}^m_+$, a space of **commodities**. Let $\mathbb{N}$, the natural numbers, index a set of **potential agents**. A finite subset of agents $N \subset \mathbb{N}$ will represent a **society**. The set of all societies is denoted $\mathcal{N}$. For a society N, a typical element is denoted $i \in N$.

For a society N, an **allocation for** N is a vector $(x_i) \in X^N$. An allocation specifies consumption for all agents in N. A binary relation over X is denoted by $\succeq$.[8] The set of **preferences** $\mathcal{R}$ on X is the set of binary relations which are complete, transitive, continuous, convex, and monotonic.[9,10]

An **economy** is a pair consisting of a society and its' preferences; that is, a pair $(N, (\succeq_i)_{i\in N})$. The set of economies is denoted $\mathcal{E}$.

Denote the set of binary relations on X^N by $\mathcal{R}^*_N$. A **social welfare ordering**, or a rule, is a mapping $\succeq^0 : \mathcal{E} \to \bigcup_{N\in\mathcal{N}} \mathcal{R}^*_N$ such that for all $(N, (\succeq_i)_{i\in N}) \in \mathcal{E}$, $\succeq^0((N, (\succeq_i)_{i\in N})) \in \mathcal{R}^*_N$. A social welfare ordering takes as input a society and their individual preferences over consumption bundles, and ranks allocations for that society.

The focus here is on a specific social welfare ordering. Let $\Delta(m)$ denote the unit simplex in $\mathbb{R}^m$, and let $\Delta_{++}(m)$ denote the strictly positive elements of the unit simplex. For $p \in \Delta(m)$, define the p-money metric for preference $\succeq$ as

$$U^p_{\succeq}(x) = \inf\{p \cdot y : y \succeq x\}.$$

Under our conditions, $U^p_{\succeq}$ is a utility representation of $\succeq$ when $p \in \Delta_{++}(m)$ (see Weymark

[8] The asymmetric part of $\succeq$ is denoted $\succ$, whereas the symmetric part is denoted $\sim$.

[9] Complete: For all $x, y \in X$, $x \succeq y$ or $y \succeq x$
Transitive: For all $x, y, z \in X$ for which $x \succeq y$ and $y \succeq z$, it follows that $x \succeq z$
Continuous: For all $x \in X$, the sets $\{y : y \succeq x\}$ and $\{y : x \succeq y\}$ are both closed
Convex: For all $x, y \in X$ and all $\alpha \in [0,1]$, if $x \succeq y$, then $\alpha x + (1-\alpha) y \succeq y$
Monotonic: For all $x, y \in X$, if $x \geq y$, then $x \succeq y$ and if $x \gg y$, then $x \succ y$

[10] Vector inequalities are as follows: $x \geq y$ if for all $k = 1, ..., m$, $x^k \geq y^k$, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if for all $k = 1, ..., m$, $x^k > y^k$.

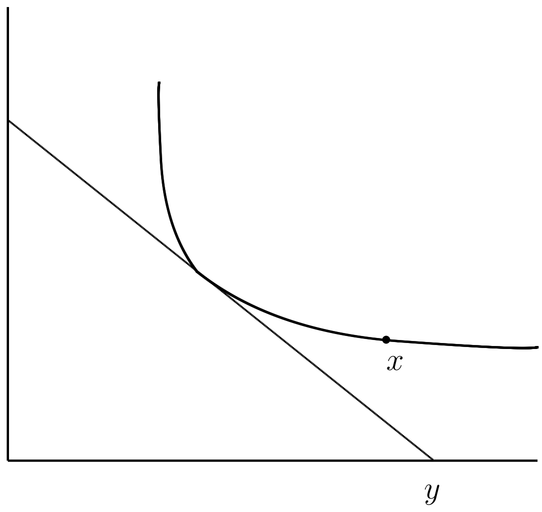

Figure 1: A money-metric utility function

[40] Proposition 1), but it need not be otherwise. The money-metric function is closely related to the expenditure function of McKenzie [30], but is a distinct mathematical object. For more on the money-metric utility, see Weymark [40]. In Figure 1, a money-metric utility is illustrated. Here, we suppose the price of the first good is normalized to one. Then the money-metric utility of x is equal to y.

For an economy $(N, (\succeq_i)_{i\in N}) \in \mathcal{E}$ and $p \in \Delta_{++}(m)$, define $F^p(N, (\succeq_i)_{i\in N}) : X^N \to \mathbb{R}$ by

$$F^p(N, (\succeq_i)_{i\in N})((x_i)_{i\in N}) = \sum_N U^p_{\succeq_i}(x_i).$$

This will be termed the p-**money metric utilitarian social welfare function**. Let π be a Borel probability measure over $\Delta_{++}(m)$, and define the **random-price money metric utilitarian social welfare function** F^π as

$$F^\pi((N, (\succeq_i)_{i\in N}))((x_i)_{i\in N}) = \int_{\Delta_{++}(m)} \sum_N U^p_{\succeq_i}(x_i) d\pi(p).$$

The function F^π induces a social welfare ordering $\succeq^\pi$ by

$$x \succeq^\pi \big(N, (\succeq_i)_{i\in N}\big)\, y$$

if and only if

$$F^\pi(N, (\succeq_i)_{i\in N})((x_i)_{i\in N}) \geq F^\pi(N, (\succeq_i)_{i\in N})((y_i)_{i\in N}).$$

The purpose of this work is to axiomatize the family of social welfare orderings consistent with random-price money-metric utilitarianism.

3 Axioms

Let us now define some properties that a social welfare ordering may have.

Weak order: For all $e \in \mathcal{E}$, $\succeq^0 (e)$ is complete and transitive.

Pareto: For all $e = (N, (\succeq_i)_{i\in N}) \in \mathcal{E}$ and all $x, y \in X^N$, if $x_i \succeq_i y_i$ for all $i \in N$, then $x \succeq^0 (e) y$. If in addition there exists $i \in N$ for which $x_i \succ_i y_i$, then $x \succ^0 (e) y$.

The following axiom is common in works axiomatizing utilitarian-like social welfare functions. It essentially first appears in Young [41, 42] and Smith [39]. It is similar to the Pareto criterion; and if a rule satisfies the requirement that when applied to an economy consisting of a single individual, the rule coincides with her preference (referred to by Young as *faithfulness*), then it in fact implies Pareto.

Reinforcement: For all $N, N' \in \mathcal{N}$ for which $N \cap N' = \varnothing$, all $x, y \in X^N$, all $z, w \in X^{N'}$, and all $(\succeq_i)_{i\in N\cup N'} \in \mathcal{R}^{N\cup N'}$, if $x \succeq^0 (N, (\succeq_i)_{i\in N}) y$ and $z \succeq^0 (N', (\succeq_i)_{i\in N'}) w$, then $(x, z) \succeq^0 (N \cup N', (\succeq_i)_{i\in N\cup N'})(y, w)$, with strict preference if either original preference is strict.

Our next axiom states that the rule should be applicable with as little information as possible. It requires that for two economies and two allocations, the ranking of the two allocations is determined by their "community indifference maps." In order to ensure that the aggregate bundle actually lies on the "community indifference map," we may require that the actual allocation under consideration is Pareto efficient. However, in the interest of simplicity, we state the axiom without this caveat.

Instead of talking about indifference maps, it is useful to talk about the upper contour sets of those maps. For an economy $e = (N, (\succeq_i)_{i \in N})$ and an allocation $x \in X^N$, define the **Scitovsky upper contour set** as

$$S(e, x) = \{\sum_{i \in N} y_i : y_i \succeq_i x_i \text{ for all } i \in N\}.$$

Thus, $S(e, x)$ is the set of aggregate bundles that could be allocated in a way as to make everybody in society weakly better off than under the allocation x. The allocation x may not be on the boundary of $S(e, x)$, but it is when it is Pareto efficient.

Aggregate independence: For all $e = (N, (\succeq_i)_{i \in N})$, $e' = (N', (\succeq_i')_{i \in N'}) \in \mathcal{E}$, all $x, y \in X^N$ for which $\sum x_i = \sum y_i$ and all $z, w \in X^{N'}$ for which $\sum z_i = \sum w_i$, if $S(e, x) = S(e', z)$ and $S(e, y) = S(e', w)$, then $x \succeq^0 (e)y$ if and only if $z \succeq^0 (e')w$.

Strong aggregate independence: For all $e = (N, (\succeq_i)_{i \in N})$, $e' = (N', (\succeq_i')_{i \in N'}) \in \mathcal{E}$, all $x, y \in X^N$ and all $z, w \in X^{N'}$, if $S(e, x) = S(e', z)$ and $S(e, y) = S(e', w)$, then $x \succeq^0 (e)y$ if and only if $z \succeq^0 (e')w$.

Let us consider the two aggregate independence axioms separately. Strong aggregate independence states that a comparison of two allocations can be made as long as we have knowledge of their Scitovsky upper contour sets. Indeed, it states that if two pairs of allocations have identical corresponding Scitovsky upper contour sets, then their rankings must coincide. The rule is not allowed to use information other than the Scitovsky upper contour sets. Aggreagte independence weakens this axiom slightly. It allows the rule to use information about the aggregate allocations under consideration as well as the Scitovsky upper contour sets, but nothing else. The axiom is one formalization of the notion that a rule should be informationally parsimonious, and should not depend on the number of agents in society or their individual preferences or consumption. Instead, it allows us to rank allocations based on functions of the individual preferences and consumption. By their very nature, Scitovsky upper contour sets are more readily observable than individual preferences.

We could ask that a rule use even *less* information about preferences than Scitovsky upper contour sets, but any less information would result in an impossibility–on this, see [17].[11] Any axiom which requires using less data about preferences will usually conflict with the Pareto principle. To this end, we have pushed the informational parsimony requirement as far as we can go without running into conflict with basic Paretian objectives.

The axiom is formally unrelated to Arrow's independence of irrelevant alternatives [2]. However, it is implied by a slightly weaker notion of Arrow's IIA studied in [23, 16, 18, 31], among other works. This weaker notion states that in determining the ranking of two alternatives, a rule is allowed to use information about the indifference surfaces passing through the two alternatives for *each individual*.[12] Our axiom requires that a rule only use information about the individual indifference surfaces after they have been collapsed into aggregate indifference surfaces. We note that while, in principle, Scitovsky sets are more readily observable than individual preferences (indeed, preference profiles are a sufficient statistic for Scitovsky sets, while the converse is not true), in practice at this stage, it may

[11] While aggregate independence appears very strong (and it is), it is satisfied by several well-known concept of economic theory and at least one which does not deal with Paretian rankings. The coefficient of resource utilization, introduced by Debreu [11], is easily seen to satisfy it. This was pointed out to us by François Maniquet.

[12] There are other notions that are meaningful in economic environments, such as independence of infeasible alternatives. For more on this, see Le Breton [26].

not be any easier to elicit Scitovsky sets than to elicit individual preferences.

Our last property rules out pathological rules that may not have a functional representation. In many practical situations, social welfare should be stated in monetary terms (on this, see Samuelson [35] or McKenzie [29]). To facilitate such a possibility, we require the following.

Continuity: For all $e \in \mathcal{E}$, $\succeq^0 (e)$ is a continuous binary relation.

4 Results

The following is our main result. It states that our axioms are satisfied if and only if a rule is a random-price money-metric utilitarian rule.

Theorem 1 $\succeq^0$ *satisfies weak order, Pareto, reinforcement, aggregate independence, and continuity if and only if there exists a probability measure* π *over* $\Delta_{++}(m)$ *for which* $\succeq^0 = \succeq^\pi$.

- For *any* list of positive prices p, and any real-valued social welfare function respecting the Pareto property, one may, for a fixed population, write this social welfare function in the Bergson-Samuelson form, so that $U(x) = W((U^p_{\succeq_i}(x_i))_{i \in N})$. The real content of the theorem is therefore the *joint* restriction on the functional form of utility *and* social welfare function.

- Most importantly, utilities are not primitive, and are not meant to be interpersonally comparable. Money-metric utilities are part of the representation of the social choice rule but have no meaning (in terms of interpersonal comparison) in and of themselves. Criticisms of the mechanism based on the representation, or in terms of interpersonal comparisons are not meaningful.

- Criticism of the axioms characterizing the rule is valid. In general, rules satisfying reinforcement tend to preclude equity considerations as they imply a type of separability across agents. Aggregate independence, moreover, precludes equity considerations as it ignores distributional considerations among agents altogether. If data comes in aggregate form; then this is unavoidable.

- Not only is $\succeq^{\pi}(e)$ continuous for all $e \in \mathcal{E}$, $F^{\pi}(e)$ is a continuous function for all $e \in \mathcal{E}$ as well.

- In the statement of the theorem, aggregate independence could be replaced by strong aggregate independence with no change to the results.

We now proceed with an intuition for why the result is true. The key axiom here is aggregate independence: this axiom states that in comparing two allocations, only the Scitovsky upper contour sets of those allocations and aggregate bundles are relevant. In fact, by constructing an appropriate economy, we can establish that a rule satisfying aggregate independence and Pareto only need consider the Scitovsky upper contour sets (the aggregate bundles become irrelevant). Hence, such a rule simply ranks Scitovsky upper contour sets. An important fact is that a given Scitovsky upper contour set can be determined uniquely by a function mapping prices into reals which, for each list of prices, specifies the sum of money-metric utilities evaluated at that list of prices. Key is to show that the ranking depends in a monotonic and linear way on these induced functions. Linearity follows from the reinforcement property (which is formally a condition of additive separability), whereas positivity follows from the Pareto property. Once we have established that such an order can be represented by a positive linear functional, we use the Reisz representation theorem to identify this positive linear functional with a probability measure. This gives the desired representation.

5 A proof of the result

Necessity of axioms:

To see Pareto, note that for $p \in \Delta_{++}(m)$, $U^p_{\succeq}$ is a utility representation for $\succeq \in \mathcal{R}$ (Weymark [40] Proposition 1). Consequently, $\int_{\Delta_{++}(m)} U^p_{\succeq}(x)\, d\pi(x)$ is a utility representation for $\succeq$ (this verifies Pareto). Reinforcement is trivially satisfied. To see that aggregate independence is satisfied, let $e = \left(N, (\succeq_i)_{i\in N}\right) \in \mathcal{E}$ and $e' = \left(N', (\succeq_i)_{i\in N'}\right) \in \mathcal{E}$, and suppose that $S(e,x) = S(e',y)$. Then for all $p \in \Delta(m)$, $\sum_N U^p_{\succeq}(x_i) = \sum_{N'} U^p_{\succeq}(y_i)$, establishing the result. For example, see Proposition 5, Proposition 7, and Proposition 8, below.

To verify continuity, we will verify that for all $\succeq \in \mathcal{R}$, $\int_{\Delta_{++}(m)} U^p_{\succeq}(x)\, d\pi(p)$ is continuous in x. So, suppose that $\{x^\nu\} \to x$. Continuity of $U^p_{\succeq}$ for all $p \in \Delta_{++}(m)$ (implied by Weymark [40] Proposition 2 or Honkapojha [25] Proposition 5) implies that for all $p \in \Delta_{++}(m)$, $U^p_{\succeq}(x^\nu) \to U^p_{\succeq}(x)$. As $\{x^\nu\}$ is a convergent sequence, there exists some $y \in X$ such that for all ν, $y \geq x^\nu$ and $y \geq x$. Consequently, by definition, for all $p \in \Delta_{++}(m)$ and all ν, $0 \leq U^p_{\succeq}(x^\nu) \leq U^p_{\succeq}(y)$; and $0 \leq U^p_{\succeq}(x) \leq U^p_{\succeq}(y)$. As $\int_{\Delta_{++}(m)} U^p_{\succeq}(y)\, d\pi(p)$ exists and is finite, by the Lebesgue dominated convergence theorem (11.21 in Aliprantis and Border [1]), we may conclude that

$$\int_{\Delta_{++}(m)} U^p_{\succeq}(x^\nu)\, d\pi(p) \to \int_{\Delta_{++}(m)} U^p_{\succeq}(x)\, d\pi(p),$$

verifying continuity of $\int_{\Delta_{++}(m)} U^p_{\succeq}(x)\, d\pi(p)$ as a function of x, and as a consequence, continuity of $F^\pi(e)$ for all $e \in \mathcal{E}$.

Sufficiency of axioms:

For $\succeq \in \mathcal{R}$ and $x \in X$, denote $U(\succeq, x) = \{y \in X : y \succeq x\}$. By our assumptions on $\mathcal{R}$, for all $x \in X$, $U(\succeq, x)$ is a convex, closed set. We will say a set $A \subset \mathbb{R}^m_+$ is **upper comprehensive** if $x \in A$ and $y \geq x$ implies $y \in A$. Denote by $\mathcal{K}$ the set of all nonempty, closed, convex, and upper comprehensive sets in $\mathbb{R}^m_+$. By our assumptions on $\mathcal{R}$, for all

$\succeq \in \mathcal{R}$ and all $x \in X$, $U(\succeq, x) \in \mathcal{K}$. We say $\succeq \in \mathcal{R}$ is **homothetic** if for all $x, y \in X$ and all $\alpha > 0$, $x \succeq y \Longleftrightarrow \alpha x \succeq \alpha y$.

Proposition 2 *For all $K \in \mathcal{K}$, there exists a homothetic $\succeq \in \mathcal{R}$ and $x \in X$ such that $K = U(\succeq, x)$.*

Proof. Let $K \in \mathcal{K}$. If $K = X$, then we may choose $u(x) = \prod_{i=1}^{m} x^i$, for example, and set $x = 0$. Otherwise, let a utility representation $u : X \to \mathbb{R}$ be defined as

$$u(w) = \inf\{\lambda : w \notin \lambda K\}.$$

As $0 \notin K$, by assumption, there exists some neighborhood ε of 0 for which $N_\varepsilon(0) \cap K = \varnothing$. Consequently, for all w, there exists λ large for which $w \notin \lambda K$. Hence, u is well-defined.

We first establish that u is homogeneous of degree zero (so that for all $\alpha > 0$ and all $w \in X$, $u(\alpha w) = \alpha u(w)$.) Thus, let $w \in X$ and let $\alpha > 0$. If $u(w) = 0$, then for all $\lambda > 0$, $w \notin \lambda K$, so that in particular, for all $\lambda > 0$, $w \notin \frac{\lambda}{\alpha} K$, or $\alpha w \notin \lambda K$, so that $u(\alpha w) = \alpha u(w)$. Otherwise, note that $w \notin \lambda K$ if and only if $\alpha w \notin \alpha \lambda K$, so that $u(\alpha w) = \alpha u(w)$.

Moreover, u is quasiconcave. That is, suppose that $u(z) \geq u(w)$ and let $\alpha \in [0, 1]$. If $u(w) = 0$, the result is trivial, so suppose that $u(w) > 0$. We claim that $w \in u(w)K$. In particular, by definition of $u(w)$, for all $0 < \varepsilon < 1$, $w \in \varepsilon u(w)K$. Hence, $\frac{w}{\varepsilon} \in u(w)K$, or, taking limits and using the fact that $u(w)K$ is closed, $w \in u(w)K$. Similarly, this demonstrates that $z \in u(z)K \subset u(w)K$. As $u(w)K$ is convex, $\alpha z + (1 - \alpha)w \in u(w)K$. Consequently, $u(\alpha z + (1 - \alpha)w) \geq u(w)$, by definition of u. Monotonicity follows similarly; that is, suppose that $z \geq w$. Again, if $u(w) = 0$, the result is trivial; otherwise, we may again conclude that $w \in u(w)K$. As K is upper comprehensive, conclude that $z \in u(w)K$, or that $u(z) \geq u(w)$. If $z \gg w$, then if $u(w) = 0$, $z \in \mathbb{R}^m_{++}$, and hence there exists $\lambda > 0$ small enough so that $z \in \lambda K$, so that $u(z) > 0$. Otherwise, $z \in \operatorname{int} u(w)K$, and consequently there exists ε for which $z \in [u(w) + \varepsilon] K$, so that $u(z) > u(w)$. Lastly, we verify that u is continuous. Suppose that $w^\nu \to w$ and let $\varepsilon < 1$. First, we show that for ν large, $u(w^\nu) > \varepsilon u(w)$. Again, if $u(w) = 0$, this is trivial, so suppose that $u(w) > 0$. Then, by homogeneity, $u(\varepsilon w) = \varepsilon u(w)$, so that we can conclude $\varepsilon w \in \varepsilon u(w)V$. Hence, by monotonicity, $w \in \operatorname{int} \varepsilon u(w)V$. Consequently, for all ν large, $w^\nu \in \operatorname{int} \varepsilon u(w)V$, so that $u(w^\nu) \geq \varepsilon u(w)$. Now, suppose $\varepsilon > 0$; we verify that for ν large, $u(w^\nu) \leq u(w) + \varepsilon$. By definition $w \notin [u(w) + \varepsilon]V$. But the complement of $[u_1(w) + \varepsilon]V$ is open; so for ν large, $w^\nu \notin [u(w) + \varepsilon]V$, so for ν large, $u(w^\nu) \leq u(w) + \varepsilon$. This verifies continuity.

Next, we show that $w \in K$ if and only if $u(w) \geq 1$. Thus, suppose $w \in K$. Then in particular, if $w \notin \lambda K$, then $\lambda > 1$. Consequently, $u(w) \geq 1$. Now, suppose that $u(w) \geq 1$. We have proved before that $w \in u(w)K$; consequently we may conclude that $w \in K$.

Let $\succeq$ be the binary relation represented by u. Furthermore, by choosing x on the boundary of K, we ensure that $u(x) = 1$; so that $U(\succeq, x) = K$. ■

Proposition 3 *If a rule satisfies Pareto and aggregate independence, then it satisfies strong aggregate independence.*

Proof. Let $e = (N, (\succeq_i)_{i \in N})$, $e' = (N', (\succeq'_i)_{i \in N'}) \in \mathcal{E}$, $x, y \in X^N$, and $z, w \in X^{N'}$ satisfy $S(e, x) = S(e', z)$ and $S(e, y) = S(e', w)$, and suppose that $x \succeq^0 (e)y$. We claim that $z \succeq^0 (e')w$–by symmetry of the definition, this will complete the proof.

We construct a new economy as follows. Let e'' consist of three agents, $N = \{1, 2, 3\}$. As $S(e, x) \in \mathcal{K}$, by Proposition 2, there exists $\succeq \in \mathcal{R}$ and $x' \in X$ such that $S(e, x) = U(\succeq, x')$. In fact, we may choose $\succeq$ (not homothetic) so that the boundary of $S(e, x)$ is equal to $\{y : y \sim x'\}$.[13] Let $\succeq_1 = \succeq$. Similarly, there exists $\succeq' \in \mathcal{R}$ and $y' \in X$ such that $S(e, y) = U(\succeq', y')$. Let $\succeq_2 = \succeq'$. Lastly, let $\succeq_3$ be a Leontief preference, represented by the utility function $u(x) = \min\{x^i\}$.

Now, let us fix a commodity, say, commodity 1. Define $\varepsilon(x) = \max\left\{\varepsilon \in \mathbb{R} : x - \varepsilon 1_{\{1\}} \in U(\succeq, x')\right\}$. Define $x'' = x - \varepsilon(x) 1_{\{1\}}$, and note by construction

[13]The argument can be established using techniques similar to those in Proposition 10 below.

that $x'' \sim_1 x'$. As $S(e,z) = S(e,x)$, we may similarly $\varepsilon(z)$ and define z'' as $z - \varepsilon(z) 1_{\{1\}}$, and note as well that $z'' \sim_1 z$. Now, consider the two allocations $\left(x'', 0, \varepsilon(x) 1_{\{1\}}\right)$ and $\left(z'', 0, \varepsilon(z) 1_{\{1\}}\right)$. Clearly $x'' \sim_1 z''$ and as $\succeq_3$ is a Leontief preference, $\varepsilon(x) 1_{\{1\}} \sim \varepsilon(z) 1_{\{1\}}$. By Pareto, therefore

$$\left(x'', 0, \varepsilon(x) 1_{\{1\}}\right) \sim^0 (e'') \left(z'', 0, \varepsilon(z) 1_{\{1\}}\right).$$

Note in particular that $x'' + \varepsilon(x) 1_{\{1\}} = x$ and $z'' + \varepsilon(z) 1_{\{1\}} = z$.

We may construct similar allocations, defining $\varepsilon'(y) = \max\left\{\varepsilon \in \mathbb{R} : y - \varepsilon 1_{\{1\}} \in U(\succeq', y')\right\}$, and $\varepsilon'(w)$ analogously. From this we also construct $y'' = y - \varepsilon'(y) 1_{\{1\}}$ and $w'' = w - \varepsilon'(w) 1_{\{1\}}$. It is then easy to see that

$$\left(0, y'', \varepsilon'(y) 1_{\{1\}}\right) \sim^0 (e'') \left(0, w'', \varepsilon'(w) 1_{\{1\}}\right).$$

Moreover, $y'' + \varepsilon'(y') 1_{\{1\}} = y$ and $w'' + \varepsilon'(w') 1_{\{1\}} = w$.

Now note that $S\left(e'', \left(x'', 0, \varepsilon(x) 1_{\{1\}}\right)\right) = S(e,x)$, $S\left(e'', \left(z'', 0, \varepsilon(z) 1_{\{1\}}\right)\right) = S(e',z)$, $S\left(e'', \left(0, y'', \varepsilon'(y) 1_{\{1\}}\right)\right) = S(e,y)$ and $S\left(e'', \left(0, w'', \varepsilon'(w) 1_{\{1\}}\right)\right) = S(e',w)$. By aggregate independence, as $x \succeq^0 (e)\, y$, we conclude that $\left(x'', 0, \varepsilon(x) 1_{\{1\}}\right) \succeq^0 (e'') \left(0, y'', \varepsilon'(y) 1_{\{1\}}\right)$. By transitivity, we therefore conclude $\left(z'', 0, \varepsilon(z) 1_{\{1\}}\right) \succeq^0 (e'') \left(0, w'', \varepsilon'(w) 1_{\{1\}}\right)$. Applying aggregate independence again, we conclude $z \succeq^0 (e')\, w$. ■

Define $K + K' = \{x + y : x \in K, y \in K'\}$.

Proposition 4 *For $K, K' \in \mathcal{K}$, $K + K' \in \mathcal{K}$.*

Proof. Let $K, K' \in \mathcal{K}$. It is trivial to verify that $K + K'$ is convex and upper comprehensive. Closure follows as each of $K, K' \subset \mathbb{R}^m_+$. That is, let $\{x^\nu\} \subset K + K'$ be a sequence where $x^\nu \to x^*$. Then for all ν, $x^\nu = x^{\nu,K} + x^{\nu,K'}$ for some $x^{\nu,K} \in K$, $x^{\nu,K'} \in K'$.

Note that

$$\begin{aligned} \|x^\nu\|^2 &= \left\|x^{\nu,K}\right\|^2 + 2x^{\nu,K}\cdot x^{\nu,K'} + \left\|x^{\nu,K'}\right\|^2 \\ &\geq \left\|x^{\nu,K}\right\|^2 + \left\|x^{\nu,K'}\right\|^2. \end{aligned}$$

The inequality follows as $x^{\nu,K}$, $x^{\nu,K'} \in \mathbb{R}^m_+$. Now, as x^ν is convergent, $\sup_\nu \|x^\nu\| < +\infty$. Moreover, for all ν, $\left\|x^{\nu,K}\right\| \leq \sup_\nu \|x^\nu\|$ and $\left\|x^{\nu,K'}\right\| \leq \sup_\nu \|x^\nu\|$. Consequently, each sequence $\left\{x^{\nu,K}\right\}$ and $\left\{x^{\nu,K'}\right\}$ lies in a compact set and hence have convergent subsequences. We may therefore without loss of generality suppose $x^{\nu,K} \to x^{*\nu,K} \in K$ and $x^{\nu,K'} \to x^{*\nu,K'} \in K'$. Thus, $x^* = x^{*\nu,K} + x^{*\nu,K'} \in K + K'$. ■

A well-known fact is that for all $e \in \mathcal{E}$, where $e = (N, (\succeq_i)_{i\in N})$, $S(e,x) = \sum_N U(\succeq_i, x_i)$. We will prove this below.

Proposition 5 *For all $e \in \mathcal{E}$, where $e = (N, (\succeq_i)_{i\in N})$, and all $x \in X^N$, $S(e,x) = \sum_N U(\succeq_i, x_i)$.*

Proof. Suppose that $y \in S(e,x)$. Then $y = \sum_N y_i$ where $y_i \succeq_i x_i$ for all $i \in N$. Consequently, $y_i \in U(\succeq_i, x_i)$ for all $i \in N$, so that $y \in \sum_N U(\succeq_i, x_i)$. The converse is easily seen to be true as well. ■

A rule satisfying our axioms is completely determined by a complete, transitive, ordering on the possible community indifference sets (or equivalently, community upper contour sets).

Proposition 6 *If a rule $\succeq^0$ satisfies weak order and strong aggregate independence, then there exists a complete, transitive binary relation $\succeq^K$ on $\mathcal{K}$ such that for all $e \in \mathcal{E}$, $e = (N, (\succeq_i)_{i\in N})$, for all $x, y \in X^N$, $x \succeq^0 (e) y$ if and only if $S(e,x) \succeq^K S(e,y)$.*

Proof. That there exists an order $\succeq^K$ on $\mathcal{K}$ such that for all $e \in \mathcal{E}$, $e = (N, (\succeq_i)_{i\in N})$, for all $x, y \in X^N$, $x \succeq^0 (e) y$ if and only if $S(e,x) \succeq^K S(e,y)$ is the definition of strong aggregate independence. To show that it is complete, let $K_1, K_2 \in \mathcal{K}$. Let $N = \{1, 2\}$. We construct $(\succeq_i)_{i \in N} \in \mathcal{R}^N$ and allocations $x, y \in X^N$ such that if $e = (N, (\succeq_i)_{i \in N})$, then $S(e, x) = K_1$ and $S(e, y) = K_2$. By Proposition 2, there exists $\succeq_1, \succeq_2 \in \mathcal{R}$ and $x_1, x_2 \in X$ for which $K_1 = U(\succeq_1, x_1)$ and $K_2 = U(\succeq_2, x_2)$. Consequently, by Proposition 5, $S(e, (x_1, 0)) = K_1$ and $S(e, (0, x_2)) = K_2$. By completeness of $\succeq^0$, either $(x_1, 0) \succeq^0 (e)\, (0, x_2)$, or $(0, x_2) \succeq^0 (e)\, (x_1, 0)$, so that either $K_1 \succeq^K K_2$ or $K_2 \succeq^K K_1$. Hence $\succeq^K$ is complete.

Now, let $K_1, K_2, K_3 \in \mathcal{K}$ and suppose that $K_1 \succeq^K K_2$ and $K_2 \succeq^K K_3$. Let $N = \{1, 2, 3\}$ and let $\succeq_1, \succeq_2, \succeq_3 \in \mathcal{R}$ and $x_1, x_2, x_3 \in X$ such that $K_1 = U(\succeq_1, x_1)$, $K_2 = U(\succeq_2, x_2)$, and $K_3 = U(\succeq_3, x_3)$ (again that this is possible follows from Proposition 2). Let $e = (N, (\succeq_i)_{i \in N})$. Then by definition of $\succeq^K$, $(x_1, 0, 0) \succeq^0 (e)(0, x_2, 0)$ and $(0, x_2, 0) \succeq^0 (e)\, (0, 0, x_3)$. Consequently, by transitivity of $\succeq^0 (e)$, $(x_1, 0, 0) \succeq^0 (e)\, (0, 0, x_3)$. Thus, by definition of $\succeq^K$, $K_1 \succeq^K K_3$, establishing transitivity of $\succeq^K$. ■

The preceding proposition allows us to study the problem from the perspective of ranking community indifference surfaces.

The **support function** $H : \mathcal{K} \times \Delta(m) \to \mathbb{R}$ is defined as

$$H(K, \lambda) = \inf_{x \in K} \lambda \cdot x.$$

It is well-known that for all $K \in \mathcal{K}$, $H(K, \lambda)$ is continuous in λ.

Proposition 7 *For all $K, K' \in \mathcal{K}$, $H(K, \cdot) = H(K', \cdot)$ implies $K = K'$; furthermore, $H(K + K', \cdot) = H(K, \cdot) + H(K', \cdot)$ and for all $\alpha > 0$, $H(\alpha K, \cdot) = \alpha H(K, \cdot)$.*

Proof. We establish that for all $K, K' \in \mathcal{K}$, $H(K, \cdot) = H(K', \cdot)$ implies $K = K'$; to do so, we show that if $K, K' \in \mathcal{K}$ satisfy $K \neq K'$, then there exists some $\lambda \in \Delta(m)$ such that $H(K, \lambda) \neq H(K', \lambda)$. So, suppose that $K \neq K'$ and without loss of generality, suppose that there exists $x \in K \backslash K'$. Then as K' is closed, there exists some $\lambda \in \mathbb{R}^m \backslash \{0\}$ such that $\lambda \cdot x + \varepsilon < \lambda \cdot y$ for all $y \in K'$ and some $\varepsilon > 0$ (see, for example, Rockafellar [33] Corollary

11.4.2.) In particular, $\lambda \in \mathbb{R}^m_+$ (if $\lambda \in \mathbb{R}^m \backslash \mathbb{R}^m_+$, then there exists some j for which $\lambda_j < 0$; consequently, by upper comprehensivity of K', one may choose $y \in K'$ for which $\lambda \cdot y < \lambda \cdot x$, a contradiction). We may of course normalize λ to lie in $\Delta(m)$. Consequently,

$$H(K, \lambda) \leq \lambda \cdot x < \inf_{y \in K'} \lambda \cdot y = H(K', \lambda).$$

The remaining parts of the Proposition are simply verified; see for example Rockafellar [33] Corollary 13.1.1 and the remaining discussion on p. 113. ■

The following proposition follows by definition.

Proposition 8 *For all $\succeq \in \mathcal{R}$ and all $x \in X$,*

$$H(U(\succeq, x), \lambda) = U^{\lambda}_{\succeq}(x).$$

These properties of the support function allow us to work with the space of functions induced as support functions. In fact, we can now define another binary relation, this time on $C(\Delta(m))$ (the space of continuous functions on $\Delta(m)$). Define

$$C_{--}(\Delta(m)) = \{f \in C(\Delta(m)) : f(s) < 0 \text{ for all } s \in \Delta(m)\}.$$

For f, $g \in C(\Delta(m))$, define $\succeq^c$ on $C(\Delta(m))$ by

$$f \succeq^c g$$

if and only if there exists $K, K' \in \mathcal{K}$ for which for $f(\cdot) = H(K, \cdot)$ and $g(\cdot) = H(K', \cdot)$ for which

$$K \succeq^K K'.$$

The order $\succeq^c$ is clearly transitive, as $\succeq^K$ is transitive. It is also complete when restricted to

$$\{H(K,\cdot) : K \in \mathcal{K}\}.$$

We now define the following subset of $C(\Delta(m))$:

$$\mathcal{F} = \{g - f : g \succeq^c f\}.$$

The first claim ensures that this set captures all that is relevant for recovering $\succeq^c$.

Proposition 9 *Let $K, K', K'', K''' \in \mathcal{K}$ such that $H(K,\cdot) - H(K',\cdot) = H(K'',\cdot) - H(K''',\cdot)$. Then*

$$H(K,\cdot) \succeq^c H(K',\cdot) \Leftrightarrow H(K'',\cdot) \succeq^c H(K''',\cdot).$$

Hence $H(K,\cdot) \succeq^c H(K',\cdot) \Longleftrightarrow H(K,\cdot) - H(K',\cdot) \in \mathcal{F}$.

Proof. Let K, K', K'', K''' be as in the hypothesis of the theorem, and suppose that $H(K,\cdot) \succeq^c H(K',\cdot)$. We claim that $H(K'',\cdot) \succeq^c H(K''',\cdot)$. So, suppose by means of contradiction that this statement is false. As $\succeq^c$ is complete on the domain of support functions, we may conclude that $H(K''',\cdot) \succ^c H(K'',\cdot)$. Note that $H(K,\cdot) + H(K''',\cdot) = H(K',\cdot) + H(K'',\cdot)$. Let $e = (N, (\succeq_i)_{i\in N}), e' = (N', (\succeq_i)_{i\in N'}) \in \mathcal{E}$ such that $N \cap N' = \varnothing$, and let $x, x' \in X^N$, $x'', x''' \in X^{N'}$ be allocations such that $S(e,x) = K$, $S(e,x') = K'$, and $S(e',x'') = K''$, $S(e',x''') = K'''$ (such economies exist by Proposition 2). By strong aggregate independence, we know that $x \succeq^0 (e)x'$ and $x''' \succ^0 (e')x''$. Let $e'' = (N \cup N', (\succeq_i)_{i\in N\cup N'})$. By reinforcement, $(x, x''') \succ^0 (e'')(x', x'')$. However, this contradicts strong aggregate independence, as $S(e'', (x, x''')) = S(e'', (x', x''))$. Hence $H(K'',\cdot) \succeq^c H(K''',\cdot)$. ■

The following is related to Proposition 2, but does not guarantee homotheticity.

Proposition 10 *Let $K, K' \in \mathcal{K}$, and suppose that $K' \subset \operatorname{int} K$ (here,* int *denotes the relative interior). Then there exists $\succeq \in \mathcal{R}$ and $w, w' \in X$ for which $U(\succeq, w) = K$ and $U(\succeq, w') = K'$.*

Proof. The strategy of the proof is to use convex analytic techniques to construct such a utility function which represents such a preference. If, in fact, $K = X$, then the result follows from Proposition 2 above. Hence, we suppose without loss of generality that $K \subset \operatorname{int} \mathbb{R}^m_+$.

Now, for all $x \notin K$, define

$$u(x) = \inf \{\lambda : x \notin \lambda K\}.$$

For all $x \in K \backslash K'$, define

$$u(x) = 1 + \inf \{\lambda \in [0,1] : x \notin \lambda K' + (1 - \lambda) K\}.$$

Finally, for all $x \in K'$, define

$$u(x) = 2 + \inf \{\lambda : x \notin \lambda K'\}.$$

Firstly, we note that u is in fact well-defined. The only case in which this needs to be verified is the case in which $x \in K'$. We claim that there exists some λ for which $x \notin \lambda K'$. To see this, note that as K' is closed, and since $0 \notin K'$, then in particular, there exists an open neighborhood about 0 not intersecting K'. Some multiple of x is in this neighborhood, say $\mu x \notin K'$ for $0 < \mu < 1$. But then $x \notin \frac{1}{\mu} K'$, so in fact u is well-defined.

Now, we work with all three cases ($x \notin K$, $x \in K \backslash K'$, and $x \in K'$) to establish the result.

Let us first show that for all $x \notin K$, $u(x) < 1$. So suppose that $x \notin K$, yet $u(x) \geq 1$.

Then in particular, for all $\varepsilon < 1$, $x \in \varepsilon K$, so that $\frac{x}{\varepsilon} \in K$, implying (by closedness of K) that $x \in K$, a contradiction.

Next let us show that for all $x \notin K'$, $u(x) < 2$. So, suppose that $x \notin K'$, yet $u(x) \geq 2$. Then, in particular, for all $\lambda < 1$, $x \in \lambda K' + (1-\lambda) K$. Consequently, $\frac{x}{\lambda} \in K' + \left(\frac{1-\lambda}{\lambda}\right) K$, but as K' is upper comprehensive, and as $\left(\frac{1-\lambda}{\lambda}\right) K \subset \mathbb{R}^m_+$, we may conclude that $\frac{x}{\lambda} \in K'$, or in particular, since λ is arbitrary and K' closed, $x \in K'$.

Without loss of generality, we let $0K = \mathbb{R}^m_+$ (as opposed to $\{0\}$). First, if $x \notin K$, then $u(x) < 1$. Now, we claim that $u(y) \geq u(x)$ if and only if $y \in u(x)K$. This result is trivial in the case $u(x) = 0$, so suppose otherwise. Note that if $y \in u(x)K$, then either $y \in K$ (in which case $u(y) \geq u(x)$), or $y \notin K$ and by definition $u(y) \geq u(x)$. Next, suppose that $u(y) \geq u(x)$. Then in particular either $y \in K$ (in which case $y \in u(x)K$), or for all $\varepsilon < 1$, $y \in \varepsilon u(x)K$. Thus, $\frac{y}{\varepsilon} \in u(x)K$, from which the result follows from closure of $u(x)K$.

Now, suppose that $x \in K \backslash K'$. Define $\lambda^*(z) = \inf\{\lambda \in [0,1] : z \notin \lambda K' + (1-\lambda) K\}$. We claim that $u(y) \geq u(x)$ if and only if $y \in \lambda^*(x)K' + (1-\lambda^*(x)) K$. To see this, suppose that $u(y) \geq u(x)$. Then it follows that $u(y) \geq 1$; in particular, this implies that either $y \in K'$ (in which case the result is obvious) or that $y \in K\backslash K'$. In the latter case, it is clear that $u(y) \geq u(x)$ implies that $\lambda^*(y) \geq \lambda^*(x)$.

First, suppose that $\lambda^*(x) = 0$. As $y \in K$, $y \in 0K' + (1-0) K$, establishing the result. Otherwise, $\lambda^*(x) > 0$. In this case, for all $\varepsilon < 1$,

$$\begin{aligned} y &\in \varepsilon\lambda^*(x)K' + (1-\varepsilon\lambda^*(x)) K \\ &= \varepsilon[\lambda^*(x)K' + (1-\lambda^*(x)) K] + (1-\varepsilon) K. \end{aligned}$$

Consequently

$$\frac{y}{\varepsilon} \in \lambda^*(x)K' + (1-\lambda^*(x)) K + \frac{(1-\varepsilon)}{\varepsilon} K.$$

But as $\lambda^*(x)K' + (1-\lambda^*(x)) K$ is upper comprehensive and all elements of $\frac{(1-\varepsilon)}{\varepsilon} K$ are elements of $\mathbb{R}^m_+$, we therefore conclude that

$$\frac{y}{\varepsilon} \in \lambda^*(x)K' + (1 - \lambda^*(x))\, K.$$

The result then follows by closedness of $\lambda^*(x)K' + (1 - \lambda^*(x))\, K$. Now, suppose that $y \in \lambda^*(x)K' + (1 - \lambda^*(x))\, K$. Then in particular, either $y \in K'$ (in which case $u(y) \geq 2 > u(x)$), or by definition $\lambda^*(y) \geq \lambda^*(x)$, in which case $u(y) \geq u(x)$.

Lastly, if $x \in K'$, define $\lambda^{**}(z) = \inf\{\lambda : z \notin \lambda K'\}$. Again, we claim that $u(y) \geq u(x)$ if and only if $y \in \lambda^{**}(x)K'$. But this is entirely analogous to the first part of the proof.

We have therefore succeeded in showing that u is both upper semicontinuous (in the sense of having closed weak upper sections) and quasiconcave. We also have established that $y \geq x \implies u(y) \geq u(x)$ (as $\mathcal{K}$ consists of upper comprehensive sets).

We now proceed to show that it is lower semicontinuous. To do so, we will show that u has open strict upper sections. First, suppose that $x \notin K$. We claim that $u(y) > u(x)$ if and only if $y \in \operatorname{int} u(x)K$. So, suppose that $u(y) > u(x)$. Then either $y \in K$, in which case the result is obvious, or $y \notin K$. However, note that by definition, $y \in \left[\frac{u(x)+u(y)}{2}\right] K$, consequently, $y \in \operatorname{int} u(x)K$ (as there are elements of $u(x)K$ which are strictly smaller than y and $u(x)K$ is upper comprehensive). Now, suppose that $y \in \operatorname{int} u(x)K$. Then either $y \in K$ (in which case $u(y) \geq 2 > u(x)$) or there exists $\varepsilon > 1$ for which $y \in \operatorname{int} \varepsilon u(x)K$, so that $u(y) \geq \varepsilon u(x) > u(x)$. (This follows as if $y \in \operatorname{int} u(x)K$, there exists some element strictly less than y in $u(x)K$; by choosing ε close enough to one, an ε multiple of this element is still strictly less than y).

Now suppose that $x \in K \backslash K'$. We claim that $u(y) > u(x)$ if and only if $y \in \operatorname{int} \lambda^*(x)K' + (1 - \lambda^*(x))\, K$. So suppose that $u(y) > u(x)$. It is clear that $u(x) < 2$ (as otherwise one could establish that $x \in K'$), so if $u(y) > u(x)$, then either $y \in K'$, in which case it is obvious that $y \in \operatorname{int} \lambda^*(x)K' + (1 - \lambda^*(x))\, K$, or $y \in K \backslash K'$. But again, by definition, we know that $y \in \left[\frac{\lambda^*(x)+\lambda^*(y)}{2}\right] K' + \left(1 - \frac{\lambda^*(x)+\lambda^*(y)}{2}\right) K$, from which we conclude that $y \in \operatorname{int} \lambda^*(x)K' + (1-\lambda^*(x))\, K$ (this follows as $K' \subset \operatorname{int} K$). Now, suppose that $y \in \operatorname{int} \lambda^*(x)K' + (1-\lambda^*(x))\, K$. Then either $y \in K'$, in which case $u(y) \geq 2 > u(x)$, or again, there exists some $\varepsilon > 1$ such that $y \in \operatorname{int} \varepsilon\lambda^*(x)K' + (1-\varepsilon\lambda^*(x))\, K$, in which case we establish that $\lambda^*(y) \geq \varepsilon\lambda^*(x) > \lambda^*(x)$, so that $u(y) > u(x)$.

The last case, in which $x \in K'$, is entirely analogous to the first, so we do not prove it here. We have therefore established that u has both open strict upper sections, and closed weak upper sections, so that it represents a continuous binary relation.

Lastly, let us show that u is monotonic in the sense that $y \gg x$ implies $u(y) > u(x)$. But this is also trivial; for example, in the case in which $x \notin K$, $x \in u(x)K$, and consequently $y \in \operatorname{int} u(x)K$, from which we establish that $u(y) > u(x)$ (using the results above). The results follow analogously for the case $x \in K \backslash K'$ and $x \in K'$.

Let $\succeq$ be the preference relation represented by u. Let w be on the boundary of K, and let w' be on the boundary of K'. It is clear to see that $U(\succeq, w) = K$ and $U(\succeq, w') = K'$. ■

Proposition 11 *The convex hull of $\mathcal{F}$ is disjoint from $C_{--}(\Delta(m))$.*

Proof. Let $f - g, f' - g' \in \mathcal{F}$. We first show that $(f + f') - (g + g') \in \mathcal{F}$. The structure of the proof is as follows. We know that $f \succeq^c g$ and $f' \succeq^c g'$. By strong aggregate independence and reinforcement, we will establish that $f + f' \succeq^c g + f'$ and $f' + g \succeq^c g' + g$. Transitivity will allow us to conclude $f + f' \succeq^c g + g'$, so that $(f + f) - (g + g') \in \mathcal{F}$.

In particular, by definition, there exist $e = (N, (\succeq_i)_{i \in N})$, $e' = (N', (\succeq_i)_{i \in N'})$ and $x, y, x', y' \in X^N$ for which $H(S(e, x), \cdot) = f(\cdot)$, $H(S(e, y), \cdot) = g(\cdot)$, $H(S(e', x'), \cdot) = f'(\cdot)$, and $H(S(e', y'), \cdot) = g'(\cdot)$, where $x \succeq^0 (e)y$ and $x' \succeq^0 (e')y'$ and $N \cap N' = \varnothing$ (that we may choose $N \cap N' = \varnothing$ follows from strong aggregate independence).

Let $e'' = (N \cup N', (\succeq_i)_{i \in N \cup N'})$. First, note that $H(S(e'', (y_N, y_{N'})), \cdot) = (g + g')(\cdot)$, and that $H(S(e'', (x_N, y_{N'})), \cdot) = (f + g')(\cdot)$. By reinforcement, $(x_N, y_{N'}) \succeq^0 (e'')(y_N, y_{N'})$, so that $f+g' \succeq^c g+g'$. Similarly, $H\left(S\left(e'',(x_N,x_{N'})\right),\cdot\right)=(f+f')\,(\cdot)$, and by reinforcement, $(x_N,x_{N'}) \succeq^0 (e'')(x_N,y_{N'})$, so that $f+f' \succeq^c f+g'$. As $\succeq^c$ is transitive, we conclude that $f+f' \succeq^c g+g'$. Consequently, $(f+f')-(g+g') \in \mathcal{F}$.

Now, we show that if $f-g \in \mathcal{F}$, then $(1/2)\,(f-g) \in \mathcal{F}$. The argument works as follows. As $f \succeq^c g$, there exists $e=(N,(\succeq_i)_{i\in N})$ and $x,y \in X^N$ and such that $f(\cdot)=H(S(e,x),\cdot)$, $g(\cdot)=H(S(e,y),\cdot)$, and $x \succeq^0 (e)y$. In particular, for all $i \in N$, either $U(\succeq_i,x_i) \subset \operatorname{int} U(\succeq_i,y_i)$, $U(\succeq_i,y_i) \subset \operatorname{int} U(\succeq_i,x_i)$, or $U(\succeq_i,x_i)=U(\succeq_i,y_i)$ (simply by definition of preference relations). We construct a preference $\succeq_i' \in \mathcal{R}$ and w_i,z_i such that $U(\succeq_i',w_i)=(1/2)\,U\left(\succeq_i,x_i\right)$ and $U(\succeq_i',z_i)=U\left(\succeq_i,y_i\right)$. That this is possible follows easily from Proposition 10, alternatively it can be seen as follows. For all $i \in N$, there exists a continuous utility representation $u_i : X \to \mathbb{R}$ of $\succeq_i$ (see for example Theorem 1 of Debreu [10]). Define $u_i'(x) \equiv u_i(2x)$ and let $w_i=(1/2)\,x_i$ and $z_i=(1/2)\,y_i$. In particular, u_i' is continuous and hence represents a continuous binary relation $\succeq_i'$; it is also obviously monotonic and quasiconcave (as u_i is). Note that $y \in U(\succeq_i',w_i)$ if and only if $u_i'(y) \geq u_i'(w_i)$, which is true if and only if $u_i(2y) \geq u_i(2w_i)=u_i(x_i)$, or if and only if $2y \in U(\succeq_i,x_i)$. But $2y \in U(\succeq_i,x_i)$ if and only if $y \in (1/2)\,U(\succeq_i,x_i)$. Consequently, $U(\succeq_i',w_i)=(1/2)\,U\left(\succeq_i,x_i\right)$. We may similarly verify that $U\left(\succeq_i',z_i\right)=(1/2)\,U\left(\succeq_i,y_i\right)$.

Let $e'=(N,(\succeq_i')_{i\in N})$. By completeness of $\succeq^0 (e')$, either $w \succeq^0 (e')z$ or $z \succ^0 (e')w$. However, note that $H(S(e',w),\cdot)=(1/2)f(\cdot)$ and $H(S(e',z),\cdot)=(1/2)g(\cdot)$. Thus, there are two cases: either $(1/2)f \succeq^c (1/2)g$ or $(1/2)g \succ^c (1/2)f$. In the second case, we establish, using the first argument of the proof of this Proposition and reinforcement (for strong preference), that $g \succ^c f$, contradicting the hypothesis that $f \succeq^c g$. Therefore, $(1/2)f \succeq^c (1/2)g$. Conclude that $\mathcal{F}$ is a cone which is closed under dyadic convex combinations.

Now, suppose that $\operatorname{conv}\mathcal{F} \cap C_{--}(\Delta(m)) \neq \varnothing$. Thus, there exists $\{f_1,...,f_K\} \subset \mathcal{F}$ and $\alpha \in \Delta(K)$ such that $\sum_{k=1}^{K} \alpha_k f_k(s) < 0$ for all $s \in \Delta(m)$. Let $\{\alpha^w\}_{w=1}^{\infty} \subset \Delta(K)$ be a series of dyadic rationals tending to α. We claim that for all $w \geq 1$, there exists $s^w \in \Delta(m)$ such that

$$\sum_{k=1}^{K} \alpha_k^w f_k(s^w) \geq 0.$$

Suppose false, so that there exists some α^w for which $\sum_{k=1}^{K} \alpha_k^w f_k(s) < 0$ for all $s \in \Delta(m)$. In particular, $\mathcal{F}$ is closed under dyadic rational combinations, so $\sum_{k=1}^{K} \alpha_k^w f_k \in \mathcal{F}$. Consequently, $\sum_{k=1}^{K} \alpha_k^w f_k = f - g$ for some $f \succeq^c g$, where $f \ll g$.

By Proposition 10, there exists $\preceq \in \mathcal{R}$ and $x, y \in X$ for which $f(\cdot) = H(U(\succeq, x), \cdot)$ and $g(\cdot) = H(U(\succeq, y), \cdot)$. In particular, in this case, $U(\succeq, y) \subset \operatorname{int} U(\succeq, x)$, from which we can conclude that $y \succ x$. Therefore, let $i \in \mathbb{N}$ and consider $e = (\{i\}, \succeq)$. By Pareto, $y \succ^0 (e)x$. But this latter statement implies that $g \succ^c f$, a contradiction. So, in fact, there exists $\{s^w\} \subset \Delta(m)$ for which $\sum_{k=1}^{K} \alpha_k^w f_k(s^w) \geq 0$. Now, simply let $s^* \in \Delta(m)$ be a cluster point of $\{s^w\}$ and note that by continuity of each f_k,

$$\sum_{k=1}^{K} \alpha_k f_k(s^*) \geq 0.$$

■

As $C_{--}(\Delta(m))$ has an internal point, Proposition 11 demonstrates the existence of a nonzero linear functional Π separating $\mathcal{F}$ from $C_{--}(\Delta(m))$ (see Aliprantis and Border 5.61 [1]). This hyperplane generated by the functional clearly passes through zero (as $C_{--}(\Delta(m))$ is a cone and $\mathcal{F}$ is closed under dyadic scalar multiplication). Assume that for all $f \in C_{--}(\Delta(m))$, $\langle f, \Pi \rangle \leq 0$ and for all $f \in \mathcal{F}$, $\langle f, \Pi \rangle \geq 0$. We claim that, in fact, for all $f \in C_{--}(\Delta(m))$, $\langle f, \Pi \rangle < 0$. First, note that there exists some $f^* \in C_{--}(\Delta(m))$ for which $\langle f^*, \Pi \rangle < 0$; otherwise, $\langle f, \Pi \rangle = 0$ for all $f \in C_{--}(\Delta(m))$, from which we conclude that Π is a zero functional (because all $g \in C(\Delta(m))$ can be expressed as $g = h - h'$ for $h, h' \in C_{--}(\Delta(m))$), which is a contradiction.

Now, for all $f \in C_{--}(\Delta(m))$, there exists $\alpha > 0$ small enough so that $\alpha f^* < f$ (this follows as $\inf_{s \in \Delta(m)} f(s) > 0$), so that $f = \alpha f^* + (f - \alpha f^*)$, so $\langle f, \Pi \rangle = \langle \alpha f^*, \Pi \rangle +$

$\langle f - \alpha f^*, \Pi \rangle < 0$, as $f - \alpha f^* \in C_{--}(\Delta(m))$.

Consequently, for all $f \in C_{--}(\Delta(m))$, $\langle f, \Pi \rangle < 0$.

We now claim that for all $f \leq 0$, $\langle f, \Pi \rangle \leq 0$. So let $f \leq 0$. Let $g \in C_{--}(\Delta(m))$ and note that $g + f \leq g$. Let $\alpha < 1$; then in particular, $g + f \ll \alpha g$, so that $\langle g + f, \Pi \rangle < \langle \alpha g, \Pi \rangle = \alpha \langle g, \Pi \rangle$. As α is arbitrary, we conclude that $\langle g + f, \Pi \rangle \leq \langle g, \Pi \rangle$, so that $\langle f, \Pi \rangle \leq 0$. Consequently, Π is monotonic.

By the Reisz Representation Theorem (Theorem 14.12 of Aliprantis and Border [1]), there exists a nonnegative, countably additive measure π representing Π which can be normalized to be a probability measure by rescaling, for which if $f \succeq^c g$, then

$$E_\pi [f] \geq E_\pi [g].$$

We show now that if $f \succ^c g$, then $E_\pi [f] > E_\pi [g]$. To see this, we first suppose without loss of generality that $f, g \gg 0$. For, we know that $f, g \geq 0$; and if they are not both strictly positive, then we may choose some $h \gg 0$, and observe that by reinforcement, if $f \succ^c g$, then $f + h \succ^c g + h$; where $f + h, g + h \gg 0$. If we then establish that $E_\pi [f + h] > E_\pi [g + h]$, it follows that $E_\pi [f] > E_\pi [g]$. So assume without loss that $f, g \gg 0$ and that $f \succ^c g$. In particular, let $N = \{1, 2\}$, and let $\succeq_1, \succeq_2 \in \mathcal{R}$ be homothetic and let $x_1, x_2 \in X$ for which $H(U(\succeq_1, x_1), \cdot) = f(\cdot)$ and $H(U(\succeq_2, x_2), \cdot) = g(\cdot)$, and let $e = \left(N, (\succeq_i)_{i \in N}\right)$ (That this is possible follows by Proposition 2); clearly $x_1 \gg 0$. Then $f(\cdot) = H(S(e, (x_1, 0)), \cdot)$ and $g(\cdot) = H(S(e, (0, x_2)), \cdot)$ (for example, by Proposition 5). Consequently, $(x_1, 0) \succ^0 (e)\, (0, x_2)$. In particular, by continuity, we may choose ε small enough so that $(x_1 - \varepsilon \mathbf{1}, 0) \succ^0 (e)\, (0, x_2)$.[14] By Pareto, $(x_1, 0) \succ^0 (e)\, (x_1 - \varepsilon \mathbf{1}, 0)$; and further

[14]Here, $\mathbf{1}$ denotes a vector of ones.

as $\succeq_1$ is homothetic, $H(S(e,(x_1-\varepsilon\mathbf{1},0)),\cdot) \ll H(S(e,(x_1,0)),\cdot)$. Consequently,

$$\begin{aligned} E_\pi[f] &= E_\pi[H(S(e,x),\cdot)] \\ &> E_\pi[H(S(e,(x_i-\varepsilon,x_{-i})),\cdot] \\ &\geq E_\pi[H(S(e,y),\cdot)] = E_\pi[g]. \end{aligned}$$

The strict inequality follows from strict positivity of π.

Lastly, let $e = (N,(\succeq_i)_{i\in N}) \in \mathcal{E}$, and let $x,y \in X^N$. Then $x \succeq^0 (e)y$ implies $H(S(e,x),\cdot) \succeq^c H(S(e,y),\cdot)$, which in turn implies $E_\pi[H(S(e,x),\cdot)] \geq E_\pi[H(S(e,x),\cdot)]$. Moreover,

$$\begin{aligned} &\int_{\Delta(m)} \sum_N U^p_{\succeq_i}(x_i)d\pi(p) \\ &= E_\pi[\sum_N H(U(\succeq_i,x_i),\cdot)] \\ &= E_\pi[H(S(e,x),\cdot)], \end{aligned}$$

where the first equality follows from Proposition 8. So in fact, $\int_{\Delta(m)}\sum_N U^p_{\succeq_i}(x_i)d\pi(p) \geq \int_{\Delta(m)}\sum_N U^p_{\succeq_i}(y_i)d\pi(p)$. A similar statement holds for strict preference.

Lastly, we establish that $\pi(\Delta(m)\backslash\Delta_{++}(m)) = 0$.

Let $h:\mathbb{R}\to[0,1]$ be a monotonic and continuous function for which $h(\alpha)<1$ for all $\alpha<1$ and $h(\alpha)=1$ for all $\alpha\geq 1$. Define $u: X\to\mathbb{R}$ by

$$u(x) = \inf\left\{\alpha : \left(h(\alpha)1_j + (1-h(\alpha))\frac{1}{m}\mathbf{1}\right)\cdot x \geq \alpha \text{ for all } j\right\}.$$

Here, $1_j \in X$ is one unit of commodity j, and zero units of all remaining commodities. It can be verified that u is continuous (it is in fact a consequence of the Maximum Theorem of Berge [4]) and monotonic. Convexity follows as well by construction. Let $\succeq^*\in\mathcal{R}$ be the

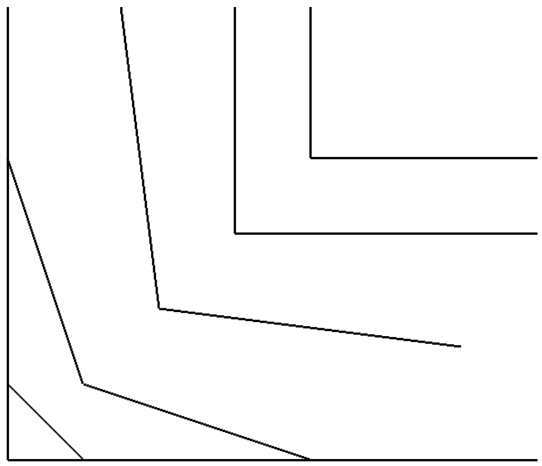

Figure 2: The preference $\succeq^{*}$

preference represented by u. The important fact here is that for all $p \in \Delta(m) \backslash \Delta_{++}(m)$ and all $\alpha < 1$, $U^{p}_{\succeq^{*}}(\alpha \mathbf{1}) = 0$, and for all $p \in \Delta(m)$, $U^{p}_{\succeq^{*}}(\mathbf{1}) = 1$. Figure 2 displays an example of such a preference.

Now, let $\alpha^{\nu} < 1$ be a sequence such that $\alpha^{\nu} \to 1$. Then

$$\begin{aligned} \int_{\Delta(m)} U^{p}_{\succeq^{*}}(\alpha^{\nu}\mathbf{1})\, d\pi(p) &= \int_{\Delta(m)\backslash\Delta_{++}(m)} U^{p}_{\succeq^{*}}(\alpha^{\nu}\mathbf{1})\, d\pi(p) + \int_{\Delta_{++}(m)} U^{p}_{\succeq^{*}}(\alpha^{\nu}\mathbf{1})\, d\pi(p) \\ &= \int_{\Delta_{++}(m)} U^{p}_{\succeq^{*}}(\alpha^{\nu}\mathbf{1})\, d\pi(p). \end{aligned}$$

We can show (as we did in verifying that $\succeq^{0}(e)$ is continuous for all $e \in \mathcal{E}$), that

$$\int_{\Delta_{++}(m)} U^{p}_{\succeq^{*}}(\alpha^{\nu}\mathbf{1})\, d\pi(p) \to \int_{\Delta_{++}(m)} U^{p}_{\succeq^{*}}(\mathbf{1})\, d\pi(p) = \pi(\Delta_{++}(m)).$$

Now, let $\succeq^{**} \in \mathcal{R}$ be a Leontief preference.[15] Let $e = \left(\{1,2\}, (\succeq_i)_{i \in \{1,2\}}\right)$, where $\succeq_1 = \succeq^*$ and $\succeq_2 = \succeq^{**}$. Let $x^\nu \in X^N$ be given by

$$x^\nu = \left(\alpha^\nu \mathbf{1}, \left(1 - \int_{\Delta_{++}(m)} U^p_{\succeq^*}(\alpha^\nu \mathbf{1})\, d\pi(p)\right) \mathbf{1}\right).$$

Note that for all $p \in \Delta(m)$,

$$U^p_{\succeq^{**}}\left(\left(1 - \int_{\Delta_{++}(m)} U^p_{\succeq^*}(\alpha^\nu \mathbf{1})\, d\pi(p)\right) \mathbf{1}\right) = 1 - \int_{\Delta_{++}(m)} U^p_{\succeq^*}(\alpha^\nu \mathbf{1})\, d\pi(p).$$

Now, $F^\pi(e)(x^\nu) = 1$ for all ν. Further, note that $x^\nu \to (\mathbf{1}, (1 - \pi(\Delta_{++}(m)))\mathbf{1})$. Consequently, as $x^\nu \sim^0 (e)\, x^{\nu'}$ for all ν, ν'; and since $\succeq^0 (e)$ is continuous, it follows that $F^\pi(e)(\mathbf{1}, (1 - \pi(\Delta_{++}(m)))\mathbf{1}) = 1$ as well; but an explicit calculation obtains:

$$F^\pi(e)(\mathbf{1}, (1 - \pi(\Delta_{++}(m)))\mathbf{1}) = 1 + (1 - \pi(\Delta_{++}(m))).$$

Consequently, $1 + (1 - \pi(\Delta_{++}(m))) = 1$, or $\pi(\Delta_{++}(m)) = 1$.

6 Discussion and concluding remarks

6.1 On the ethical value of money metric utilitarianism

Obviously, the main problem with money metric utilitarianism as a social welfare indicator lies in its complete lack of ethical considerations. This point has been recognized by many authors. Samuelson [36] himself cautions against attributing any ethical value to the rule:

Whatever the merits of the money-metric utility concept developed here, a warn-

[15] So that it has utility representation

$$u(x) = \min_{j \in \{1,\dots,m\}} x_j.$$

ing must be given against its misuse. Since money can be added across people, those obsessed by Pareto-optimality in welfare economics as against interpersonal equity may feel tempted to add money-metric utilities across people and think that there is ethical warrant for maximizing the resulting sum. That would be an illogical perversion, and any such temptation should be resisted.

In this regard, aggregate independence has a dual interpretation, which can be understood as formally confirming the intuition of Samuelson. In fact, the axiom can almost be viewed as a requirement that the *distribution* of resources is typically *completely irrelevant* for the application of this rule. Unless one can infer something about the distribution of resources from the Scitovsky set and aggregate bundle under consideration (typically a very difficult–often impossible–thing to do), then aggregate independence rules out consideration of distributional issues. For a simple example when distribution of resources is irrelevant for this rule, consider a two agent economy, where the two agents have identical linear preferences. Suppose we wish to rank all non-wasteful allocations of some aggregate bundle among the two agents. All such allocations have the same Scitovsky set and thus must be considered indifferent.

6.2 Alternative methods of choosing allocations

We study a particular condition on the ranking of economic allocations. An alternative approach is to simply recommend an allocation for a given economy, subject to some constraints. The approach followed here allows us to make recommendations for broad classes of allocation problems. That is, let $e \in \mathcal{E}$, and suppose that $A \subset X^N$ is a set of allocations. We may define an allocation rule by:

$$\arg\max_{x \in A} F^{\pi}(e)(x).$$

Allocation rules are slightly more general as they need not be generated by the maximization of any function.

6.3 Optimal allocations

In the case when there is some fixed amount of resources to divide among an economy, it is natural to ask what is the optimal allocation. This is usually difficult to determine, except in the case of a non-random money-metric utilitarian rule. If the rule is non-random, then there exists an associated "price," p. The maximal elements are then those efficient allocations with supporting prices p (when they exist). Again, if such efficient allocations do not exist, the rule recommends significantly more complicated allocations.

6.4 Choosing the right probability

While the money-metric utilitarian rules depend on a parameter which can be interpreted as a probability distribution over prices, this interpretation is not implied by the data of the problem. The probability can be tied to the outside world, but it need not be. In fact, it need not have any significance in terms of observable data at all. Presumably the probability measure should be chosen carefully, to provide rankings which are either intuitive in specific contexts, or which allow other general normative properties to be satisfied.

6.5 Allowing more information into the model

We finish with a remark on what to do when there is more information available than just the Scitovsky upper contour sets of the allocations. The current work is to be understood as a first step in considering aggregate data in social choice. However, one can easily imagine obtaining the Scitovsky upper contour sets of different sectors of the economy. For example, one might have a set of **types** $\mathcal{T}$, and a mapping associating a type to each agent in an economy. Aggregate independence could then be reformulated so that if the Scitovsky upper contour sets and aggregate allocations corresponding to different types are indistinguishable across two economies and corresponding allocations, the rankings should not change either.[16] The results would not largely change in this environment; namely, for each type, we would have a corresponding probability measure π_t and a positive weight $\lambda(t)$ such that, letting $t(i)$ be the type of agent i in economy e, $F(e)(x)$ could be expressed as:

$$\sum_N \lambda_{t(i)} \int U^p_{\succeq_i}(x_i)\, d\pi_{t(i)}(p).$$

The only requirement to obtain such a result is that types not be correlated with consumption; that is, any type would be allowed to consume any amount (at least hypothetically). Hence, it may not make sense to consider "poor" and "wealthy" agents as distinct types.

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