

# Ordinal Aggregation and Quantiles

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## Abstract

Consider the problem of aggregating a profile of interpersonally comparable utilities into a social utility. We require that the units of measurement of utility used for agents is the same as the units of measurement for society (ordinal covariance) and a mild Pareto condition (monotonicity). We provide several representations of such social aggregation operators: a canonical representation, a Choquet expectation representation, a minimax representation, and a quantile representation (with respect to a possibly non-additive set function on the agents). We also isolate an additional condition that gives us a quantile representation with respect to a probability measure, in both the finite and infinite agents case. Keywords: Ordinal, quantile, axiom, Choquet integral, probability, Rawlsian, social welfare. JEL classification: D31, D63, D71, D81.

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# 1 Introduction

Consider the problem of aggregating the utilities of a collection of agents into a social utility. Utilities are assumed to have ordinal content (both across individuals and alternatives), but utility itself only has meaning as a cardinal index of this ordinal ranking. To this end, aggregation of individual utilities should be independent of the index in question. Thus, the common unit of measurement of utility for the agents should also be the unit of measurement for the social utility.

A utility profile is a mapping carrying agents into the reals, say  $f$ . This utility profile represents the utility each individual in society derives from some social alternative. A social utility can be arrived at through some aggregation operator, which in this paper we will refer to as  $T$ . Our main criterion for this social utility is that it should be equivariant with respect to arbitrary continuous transformations of utilities. Thus, suppose that  $\varphi$  is strictly increasing and continuous. We will require that  $\varphi(Tf) = T(\varphi \circ f)$ .  $Tf$  is the social utility in the original measurement units;  $\varphi(Tf)$  is the transformed social utility.  $T(\varphi \circ f)$  is the social utility of transformed individual utilities. This captures requirement that the index of utility used for the agents should be the same as the scale used for society, but that utility otherwise has no cardinal content. We will refer to this property of  $T$  as *ordinal covariance*.

A social utility should minimally also satisfy some Pareto-like criterion. For two utility profiles,  $f$  and  $g$ , if  $f$  is everywhere less or equal to  $g$ ,  $Tf$  should be less than or equal to  $Tg$ . We will refer to this property as *monotonicity*.<sup>1</sup>

The canonical example of operators satisfying the two properties is the *quantile*. The quantile generalizes the notion of median, and makes sense when agents are given weights which sum to one. Fix a number  $\alpha$  in between zero and one. The  $\alpha$ -quantile of a function returns the smallest value of the function for which the set of agents receiving at most that value has weight less than or equal to  $\alpha$ . A median is simply a 1/2-quantile. The quantile is one of the most common statistics of wealth distribution.

Given the prevalence of quantiles in the statistical and economic literature, some motivation is needed for the departure from this classical example. We submit that a general theory of ordinally covariant and monotonic functionals allows us to analyze important economic concepts, such as fairness. Consider an environment in which there are agents of two types. Type A agents are historically and systematically discriminated against relative to type B agents in terms of income. A statistic whose maximization would tend to equate the median incomes of the two groups is the minimum of the median incomes of the two groups. This statistic is indeed ordinally covariant and monotonic, but it is *not a quantile with respect to a probability measure*.

We provide a general representation theorem for all ordinally covariant and monotonic functionals. We should clarify that we do not work with a finite

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<sup>1</sup>Note that by working with utility profiles, we implicitly assume *welfarism*—the only characteristics of social alternatives relevant in making social decisions are the welfares of the agents. This is a type of Pareto-indifference criterion.

set of agents, but rather an arbitrary measurable space. The intuition for our characterization result actually comes from several important theorems from the social choice literature. Theorems found in [2, 10, 11, 18, 19] use concepts similar to ordinal covariance to characterize leximin rankings and rank-order dictatorships (see also Sprumont [25] for a related condition in the cost-sharing literature).

We show that any ordinally covariant and monotonic functional is characterized by a particular collection of coalitions of agents. In the case of quantiles, these coalitions are those which have weight weakly less than  $\alpha$ . In the general case, it is any family of coalitions satisfying two conditions: *i*) the collection contains the empty coalition and does not contain the grand coalition, and *ii*) if a coalition is contained in the collection, then so is any subcoalition. The social utility then finds the smallest utility such that the set of agents receiving at least that utility is contained in the collection.

This is our canonical representation. However, there are several other natural representations. One is a type of expectation operator. Any such aggregation operator is the expectation operator associated with a possibly non-additive probability (a capacity) which takes values in only zero and one. A capacity thus weights coalitions with either zero or one (the weights here are non-additive). Such an expectation operator is referred to as a Choquet integral. This class of operators was axiomatized in the mathematics literature by Schmeidler [23]. Here, our axiomatization singles out a subclass of these operators.

Aside from the utilitarian operator, the most important social choice operator is the maxmin, or Rawlsian [17], operator. The maxmin operator fits neatly into our framework. We provide a “generalized Rawlsian” representation. Fix any family of coalitions. Given a utility profile, for every coalition in the family, find the utility of the agent who is most well-off. Then, take the smallest such utility across all coalitions in the family. Such a functional is ordinally covariant and monotonic, and any ordinally covariant and monotonic function can be so represented. We therefore have a foundation for a type of Rawlsianism across coalitions. We should point out that this particular representation of this class of functionals is provided in the mathematics literature for the finite agents case (for example, see Marichal and Mathonet [14] and Marichal [15]). However, the proof of this characterization theorem in the mathematics literature critically relies on the fact that the underlying function space is finite-dimensional. Our main theorem yields this result as a corollary in the general case.

Lastly, we return to quantiles. We show that any such aggregation operator can also be characterized as a quantile, but not necessarily as a quantile with respect to a probability measure. An aggregation operator satisfies ordinal covariance and monotonicity if and only if there exists some capacity  $\nu$  and some  $\alpha < 1$  for which the operator finds the smallest value for which the set of agents receiving at least this utility has weight weakly less than  $\alpha$ .

A natural and important question to ask is when does there exist a quantile representation with respect to a probability measure? If we can assign a probability measure to agents, then we can without loss of generality talk about a “utility distribution,” and work with cumulative distribution functions. As

far as we know, nobody has used quantiles with respect to non-additive probabilities in empirical analysis; therefore, it is important to understand exactly the implicit assumptions being made when probabilistic quantiles are used. For finite sets of agents, this is easy enough to do using simple duality results. We provide a condition, which we call betting consistency, that allows us to write the aggregation operator as a quantile with respect to some probability measure. This type of condition is well-known, especially in the theory of simple games [26]. The general (infinite agents) case is more difficult to answer and the axiom ( $\sigma$ -betting consistency) used to characterize those aggregation operators which are probabilistic quantiles is more difficult to describe. However, this condition and characterization, as far as we know, is novel.

There is a connection between our work and a series of classical papers by Bickel and Lehmann [3, 4, 5, 6]. These papers are concerned with functions mapping from random variables to real numbers satisfying several invariance properties. Although they mention our ordinal covariance axiom, ([4], p. 1048, property (v)), they provide no characterizations based on it. They do, however, utilize a monotonicity condition throughout their works, although it is somewhat stronger than ours. They are interested in the case in which a probability measure is exogenously specified, and identify any two random variables which possess the same induced distribution. This is a departure from our methodology.

Lastly, it is important to point out the recent work of Rostek [21]. She works with order structures defined on function spaces which need not be real-valued, but take values in some abstract space, interpreted as Savage-style acts [22]. She provides a Savage-style representation theorem for quantile maximizers in a decision-theoretic framework. Of primary concern in her work is the notion of “probabilistic sophistication.” A decision maker in her framework is a quantile maximizer if there is an ordinally unique utility index over outcomes, and a unique probability measure over the underlying state space, for which the utility of an act is some  $\alpha$ -quantile of the utility index. Her concern is not with the axioms of ordinal covariance and monotonicity, but rather with the complete testable implications of quantile maximization. Indeed, her axiomatic system differs significantly from ours and produces representation theorems of a very different type. The main similarity is that our works are perhaps the first two dealing with axiomatic characterizations of quantile related subjects.

Section 2 provides the model and representation theorems. Section 3 concludes.

## 2 Representations for ordinally covariant and monotonic functionals

Let  $(\Omega, \Sigma)$  be a measurable space, and let  $B(\Omega, \Sigma)$  be the vector space of real-valued, bounded,  $\Sigma$  measurable functions. The set  $\Omega$  can be thought of as representing a set of agents, and  $B(\Omega, \Sigma)$  might represent a set of possible utility

profiles. We will study functionals (aggregation operators)  $T : B(\Omega, \Sigma) \rightarrow \mathbb{R}$ .

We wish to characterize those operators satisfying the following two conditions:

**Ordinal covariance:** For all  $f \in B(\Omega, \Sigma)$  and all strictly increasing and continuous  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ ,  $T(\varphi \circ f) = \varphi(T(f))$ .

**Monotonicity:** For all  $f, g \in B(\Omega, \Sigma)$ , if  $f \leq g$ , then  $Tf \leq Tg$ .

First, we provide the following lemma:

**Lemma 1:** Suppose that  $T$  satisfies ordinal covariance. Then for all  $E \in \Sigma$ ,  $T1_E \in \{0, 1\}$ .

**Proof.** Let  $E \in \Sigma$ . Let  $\varphi$  be defined as

$$\varphi(x) \equiv \begin{cases} \frac{3x}{2} & \text{for } x < 1/2 \\ \frac{x+1}{2} & \text{for } x \geq 1/2 \end{cases}.$$

Note that  $\varphi(0) = 0$  and  $\varphi(1) = 1$ . Hence,  $\varphi \circ 1_E = 1_E$ . By ordinal covariance, conclude  $\varphi(T1_E) = T(\varphi \circ 1_E) = T1_E$ . But the only fixed points of  $\varphi$  are 0 and 1. Hence,  $T1_E \in \{0, 1\}$ . ■

Note that the only transformation  $\varphi$  used in this lemma is piecewise linear in two pieces and concave.

## 2.1 The primary representation

Say a collection of sets  $\mathcal{E} \subset \Sigma$  is a **downset** if  $A \in \mathcal{E}$  and  $B \subset A$  and  $B \in \Sigma$  implies that  $B \in \mathcal{E}$ .

Our main representation theorem characterizes ordinally covariant and monotonic functionals in the following way: there exists a downset for which for every  $f \in B(\Omega, \Sigma)$ ,  $Tf$  returns the smallest value  $x$  for which the weak upper contour set of  $x$  according to  $f$  lies in the downset. The proof is relatively straightforward. If a functional satisfies the two axioms, then one may construct its corresponding downset as those sets whose indicator functions return a value of zero. From here, it is a matter of using monotonicity to “squeeze” any arbitrary function between two scaled indicator functions whose values under  $T$  are known, so that the monotonicity axiom can be applied.

**Theorem 1:** A functional  $T$  satisfies ordinal covariance and monotonicity if and only if there exists a unique downset  $\mathcal{E} \subset \Sigma$  such that  $\emptyset \in \mathcal{E}$  and  $\Omega \notin \mathcal{E}$  for which

$$Tf = \inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}.$$

**Proof.** Suppose first that the functional  $T$  can be represented as  $Tf = \inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}$  for some downset  $\mathcal{E}$ . To see that it is ordinally

covariant, let  $f \in B(\Omega, \Sigma)$  and let  $\varphi$  be strictly increasing and continuous. Then

$$\begin{aligned}
\varphi(Tf) &= \varphi(\inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}) \\
&= \inf \{\varphi(x) : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\} \\
&= \inf \{x : \{\omega : f(\omega) \geq \varphi^{-1}(x)\} \in \mathcal{E}\} \\
&= \inf \{x : \{\omega : \varphi(f(\omega)) \geq x\} \in \mathcal{E}\} \\
&= T(\varphi \circ f).
\end{aligned}$$

To see that it is monotonic, suppose that  $f \geq g$ . For all  $x \in \mathbb{R}$ ,  $\{\omega : g(\omega) \geq x\} \subset \{\omega : f(\omega) \geq x\}$ . Hence  $\{\omega : f(\omega) \geq x\} \in \mathcal{E}$  implies  $\{\omega : g(\omega) \geq x\} \in \mathcal{E}$ . This implies that  $\inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\} \geq \inf \{x : \{\omega : g(\omega) \geq x\} \in \mathcal{E}\}$ , or that  $Tf \geq Tg$ .

Conversely, suppose that  $T$  is ordinally covariant and monotonic. Define the family  $\mathcal{E} \equiv \{E \in \Sigma : T1_E = 0\}$ . By Lemma 1, for all  $E \in \Sigma$ ,  $T1_E \in \{0, 1\}$ . Denoting the constant function taking value 0 everywhere as 0, it is clear that  $T0 = 0$  (this follows as  $T0 = T(a0) = aT0$  for any  $a > 0$ ). Moreover, from ordinal covariance, we may also conclude therefore that  $T1 = 1$ . As  $T1_\emptyset = T0 = 0$ , and as  $T1_\Omega = T1 = 1$ ,  $\emptyset \in \mathcal{E}$  and  $\Omega \notin \mathcal{E}$ . Further, if  $A \in \mathcal{E}$  and  $B \subset A$ , then  $1_B \leq 1_A$ , so that  $T1_B \leq T1_A = 0$ , from whence we conclude that  $B \in \mathcal{E}$ . Hence  $\mathcal{E}$  is a downset.

We verify that for all  $E \in \Sigma$ ,  $T1_E = \inf \{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\}$ . Let  $E \in \mathcal{E}$ . For all  $x > 0$ ,  $\{\omega : 1_E(\omega) \geq x\} \subset E$ , so that  $\{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}$ . Therefore,  $\inf \{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\} \leq 0$ . However, for all  $x < 0$ ,  $\{\omega : 1_E(\omega) \geq x\} = \Omega \notin \mathcal{E}$ . Hence, we may conclude that  $\inf \{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\} = 0$ , so that  $T1_E = \inf \{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\}$ . Suppose instead that  $E \notin \mathcal{E}$ . Then for all  $x > 1$ ,  $\{\omega : 1_E(\omega) \geq x\} = \emptyset$ , so that  $\inf \{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\} \leq 1$ . But for all  $x < 1$ ,  $E \subset \{\omega : 1_E(\omega) \geq x\}$ , so that  $\{\omega : 1_E(\omega) \geq x\} \notin \mathcal{E}$ . Hence  $\inf \{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\} = 1$ . Therefore,  $T1_E = \inf \{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\}$ . Next, we extend the result from indicator functions to all functions. Uniqueness of  $\mathcal{E}$  is obvious from this paragraph.

Let  $f \in B(\Omega, \Sigma)$  be arbitrary, and set  $x^*(f) = \inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}$ . Let  $\varepsilon > 0$  be arbitrary. Then  $\{\omega : f(\omega) \geq x^*(f) + \varepsilon\} \in \mathcal{E}$  by definition of  $x^*(f)$ . Let  $g^\varepsilon \in B(\Omega, \Sigma)$  be defined as

$$g^\varepsilon \equiv \begin{cases} \sup f \text{ for } \omega : f(\omega) \geq x^*(f) + \varepsilon \\ x^*(f) + \varepsilon \text{ otherwise} \end{cases}.$$

Then  $f \leq g^\varepsilon$ , so that by monotonicity,  $Tf \leq Tg^\varepsilon$ . Note that  $\{\omega : f(\omega) \geq x^*(f) + \varepsilon\} \in \mathcal{E}$ , so that  $\{\omega : g^\varepsilon(\omega) \geq \sup f\} \in \mathcal{E}$ . As  $g^\varepsilon$  is an ordinal transformation of the indicator function of  $\{\omega : f(\omega) \geq x^*(f) + \varepsilon\}$ , we may conclude  $Tg^\varepsilon = x^*(f) + \varepsilon$ . As  $\varepsilon$  is arbitrary,  $Tf \leq x^*(f)$ .

Let  $\varepsilon > 0$  be arbitrary. Let  $h^\varepsilon \in B(\Omega, \Sigma)$  be defined as

$$h^\varepsilon \equiv \begin{cases} \inf f \text{ for } \omega : f(\omega) < x^*(f) - \varepsilon \\ x^*(f) - \varepsilon \text{ otherwise} \end{cases}.$$

Then  $f \geq h^\varepsilon$ . Moreover,  $\{\omega : f(\omega) \geq x^*(f) - \varepsilon\} \notin \mathcal{E}$ . But  $f(\omega) \geq x^*(f) - \varepsilon$  if and only if  $h^\varepsilon(\omega) \geq x^*(f) - \varepsilon$ . Therefore,  $\{\omega : h^\varepsilon(\omega) \geq x^*(f) - \varepsilon\} \notin \mathcal{E}$ . As  $h^\varepsilon$  is an ordinal transformation of the indicator function of  $\{\omega : f(\omega) \geq x^*(f) - \varepsilon\}$ , we may conclude  $Th^\varepsilon = x^*(f) - \varepsilon$ . By monotonicity,  $Tf \geq x^*(f) - \varepsilon$ . As  $\varepsilon$  is arbitrary,  $Tf \geq x^*(f)$ .

Therefore  $Tf = \inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}$ . ■

- With this representation, we can easily characterize subfamilies of functionals satisfying additional axioms. For example, suppose that  $T$  satisfies **negation covariance**, so that for all  $f \in B(\Omega, \Sigma)$ ,  $T(-f) = -Tf$ . This additional axiom is satisfied if and only if the associated downset  $\mathcal{E}$  is **strong**: so that  $E \in \mathcal{E}$  if and only if  $\Omega \setminus E \notin \mathcal{E}$ . Suppose that  $T$  is **subadditive**, so that for all  $f, g \in B(\Omega, \Sigma)$ ,  $T(f + g) \leq Tf + Tg$ . This additional axiom is satisfied if and only if the associated downset  $\mathcal{E}$  is an **ideal**, so that  $E, F \in \mathcal{E} \implies E \cup F \in \mathcal{E}$ .
- $T$  need not be covariant with respect to all continuous and increasing functions; indeed, it is enough that  $T$  is covariant with respect to all affine and increasing functions and *at least one* function which is increasing, piecewise linear (in two pieces), and concave. Requiring invariance under only strictly increasing affine functions is not enough to prove the characterization theorem.
- For a downset  $\mathcal{E}$ ,

$$\inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\} = \inf \{x : \{\omega : f(\omega) > x\} \in \mathcal{E}\}.$$

There is no distinction between working with strict upper contour sets of  $f$  and weak upper contour sets of  $f$ . Moreover, a similar representation in terms of lower contour sets of functions is also possible.

Following are some examples:

**Example 1 (Dictatorship):** Fix any agent  $\omega^* \in \Omega$  and define  $Tf \equiv f(\omega^*)$ . Then it is clear that  $T$  is both ordinally covariant and monotonic. The associated downset is  $\mathcal{E} \equiv \{E \in \Sigma : \omega^* \notin E\}$ .

**Example 2 (Rawlsian rules):** Another example is that of the supremum, whereby  $Tf \equiv \sup_\omega f(\omega)$ , resulting when  $\mathcal{E} \equiv \{\emptyset\}$ . The infimum,  $Tf \equiv \inf_\omega f(\omega)$ , results when  $\mathcal{E} \equiv \Sigma \setminus \Omega$ . The essential supremum on a probability space  $(\Omega, \Sigma, p)$  results when  $\mathcal{E} \equiv \{E \in \Sigma : p(E) = 0\}$ . The essential infimum results when  $\mathcal{E} \equiv \{E \in \Sigma : p(E) < 1\}$ .

**Example 3 (Rawlsian rules for an oligarchy):** For any fixed  $F \in \Sigma$ ,  $Tf \equiv \sup_{\omega \in F} f(\omega)$  results from  $\mathcal{E} \equiv \{E \in \Sigma : E \subset \Omega \setminus F\}$ . Further, for any fixed  $F \in \Sigma$ ,  $Tf \equiv \inf_{\omega \in F} f(\omega)$  results from  $\mathcal{E} \equiv \{E \in \Sigma : (\Omega \setminus E) \cap F \neq \emptyset\}$ .

**Example 4 (Rawlsian rules for “invisible,” or limiting oligarchies):**

Suppose that  $\Omega \equiv \mathbb{N}$  and that  $\Sigma \equiv 2^{\mathbb{N}}$ . Define  $Tf \equiv \liminf_{n \rightarrow \infty} f(n)$ . Then it is clear again that  $T$  satisfies each of our axioms. Then  $\mathcal{E} \equiv \{E \in \Sigma : |\Omega \setminus E| = +\infty\}$ . Also, the functional  $Tf \equiv \limsup_{n \rightarrow \infty} f(n)$  satisfies each of our axioms. Then  $\mathcal{E} \equiv \{E \in \Sigma : |E| < +\infty\}$ .

## 2.2 An integral representation

A **capacity** is a set function  $v : \Sigma \rightarrow \mathbb{R}$  such that  $v(\emptyset) = 0$ ,  $v(\Omega) = 1$ , and  $A \subset B$  implies  $v(A) \leq v(B)$ . Capacities need not be additive. Capacities are often used to represent subjective notions of belief. In particular, Schmeidler [24] pioneered this idea in economics. Our results bear little resemblance to the Schmeidler axiomatizations of decision making under uncertainty; he uses a type of additivity within certain subclasses of functions, which translates into a weak version of the independence axiom of decision theory when applied to choices over acts. The independence axiom there is only required to apply across acts which, informally speaking, feature similar uncertainty. One might say that it applies only across acts for which subjective ambiguity should not reveal itself.

Our ordinal covariance axiom, on the other hand, cannot be similarly translated into an independence condition. However our results could easily be reformulated to construct a decision theoretic model.<sup>2</sup> However, it is not clear that such an axiomatization would have anything at all to do with subjective ambiguity. One would be justified in saying that the main distinction between Schmeidler’s axiomatization of the Choquet integral and our axiomatization is that Schmeidler’s axiomatization relies on an additivity condition which implies certain invariance conditions, whereas ours relies on a very strong invariance condition alone.

Any ordinally covariant and monotonic functional  $T$  is a Choquet integral with respect to a particular type of capacity. Hence,

$$Tf = \int_{\Omega} f(\omega) d\nu(\omega),$$

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<sup>2</sup>A natural way of constructing such a decision theory presents itself, for preferences  $\succeq$  over acts taking values in some abstract space  $X$ . The substantive axioms required are as follows. First, we would need a weak statewise dominance condition: if  $f(\omega) \succeq g(\omega)$  for all  $\omega$ , then  $f \succeq g$ . To translate our ordinal covariance condition, we would postulate that for any four acts,  $f_1, f_2, g_1, g_2$ , if for all  $\omega, \omega' \in \Omega$  and all  $i, j \in \{1, 2\}$

$$f_i(\omega) \succeq f_j(\omega') \iff g_i(\omega) \succeq g_j(\omega'),$$

then

$$f_1 \succeq f_2 \iff g_1 \succeq g_2.$$

Together with non-degeneracy and representability requirements (perhaps some form of continuity), we would obtain a characterization of Choquet expected utility with the additional requirement that the capacity takes values only in zero and one.



where the integration is in the sense of Choquet.<sup>3</sup> The classic reference to representation theorems for Choquet integrals is Schmeidler [23]. Here, we have simply provided a new method of characterizing a subclass of the Choquet integrals.

**Corollary 1:** A functional  $T$  is ordinally covariant and monotonic if and only if there exists a capacity  $\nu$  taking values in  $\{0, 1\}$  such that

$$Tf = \int_{\Omega} f(\omega) d\nu(\omega).$$

**Proof.** Let  $T$  be ordinally covariant and monotonic, and let  $\mathcal{E}$  be its associated downset. Define  $\nu(E) = 1$  if  $E \notin \mathcal{E}$ , 0 otherwise. Clearly  $\nu$  is a normalized capacity taking values in  $\{0, 1\}$ . We claim that  $Tf = \int_{\Omega} f(\omega) d\nu(\omega)$ . First, suppose that  $Tf \geq 0$ . Then, for all  $x < 0$ ,  $\{\omega : f(\omega) \geq x\} \notin \mathcal{E}$ . In particular, this implies that for all  $x < 0$ ,  $\nu(\{\omega : f(\omega) \geq x\}) = 1$ . Hence  $\int_{-\infty}^0 [\nu(\{\omega : f(\omega) \geq x\}) - 1] dx = \int_{-\infty}^0 0 dx = 0$ . Moreover, for all  $x > Tf$ ,  $\{\omega : f(\omega) \geq x\} \in \mathcal{E}$ , again by definition. Finally, for all  $x < Tf$ ,  $\{\omega : f(\omega) \geq x\} \notin \mathcal{E}$ . Therefore,  $\int_0^{\infty} \nu(\{\omega : f(\omega) \geq x\}) dx = \int_0^{Tf} 1 dx = Tf$ . The case for which  $Tf < 0$  is proved similarly.

For the converse direction, observe that if  $\nu$  is a normalized capacity taking values in  $\{0, 1\}$ , then  $\int_{\Omega} f(\omega) d\nu(\omega) = \inf \{x : \nu(\{\omega : f(\omega) \geq x\}) = 0\}$ . Therefore, we define  $\mathcal{E} \equiv \{E \in \Sigma : \nu(E) = 0\}$  and we are done by Theorem 1. ■

The preceding corollary implies, in particular, that any ordinally covariant and monotonic functional is comonotonically additive. This means that for any  $f, g \in B(\Omega, \Sigma)$  which are **comonotonic** (i.e. for all  $\omega, \omega' \in \Omega$ ,  $(f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0$ ),  $T(f + g) = Tf + Tg$ . See, for example, Schmeidler [23].

### 2.3 Minimax and maximin representations

The following characterization, known in the mathematics literature (Marichal and Mathonet [14] and Marichal [15]) for finite  $\Omega$ , follows from our general result. It gives us a characterization of the set of ordinally covariant and monotonic functions in terms of the supremum and infimum operators; hence, it lends to interpretation of these functionals as generalized Rawlsian aggregation operators. It might be interpreted as Rawlsianism across coalitions, or a “maximinimax” rule. Corollary 2 also tells us that the set of all such social aggregation operators is a complete lattice. This allows us to construct new social aggregation operators out of known ones.

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<sup>3</sup>The Choquet integral is defined as:

$$\int_{\Omega} f d\nu = \int_0^{\infty} \nu(\{\omega : f(\omega) \geq x\}) dx + \int_{-\infty}^0 \nu[(\{\omega : f(\omega) \geq x\}) - 1] dx.$$

Here, the integrals on the right hand side are Riemann integrals.

**Corollary 2:** The set of ordinally covariant, monotonic functionals is a complete lattice under the pointwise ordering.<sup>4</sup> Moreover, a functional  $T$  satisfies ordinal covariance and monotonicity if and only if there exists a nonempty family  $\{E_\lambda\}_{\lambda \in \Lambda} \subset \Sigma$  such that for all  $\lambda \in \Lambda$ ,  $E_\lambda \neq \emptyset$ , for which

$$Tf \equiv \inf_{\lambda \in \Lambda} \left\{ \sup_{\omega \in E_\lambda} f(\omega) \right\}.$$

Lastly, a functional  $T$  satisfies ordinal covariance and monotonicity if and only if there exists a nonempty family  $\{E_\gamma\}_{\gamma \in \Gamma} \subset \Sigma$  such that for all  $\gamma \in \Gamma$ ,  $E_\gamma \neq \emptyset$ , for which

$$Tf \equiv \sup_{\gamma \in \Gamma} \left\{ \inf_{\omega \in E_\gamma} f(\omega) \right\}.$$

**Proof.** Suppose that  $\{T_\lambda\}_{\lambda \in \Lambda}$  is a collection of ordinally covariant, monotonic functionals with corresponding downsets  $\{\mathcal{E}_\lambda\}_{\lambda \in \Lambda}$ . Then the pointwise infimum,  $\bigwedge_{\lambda \in \Lambda} T_\lambda$ , is ordinally covariant and monotonic. To see this, for strictly increasing continuous  $\varphi$ , and  $f \in B(\Omega, \Sigma)$

$$\begin{aligned} \varphi \left( \left( \bigwedge_{\lambda \in \Lambda} T_\lambda \right) (f) \right) &= \varphi \left( \inf_{\lambda \in \Lambda} T_\lambda f \right) = \inf_{\lambda \in \Lambda} \varphi(T_\lambda f) \\ &= \inf_{\lambda \in \Lambda} T_\lambda (\varphi \circ f) = \left( \bigwedge_{\lambda \in \Lambda} T_\lambda \right) (\varphi \circ f), \end{aligned}$$

so that ordinal covariance is satisfied. Let  $f, g \in B(\Omega, \Sigma)$  such that  $f \leq g$ . Then for all  $\lambda \in \Lambda$ ,  $T_\lambda f \leq T_\lambda g$  so that  $\left( \bigwedge_{\lambda \in \Lambda} T_\lambda \right) (f) \leq \left( \bigwedge_{\lambda \in \Lambda} T_\lambda \right) (g)$ . Hence, monotonicity is satisfied. It is easy to verify (using indicator functions) that the downset associated with  $\bigwedge_{\lambda \in \Lambda} T_\lambda$  is  $\bigcup_{\lambda \in \Lambda} \mathcal{E}_\lambda$ . The pointwise supremum,  $\bigvee_{\lambda \in \Lambda} T_\lambda$  can also be shown to be ordinally covariant and monotonic, with associated downset  $\bigcap_{\lambda \in \mathcal{E}} \mathcal{E}_\lambda$ .

To prove the representation theorem, it is straightforward to show that any functional so represented satisfies the two axioms. Conversely, let  $T$  be ordinally covariant and monotonic with associated downset  $\mathcal{E}$ . For all  $F \in \mathcal{E}$ , define  $\mathcal{E}_F \equiv \{E \in \Sigma : E \subset F\}$ . Then it is obvious that  $\mathcal{E} = \bigcup_{F \in \mathcal{E}} \mathcal{E}_F$ . Moreover, by Example 3, the functional  $T_F$  associated with downset  $\mathcal{E}_F$  is  $T_F(f) = \sup_{\omega \in \Omega \setminus F} f(\omega)$ . Hence, for all  $f \in B(\Omega, \Sigma)$ ,  $Tf = \inf_{F \in \mathcal{E}} \left\{ \sup_{\omega \in \Omega \setminus F} f(\omega) \right\}$ . The dual representation is similarly obtained. ■

<sup>4</sup>In fact, the mapping carrying functionals into their downsets is a lattice homomorphism when functionals are ordered pointwise and downsets are ordered according to reverse set inclusion.

## 2.4 Quantile representations

Quantiles with respect to probability measures are the canonical example of ordinally covariant and monotonic functionals. Thus, the understanding of quantiles is of obvious importance. Indeed; quantiles are typically defined (in the statistics literature) with respect to an exogenous probability measure.

Formally, we will say that a functional  $T$  is a **probabilistic quantile** if there exists a finitely additive probability measure  $p$  on  $(\Omega, \Sigma)$  and a real number  $\alpha \in [0, 1)$  such that

$$Tf = \inf \{x : p(\{\omega : f(\omega) \geq x\}) \leq \alpha\}.$$

Formally, this representation is a **lower quantile** with respect to  $p$ . An **upper quantile** would have a representation as

$$Tf = \inf \{x : p(\{\omega : f(\omega) \geq x\}) < \alpha\}.$$

In a *finite* measurable space, any functional  $T$  has a lower quantile representation if and only if it has an upper quantile representation. To see this, suppose that  $\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}$ . Then as  $(\Omega, 2^\Omega)$  is finite, there exists some  $\varepsilon > 0$  such that  $E \notin \mathcal{E}$  if and only if  $p(E) \geq \alpha + \varepsilon$ , in which case it is clear that  $\mathcal{E} = \{E \in \Sigma : p(E) < \alpha + \varepsilon\}$ .

It is simple to obtain a quantile representation for ordinally covariant and monotonic functionals for possibly *non-additive* set functions:

**Corollary 3:** A functional  $T$  satisfies ordinal covariance and monotonicity if and only if there exists a capacity  $v$  and a real number  $\alpha \in [0, 1)$  such that  $Tf = \inf \{x : v(\{\omega : f(\omega) \geq x\}) \leq \alpha\}$ .

**Proof.** It is clear that if  $T$  has the desired representation, then  $T$  is both ordinally covariant and monotonic.

Conversely, given the downset  $\mathcal{E}$  described in the preceding theorem, define  $v(E) = 0$  if  $E \in \mathcal{E}$  and  $v(E) = 1$  if  $E \notin \mathcal{E}$ . Let  $\alpha = 1/2$ . Then  $\mathcal{E} = \{E \in \Sigma : v(E) \leq \alpha\}$ . ■

The capacity derived in Corollary 3 is not unique; however, it is the unique capacity taking only the values zero and one.

The following example demonstrates that not every ordinally covariant and monotonic functional can be given a probabilistic quantile representation.

**Example 5:** Let  $\Omega = \{1, 2, 3, 4, 5, 6\}$ , and  $\Sigma = 2^\Omega$ . We may interpret  $\Omega$  as a set of agents. Elements of  $B(\Omega, \Sigma)$  specify the income of each agent (i.e.  $f(\omega)$  is the income of agent  $\omega$ ). Suppose there are two groups of agents,  $A = \{1, 2, 3\}$  and  $B = \{4, 5, 6\}$ . A common statistic of incomes is the median. However, if we are concerned with “fairness” across groups, a natural statistic of the income distribution  $f$  is given by

$$Tf = \min \{\text{med}\{f(1), f(2), f(3)\}, \text{med}\{f(4), f(5), f(6)\}\}.$$

Such a statistic takes the median income of each of the two groups, and then evaluates the minimum of these two medians. Maximizing such a statistic would tend to equate the median incomes of the two groups, hence such a statistic should be considered completely natural in an environment where fairness across groups is of concern. Obviously,  $T$  is both ordinally covariant and monotonic. The theorem which follows demonstrates that such a functional cannot generally be represented as a quantile with respect to a probability measure. However, this is easy enough to see directly. The claim is that if the downset associated with  $T$  is  $\mathcal{E}$ , there does not exist a probability measure  $p$  and an  $\alpha \in [0, 1)$  such that  $\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}$ . The following table illustrates a few income distributions (elements of  $B(\Omega, \Sigma)$ ) and their value under  $T$ .

	$T$
$f_1 = (1, 0, 1, 1, 0, 1)$	1
$f_2 = (0, 1, 1, 1, 1, 0)$	1
$f_3 = (1, 1, 1, 1, 0, 0)$	0
$f_4 = (0, 0, 1, 1, 1, 1)$	0

Why is it the case that such a  $p$  and  $\alpha$  do not exist? To see why, suppose that there do exist such a  $p$  and  $\alpha$ . Then  $p(\{1, 2, 3, 4\}) \leq \alpha$  and  $p(\{3, 4, 5, 6\}) \leq \alpha$ . Moreover,  $p(\{1, 3, 4, 6\}) > \alpha$  and  $p(\{2, 3, 4, 5\}) > \alpha$ . Hence,

$$\int_{\Omega} (f_1 + f_2)(\omega) dp(\omega) \leq 2\alpha < \int_{\Omega} (f_3 + f_4)(\omega) dp(\omega).$$

But note that  $f_1 + f_2 = f_3 + f_4$ , a contradiction. This example illustrates that when there is some “complementarity” across elements of  $\Omega$  (as in the case of our “fairness” example), ordinally covariant and monotonic statistics need not be representable as probabilistic quantiles. It is of considerable interest to formalize this notion of complementarity, to help understand why probabilistic quantiles have received such a large degree of attention in so many fields.

While the preceding example demonstrates that some type of complementarity across agents is responsible for the lack of a probabilistic quantile representation, it is not clear exactly what the formal statement of “complementarity” is. As quantiles are almost always defined with respect to a probability measure it is of obvious importance to understand exactly which quantiles are probabilistic. The following axiom clarifies the matter:

**Betting consistency:** Let  $\{A_1, \dots, A_n\} \subset 2^{\Omega}$  and  $\{B_1, \dots, B_n\} \subset 2^{\Omega}$  such that  $\sum 1_{A_i} \geq \sum 1_{B_i}$ . Then there exists  $i \in \{1, \dots, n\}$  such that  $T1_{A_i} \geq T1_{B_i}$ .

Formally, the question of when a collection of subsets has a probabilistic quantile representation is mathematically equivalent to the question of when a simple game is a weighted voting game. The answer to this question is known,

and the necessary and sufficient condition required is our axiom of betting consistency, appropriately translated into the language of simple games. For an account of this problem, see Taylor and Zwicker [26] and the references therein. Decision theorists may notice a similarity to the condition by Kraft, Pratt, and Seidenberg [13], which guarantees that an order structure over events on a finite set can be represented as a probability measure.<sup>5</sup>

**Theorem 2:** Suppose that  $|\Omega| < +\infty$ , and that  $\Sigma = 2^\Omega$ . A functional  $T$  satisfies ordinal covariance, monotonicity, and betting consistency if and only if it is a probabilistic quantile.

**Proof.** To see that the probabilistic quantiles satisfy betting consistency, let  $T$  be a probabilistic quantile. There exist corresponding  $p$  and  $\alpha$ . Let  $\{A_1, \dots, A_n\} \subset 2^\Omega$  and  $\{B_1, \dots, B_n\} \subset 2^\Omega$  for which  $\sum 1_{A_i} \geq \sum 1_{B_i}$ . Suppose, by means of contradiction, that for all  $i \in \{1, \dots, n\}$ ,  $T1_{B_i} > T1_{A_i}$ . This is only possible if for all  $i \in \{1, \dots, n\}$ ,  $T1_{B_i} = 1$  and  $T1_{A_i} = 0$ . Hence, for all  $i \in \{1, \dots, n\}$ ,  $p(B_i) > \alpha$  and  $p(A_i) \leq \alpha$ . As  $\sum 1_{A_i} \geq \sum 1_{B_i}$ ,  $\int_\Omega \sum 1_{A_i}(\omega) d\mu(\omega) \geq \int_\Omega \sum 1_{B_i}(\omega) d\mu(\omega)$ . But  $\int_\Omega \sum 1_{B_i}(\omega) d\mu(\omega) > n\alpha$  and  $n\alpha \geq \int_\Omega \sum 1_{A_i}(\omega) d\mu(\omega)$ , a contradiction.

Conversely, let  $T$  be any functional satisfying the axioms. Let  $\mathcal{E}$  be the downset from the representation in Theorem 1. We want to show that there exists a probability measure  $p$  and a number  $\alpha \in [0, 1]$  such that  $\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}$ . The existence of such a probability measure is equivalent to the existence of a  $[\alpha, p]$  solution of the following system of linear inequalities: for all  $E \in \mathcal{E}$ ,  $(1, -1_E) \cdot [\alpha, p] \geq 0$ ; for all  $E \notin \mathcal{E}$ ,  $(-1, 1_E) \cdot [\alpha, p] > 0$ ; for all  $\omega$ ,  $(0, 1_{\{\omega\}}) \cdot [\alpha, p] \geq 0$ , and  $(0, 1_\Omega) \cdot [\alpha, p] > 0$ . If this system of linear inequalities does not have a solution, then there must exist (see, for example, Rockafellar [20], Theorem 22.2) nonnegative integers for each of the preceding constraints, so that  $\sum_{E \in \mathcal{E}} n_E (1, -1_E) + \sum_{E \notin \mathcal{E}} n_E (-1, 1_E) + \sum_\omega n_\omega (0, 1_{\{\omega\}}) + m (0, 1_\Omega) = 0$ . Furthermore, one of the integers associated with one of the strict inequalities must be positive.

Therefore, we may also conclude that at least one of the  $n_E$  corresponding to an  $E \in \mathcal{E}$  must be positive. Moreover, in order to equal zero,  $\sum_{E \in \mathcal{E}} n_E = \sum_{E \notin \mathcal{E}} n_E$ . Define  $n \equiv \sum_{E \in \mathcal{E}} n_E$ , and list out all of the sets  $E \in \mathcal{E}$  a total of  $n_E$  times each to form a sequence  $\{A_1, \dots, A_n\} \subset \mathcal{E}$ . List out all of the sets  $E \notin \mathcal{E}$  a total of  $n_E$  times each to form a sequence  $\{B_1, \dots, B_n\} \subset 2^\Omega \setminus \mathcal{E}$ . By definition of  $\mathcal{E}$ , for all  $i = 1, \dots, n$ ,  $T1_{A_i} < T1_{B_i}$ . As the constraints sum to zero, and the weights  $n_\omega$  and  $m$  are nonnegative,  $\sum_i 1_{A_i} \geq \sum_i 1_{B_i}$ . This contradicts betting consistency.

Therefore, there exists a probability measure  $p$  and a number  $\alpha$  such that  $\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}$ . ■

<sup>5</sup>The reason these conditions appear similar is that they are both derived using versions of the Farkas' lemma for integer linear constraints, and characterizing certain collections of equivalence classes of ordinal probability measures. (The general solution to the problem of which order structures are representable by probability measures on Boolean algebras was provided by Chateaufneuf [7]).

Note that the representing probability measure need not be unique. Indeed, it is even possible that two probability measures which represent *different order structures* over the event space may result in the same quantile function. We are here not concerned with ordinal relations over the event space, so this probability measure does not represent anything in particular. For a decision theory, the lack of uniqueness may be more problematic. Rostek [21] defines a likelihood relation over events which is representable by a unique probability measure. She obtains this uniqueness by non-atomicity of the probability measure.

We now turn to infinite environments. Betting consistency is no longer strong enough to characterize the probabilistic quantiles. The issue has to do with separation theorems for convex sets in infinite-dimensional spaces. The following example exhibits an ordinally covariant and monotonic functional on an infinite set satisfying betting consistency which is *not* a probabilistic quantile, either upper or lower.

**Example 6:** Let  $\Omega = [0, 1]$  and  $\Sigma$  is the Lebesgue measurable sets. Define  $\mathcal{E} = \{E \in \Sigma : \lambda(E) < 1/2 \text{ or } \lambda(E) = 1/2 \text{ and } 0 \notin E\}$ , where  $\lambda$  is the Lebesgue measure. Let  $T$  be the associated functional.  $T$  satisfies betting consistency. Thus, let  $\{A_1, \dots, A_n\} \subset 2^\Omega$  and  $\{B_1, \dots, B_n\} \subset 2^\Omega$  and  $A_i \in \mathcal{E}$ ,  $B_i \in \Sigma \setminus \mathcal{E}$  for all  $i$ . We claim that there exists  $\omega$  for which  $\sum_{i=1}^n 1_{B_i}(\omega) > \sum_{i=1}^n 1_{A_i}(\omega)$ . This is obvious if there exists  $A_i$  for which  $\lambda(A_i) < 1/2$  or if there exists  $B_i$  for which  $\lambda(B_i) > 1/2$ . So, suppose that  $\lambda(A_i) = 1/2$  and  $\lambda(B_i) = 1/2$  for all  $i = 1, \dots, n$ . Then clearly,  $0 \notin A_i$  for all  $i = 1, \dots, n$ , and  $0 \in B_i$  for all  $i = 1, \dots, n$ . Hence  $\sum_{i=1}^n 1_{B_i}(0) = n > 0 = \sum_{i=1}^n 1_{A_i}(0)$ . So betting consistency is satisfied. Now, we establish that  $T$  cannot be a probabilistic quantile (lower quantile). If it were, then there would exist  $p$  and  $\alpha$  for which  $\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}$ . Suppose by means of contradiction that such a  $p$  and  $\alpha$  exist. Then it is clear that  $\alpha = 1/2$ . To see this, let  $n$  be odd and consider a partition of  $[0, 1]$  into Lebesgue measurable sets  $\{E_i\}_{i=1}^n$ , each of which satisfies  $\lambda(E_i) = 1/n$ . Then for any subcollection  $\{E_i\}_{i \in G}$ , where

$|G| = \frac{n-1}{2}$ , we have  $\bigcup_{i \in G} E_i \in \mathcal{E}$  (since  $\lambda\left(\bigcup_{i \in G} E_i\right) \leq 1/2$ ). Hence, as

$\sup_{\{G: |G| = \frac{n-1}{2}\}} p\left(\bigcup_{i \in G} E_i\right) \geq \frac{n-1}{2n}$ , due to the fact that  $p$  is a probability

measure, we may conclude that  $\alpha \geq \frac{n-1}{2n}$ . But as  $n$  is arbitrary,  $\alpha \geq 1/2$ .

Similarly, for any  $G$  for which  $|G| = \frac{n+1}{2}$ , we have  $\bigcup_{i \in G} E_i \notin \mathcal{E}$  (since

$\lambda\left(\bigcup_{i \in G} E_i\right) > 1/2$ ). Hence, as  $\inf_{\{G: |G| = \frac{n+1}{2}\}} p\left(\bigcup_{i \in G} E_i\right) \leq \frac{n+1}{2n}$ , due to

the fact that  $p$  is a probability measure, we may conclude that  $\alpha \leq \frac{n+1}{2n}$ .

But as  $n$  is arbitrary,  $\alpha \leq 1/2$ . Hence  $\alpha = 1/2$ . We now claim that if  $\lambda(E) \leq 1/2$ , then  $p(E) \leq 1/2$ . But for  $\lambda(E) = 1/2$  for which  $0 \in E$ , we know that  $E \notin \mathcal{E}$ , so that it is impossible that  $\mathcal{E} = \{E \in \Sigma : p(E) \leq 1/2\}$ .

Thus, let  $E \in \Sigma$  satisfy  $\lambda(E) \leq 1/2$ . For  $n$  odd, let  $\{E_i\}_{i=1}^n$  be a partition of  $[0, 1]$  for which  $\lambda(E_i) = \frac{1}{n}$  for all  $i$ , and for which  $E \subset \bigcup_{i=1}^{\frac{n+1}{2}} E_i$ . Existence of such a partition is easily verified as  $\lambda(E) \leq 1/2$ . Note that for all  $j = 1, \dots, \frac{n+1}{2}$ ,  $p\left(\bigcup_{i=1, i \neq j}^{\frac{n+1}{2}} E_j\right) \leq 1/2$ , as  $\lambda\left(\bigcup_{i=1, i \neq j}^{\frac{n+1}{2}} E_j\right) = \frac{n-1}{2n} < 1/2$  and hence  $\bigcup_{i=1, i \neq j}^{\frac{n+1}{2}} E_j \in \mathcal{E}$ . Thus,  $\sum_{i=1, i \neq j}^{\frac{n+1}{2}} p(E_i) \leq 1/2$ . The result is then established by summing over all  $j = 1, \dots, \frac{n+1}{2}$  and noting that  $n$  is arbitrary. By similar reasoning, we can also show that this particular functional cannot be represented as an upper quantile.

All probabilistic quantiles satisfy betting consistency. Therefore, betting consistency needs to be strengthened. The following stronger condition suggests itself:

**Strong betting consistency:**  $\inf \sup_{\omega \in \Omega} \frac{\sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega)}{n} > 0$ , where the inf is taken over all finite sequences  $\{A_1, \dots, A_n\} \subset \Sigma$  and  $\{B_1, \dots, B_n\} \subset \Sigma$  for which  $T1_{B_i} > T1_{A_i}$  for all  $i$ .

Given our downset  $\mathcal{E}$ , it is clear that strong betting consistency is satisfied if there exists  $\alpha$  and  $\varepsilon > 0$  for which  $\mathcal{E} \subset \{E \in \Sigma : p(E) \leq \alpha\}$  and  $\Sigma \setminus \mathcal{E} \subset \{E \in \Sigma : p(E) \geq \alpha + \varepsilon\}$ . Indeed, this is closely related to the work of Einy and Lehrer [9], who have provided a related characterization of those downsets  $\mathcal{E}$  for which there exists a finitely additive probability measure  $p$ , some  $\alpha$  and some  $\varepsilon > 0$  for which  $\mathcal{E} \subset \{E \in \Sigma : p(E) \leq \alpha\}$  and  $\Sigma \setminus \mathcal{E} \subset \{E \in \Sigma : p(E) \geq \alpha + \varepsilon\}$ . When added to ordinal covariance and monotonicity, strong betting consistency will imply that  $T$  is a probabilistic quantile with downset of this type (this will become evident in the proof of the next theorem). However, strong betting consistency is not always satisfied in the general case of a probabilistic quantile as defined above. Consider the following simple example:

**Example 7:** Let  $\Omega = [0, 1]$  and  $\Sigma$  is the Lebesgue measurable sets. Define  $\mathcal{E} = \{E \in \Sigma : \lambda(E) = 0\}$ , where  $\lambda$  is the Lebesgue measure. Consider the infinite sequences  $\{A_i\}_{i=1}^\infty \subset \mathcal{E}$  and  $\{B_i\}_{i=1}^\infty \subset \Sigma \setminus \mathcal{E}$  defined by  $A_i = \emptyset$  for all  $i$  and  $B_i = [1 - (\frac{1}{2})^i, 1 - (\frac{1}{2})^{i+1})$ . Then clearly for all  $n$ ,

$$\sup_{\omega \in \Omega} \frac{\sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega)}{n} = \frac{1}{n},$$

so that

$$\inf \sup_{\omega \in \Omega} \frac{\sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega)}{n} = 0,$$

contradicting strong betting consistency.

The above examples indicate that whatever condition is used in addition to ordinal covariance and monotonicity must lie strictly in between betting consistency and strong betting consistency. We use a construction related to one due to Kelley [12] which allows us to characterize those downsets which are identified with lower contour sets of some probability measure.

**$\sigma$ -Betting consistency:** For  $j = 1, \dots$ , there exists  $\mathcal{H}_j \subseteq \mathcal{E}$  for which  $\bigcup_{j=1}^{\infty} \mathcal{H}_j = \{E \in \Sigma : T1_E = 1\}$ , where for all  $j = 1, \dots$ ,

$$\inf \sup_{\omega \in \Omega} \frac{\sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega)}{\sum_{i=1}^n 1_{\{B_i \in \mathcal{H}_j\}}} > 0,$$

where the inf is taken over all finite subsequences  $\{A_i\}_{i=1}^n$  such that  $T1_{A_i} = 0$  and  $\{B_i\}_{i=1}^n$  such that  $T1_{B_i} = 1$  (with the convention that division by zero results in  $\infty$ ).

The axiom of  $\sigma$ -betting consistency requires that the complement of the downset  $\Sigma \setminus \mathcal{E}$  can be partitioned into a *countable* collection of sets, each of which can be used to deliver a condition similar to strong betting consistency.

This type of idea, especially the partitioning of a set of events into a countable collection of sets of events, is introduced by Kelley [12], Theorem 4.

**Theorem 3:** A functional  $T$  satisfies ordinal covariance, monotonicity, and  $\sigma$ -betting consistency if and only if it is a probabilistic quantile.

**Proof.** Suppose that  $T$  is a probabilistic quantile. We already know that it is ordinally covariant and monotonic. We will show that it satisfies  $\sigma$ -betting consistency. To this end, suppose without loss of generality that there exists some finitely additive probability measure  $p$  and some  $\alpha$  such that  $\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}$ . For all  $j$ , let  $\mathcal{H}_j = \{E \in \Sigma : p(E) \geq \alpha + \frac{1}{j}\}$ . Then  $\Sigma \setminus \mathcal{E} = \bigcup_{j=1}^{\infty} \mathcal{H}_j$ . Fix an arbitrary  $\mathcal{H}_j$ , and fix arbitrary sequences  $\{A_1, \dots, A_n\} \subset \mathcal{E}$  and  $\{B_1, \dots, B_n\} \subset \Sigma \setminus \mathcal{E}$ . Then

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega) dp(\omega) \\ & \geq \sum_{i=1}^n (p(B_i) - p(A_i)) \\ & \geq \sum_{i=1}^n [1_{\{B_i \in \mathcal{H}_j\}} (p(B_i) - p(A_i))] \\ & \geq \frac{1}{j} \sum_{i=1}^n 1_{\{B_i \in \mathcal{H}_j\}}. \end{aligned}$$



Therefore, (as  $p$  is a probability measure),

$$\sup_{\omega \in \Omega} \sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega) \geq \frac{1}{j} \sum_{i=1}^n 1_{\{B_i \in \mathcal{H}_j\}}.$$

Conclude that

$$\sup_{\omega \in \Omega} \frac{\sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega)}{\sum_{i=1}^n 1_{\{B_i \in \mathcal{H}_j\}}} \geq \frac{1}{j},$$

so that

$$\inf_{\omega \in \Omega} \sup_{\omega \in \Omega} \frac{\sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega)}{\sum_{i=1}^n 1_{\{B_i \in \mathcal{H}_j\}}} \geq \frac{1}{j} > 0.$$

Therefore,  $\sigma$ -betting consistency is satisfied.

Conversely, suppose that  $T$  satisfies ordinal covariance, monotonicity, and  $\sigma$ -betting consistency. Let  $\{\mathcal{H}_j\}_{j=1}^{\infty}$  be as in the statement of  $\sigma$ -betting consistency, and choose some arbitrary  $\mathcal{H}_j$ .

We will construct a probability measure  $p^j$  such that for all  $A \in \mathcal{E}$  and all  $B \in \mathcal{H}_j$ ,  $p^j(B) > p^j(A)$ , and for which for all  $A \in \mathcal{E}$  and all  $B \in \Sigma \setminus \mathcal{E}$ ,  $p^j(B) \geq p^j(A)$ .

Let

$$I(\mathcal{H}_j) \equiv \inf_{\omega \in \Omega} \sup_{\omega \in \Omega} \frac{\sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega)}{\sum_{i=1}^n 1_{\{B_i \in \mathcal{H}_j\}}},$$

where the inf is taken over all finite sequences  $\{A_1, \dots, A_n\} \subset \mathcal{E}$ ,  $\{B_1, \dots, B_n\} \subset \Sigma \setminus \mathcal{E}$ . By  $\sigma$ -betting consistency,  $I(\mathcal{H}_j) > 0$ . Let

$$\mathcal{B}(I(\mathcal{H}_j)) \equiv \left\{ f \in B(\Omega, \Sigma) : -I(\mathcal{H}_j) \leq \inf_{\omega \in \Omega} f(\omega) \right\}.$$

For any pair of sequences  $\{A_1, \dots, A_n\} \subset \mathcal{E}$  and  $\{B_1, \dots, B_n\} \subset \Sigma \setminus \mathcal{E}$  (for which at least one  $B_i$  is an element of  $\mathcal{H}_j$ ),

$$\sup_{\omega \in \Omega} \frac{\sum_{i=1}^n (1_{B_i} - 1_{A_i})(\omega)}{\sum_{i=1}^n 1_{\{B_i \in \mathcal{H}_j\}}} \geq I(\mathcal{H}_j),$$

which implies that

$$\sup_{\omega \in \Omega} \frac{\sum_{i=1}^n [(1_{B_i} - 1_{A_i})(\omega) - I(\mathcal{H}_j) 1_{\{B_i \in \mathcal{H}_j\}}]}{n} \geq 0.$$

Hence, if for all  $i = 1, \dots, n$ ,  $g_i \in \mathcal{B}(I(\mathcal{H}_j))$ ,

$$\sup_{\omega \in \Omega} \frac{\sum_{i=1}^n [(1_{B_i} - 1_{A_i})(\omega) + g_i(\omega) 1_{\{B_i \in \mathcal{H}_j\}}]}{n} \geq 0. \quad (1)$$

Let

$$C \equiv \text{conv} \{1_B - 1_A + g : B \in \mathcal{H}_j, A \in \mathcal{E}, g \in \mathcal{B}(I(\mathcal{H}_j))\}.$$

Let

$$D \equiv \text{conv} \{1_B - 1_A : B \in \Sigma \setminus \mathcal{E}, A \in \mathcal{E}\}.$$

Suppose that  $f \in C$  and  $g \in D$ , and let  $\lambda \in [0, 1]$ . By (1) and taking limits,  $\sup_{\omega \in \Omega} \lambda f + (1 - \lambda)g \geq 0$ .

Let  $E$  be the convex cone generated by the union of  $C$ ,  $D$ , and  $B(\Omega, \Sigma)_+$  (the nonnegative functions). Then for all  $f \in E$ ,  $\sup_{\omega \in \Omega} f \geq 0$ . In particular,  $E$  is convex and is disjoint from  $B(\Omega, \Sigma)_{--}$  (the strictly negative functions, which is also a convex cone). Moreover,  $B(\Omega, \Sigma)_{--}$  contains an internal point, say  $-1_X$  (Theorem 5.46 of Aliprantis and Border [1]). Since both  $E$  and  $B(\Omega, \Sigma)_{--}$  are cones there exists a positive linear functional  $\psi$  separating them. Without loss of generality, normalize  $\psi(-1) = -1$ .

Now, define  $p^j(A) \equiv \psi(1_A) \geq 0$ . Then  $p^j$  is a finitely additive probability measure. Let  $A \in \mathcal{E}$  and  $B \in \Sigma \setminus \mathcal{E}$ . Then  $1_B - 1_A \in D$ , so that  $\psi(1_B - 1_A) \geq 0$ , and hence  $p^j(B) \geq p^j(A)$ . Finally, let  $A \in \mathcal{E}$  and  $B \in \mathcal{H}_j$ . Then  $1_B - 1_A - I(\mathcal{H}_j) \in C$ , so that  $\psi(1_B - 1_A - I(\mathcal{H}_j)) \geq 0$ . In particular,  $p^j(B) - p^j(A) \geq I(\mathcal{H}_j) > 0$ , so that  $p^j(B) \geq p^j(A) + I(\mathcal{H}_j)$ . Therefore, we have constructed a probability measure  $p^j$  satisfying the appropriate properties.

Define  $p \equiv \sum_{j=1}^{\infty} \frac{1}{2^j} p^j$ . Then  $p$  is a probability measure. Clearly, for all  $B \in \Sigma \setminus \mathcal{E}$  and all  $A \in \mathcal{E}$ ,  $p(B) \geq p(A)$ . Let  $\alpha \equiv \sup_{A \in \mathcal{E}} p(A)$ . For all  $B \in \Sigma \setminus \mathcal{E}$ , there exists some  $j$  such that  $B \in \mathcal{H}_j$ , from which we conclude that for all  $A \in \mathcal{E}$ ,  $p(B) - p(A) > \frac{1}{2^j} [p^j(B) - p^j(A)] \geq \frac{1}{2^j} I(\mathcal{H}_j)$ . Conclude that  $p(B) \geq \alpha + \frac{1}{2^j} I(\mathcal{H}_j)$ , so that  $p(B) > \alpha$ . This allows us to conclude that

$$\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}.$$

Hence  $T$  has the desired representation. ■

### 3 Conclusion

A few issues remain undiscussed. Firstly, we have presupposed the existence of a functional  $T$  at all times. However, in spirit with much of the economics literature, it would be useful to understand which *order structures*  $\preceq$  over  $B(\Omega, \Sigma)$  which can be represented by functionals  $T$  satisfying our primary axioms. Such an axiomatization is trivial to provide, and is somewhat related to the details of footnote 2. Monotonicity of the order structure is easy enough to translate—if  $f \geq g$ , then  $f \succeq g$ . Ordinal covariance is translated into an invariance concept: for all strictly increasing continuous  $\varphi$  and all  $f, g \in B(\Omega, \Sigma)$ ,  $f \succeq g \iff \varphi \circ f \succeq \varphi \circ g$ . We of course need an axiom ruling out constant functionals, so suppose a non-degeneracy condition: there exists  $f, g \in B(\Omega, \Sigma)$  for which  $f \succ g$ . Lastly, for general order structures, representation by a functional is not guaranteed. In particular, there is an interesting class of examples, the lexicographic quantiles, which satisfy the remaining axioms but which are not representable by functionals. In order to maintain that a functional indeed exists, it is necessary to postulate an additional condition. A natural condition

is solvability: for every  $f \in B(\Omega, \Sigma)$ , there exists  $x \in \mathbb{R}$  for which  $f \sim x$ .<sup>6</sup> It turns out that these axioms are necessary and sufficient for representation by a functional satisfying the ordinal covariance and monotonicity properties. A study of orders which need not satisfy the solvability condition is an interesting and important direction for future research (note that the leximin orderings of social choice are examples of lexicographic quantiles).

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<sup>6</sup>Here,  $x$  also denotes the constant function which always returns value  $x$ . Solvability therefore postulates the existence of a “certainty equivalent” for every function  $f$ .

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