SPATIAL IMPLEMENTATION

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ABSTRACT. In a spatial model with Euclidean preferences, we introduce a new rule, the *geometric median*, and characterize it as the smallest rule (with respect to set inclusion) satisfying a collection of axioms. The geometric median is independent of the choice of coordinates and is Nash implementable.

1. INTRODUCTION

Our purpose is to study the implementation problem in spatial environments. We have in mind a spatial model where agents have Euclidean preferences. However, the ideal points or "peaks" of these preferences are unknown to the social planner. We study mechanisms that properly incentivize agents to reveal their peaks to the social planner, and which also satisfy other basic properties. The basic application we have in mind is the location of a public facility.

Ideally, the mechanism should achieve some very basic goals: some sort of social compromise should be reached and the participating agents should always have an incentive to reveal their true preferences over possible outcomes. The well-known Gibbard-Satterthwaite Theorem does not hold in this setting (Gibbard (1973) Satterthwaite (1975)). Many papers study the strategy-proofness property in spatial environments with preferences that are "single-peaked."[1] Moulin (1980) provides the seminal result in this literature, focusing on the single-dimensional case. He establishes that the only rules which are efficient, anonymous, and strategy-proof select a "generalized" median of the announced peaks of the agents. Multi-dimensional extensions establish characterizations of coordinate-wise medians which hold even in the Euclidean

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[1] Massó and Moreno de Barreda (2011) provide a related characterization when preferences are also required to be symmetric.

case (see Border and Jordan (1983); Kim and Roush (1984); Barberà, Gul, and Stacchetti (1993); Barberà, Massó, and Serizawa (1998); Le Breton and Sen (1999); Le Breton and Weymark (1999); Bordes, Laffond, and Le Breton (2012)). Coordinate-wise medians are intuitive, but suffer from the drawback that the choice of coordinates determines the rule. In spatial environments involving land or distances, however, there is often no natural choice of coordinate. The notion of a median in all directions, or *total median*, first studied in the economics literature by Plott (1967), generalizes the ideas of Hotelling (1929) and Black (1948) to multiple dimensions. The median in all directions is independent of choice of coordinates. However, it seldom exists.

Strategy-proofness is in general incompatible with the notion that the outcome of a rule does not depend on choice of coordinates together with any reasonable anonymity properties. Hence, we ask for the weaker notion of Maskin Monotonicity (Maskin (1999)). Maskin monotonicity is known to be necessary for Nash implementability. Using this property, we will characterize a rule which turns out to be Nash implementable.

Our goal in this work is to investigate rules which do not necessarily depend on choice of coordinates, have reasonable incentive properties, and exist quite generally. We investigate several axioms, and establish that a new rule emerges, which naturally generalizes the classical median in one dimension. This rule is called the *geometric median*.[2] To understand its properties, consider the one-dimensional case. With an odd number of agents, the median is known to minimize the total absolute deviation from the ideal points: in our setting, it is a utilitarian optimum when agents preferences are represented by the Euclidean distance. A natural idea is to generalize this optimization problem to multiple dimensions. The result of this optimization in a multi-dimensional framework constitutes the geometric median. In general, there may not exist a unique point minimizing the total absolute deviation from the ideal points. Hence, the geometric median is a set-valued concept.

We investigate the geometric median in a variable population framework. In the case of an even number of agents, we know that there is not necessarily a unique geometric median. So, in this case, we characterize a set-valued concept. Our axiomatization is based on a concept we call *mean equality*, which we believe is new to the literature. It is based on the idea that if the

[2]Despite its intuitive appeal in the spatial setting, the geometric median has received little attention in the social choice literature. See Chung and Duggan (2014) for a more general concept in the spatial model of voting. Cervone, Dai, Gnoutcheff, Lanterman, Mackenzie, Morse, Srivastava, and Zwicker (2012) investigate the geometric median in a preference aggregation framework. Finally, Baranchuk and Dybvig (2009) provide an application to corporate board consensus.

mean is equidistant from every agent's ideal point, then it should be selected. This axiom incorporates the idea that the main reason the mean might not be selected is because it "unfairly" weights certain agents. The axiom suggests that when all agents are treated symmetrically by the mean, then the mean should be selected.

Our result also relies on two classical axioms. The first, *reinforcement*, is due to Young (1975).[3] Reinforcement requires that when two disjoint societies each select the same alternative, then the society which results from collecting the agents together should also select the same alternative. The other variable population axiom, which is related to reinforcement is *replication invariance*, which makes a first appearance in Debreu and Scarf (1963). The axiom requires that in replicating a society, the set of solutions should not change. Note that reinforcement already implies in replicating an economy all solutions of the original society should also be chosen for the replica. For our purposes, the important part of replication invariance is the converse: if an alternative is chosen for the replica, it must also be chosen for the original economy.

Finally, we postulate a technical axiom, which is formally an upper hemi-continuity axiom; we refer to it simply as continuity. We show that any rule satisfying these axioms must contain the set of geometric medians as possible solutions; further, the set of geometric medians viewed as a rule satisfies all of the axioms. Hence, the set of geometric medians is the smallest rule (with respect to set inclusion) satisfying the axioms.

Importantly, because the geometric median satisfies no veto-power, we know it is Nash implementable when there are at least three agents; see Maskin (1999). It is obviously Nash implementable when there is one agent, and it is straightforward to verify that the properties of Moore and Repullo (1990) or Dutta and Sen (1991) are satisfied, so that it is Nash implementable when there are only two agents. Hence, no matter the cardinality, it is a Nash implementable rule.

The paper is organized as follows. Section 2 presents the model and our main result. Section 3 concludes.

2. The model

Here we provide an axiomatization for a variable population model and show that a social choice rule selecting the (set of) geometric median(s) is the

[3]For classical works using this property, see also Smith (1973), which deals with a preference aggregation environment, and the papers of Young characterizing specific rules; Young (1974); Young and Levenglick (1978).

smallest rule satisfying our axioms, in the sense that any other rule satisfying our axioms must contain the set of geometric medians for any economy.

Let $X = \mathbb{R}^d$ be the policy space. For any $x, y \in X$ let $\|x - y\|$ denote the Euclidean distance.

Preferences are assumed to be Euclidean. Hence, any preference $\succsim$ can be represented by an "ideal point", $z \in X$, with the property that for any $x, y \in X$, $x \succsim y$ if and only if $\|x - z\| \leq \|y - z\|$.

Let $\mathbb{N}$ index the potential agents and $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$. An *economy* (N, Z) is a set of agents $N \in \mathcal{N}$ and their ideal points $Z \in X^N$. Since there is a one-to-one relationship between preference profiles and a set of points in X, we will use the notation $Z \in X^N$ to indicate a preference profile of the agents that is represented by the points $Z = (z_i)_{i \in N}$ where $z_i \in X$ for each i.

Let the set of all possible economies be $\mathcal{E}$. A social choice rule is a correspondence $\varphi : \mathcal{E} \rightrightarrows X$ such that $\varphi(N, Z) \subseteq X$ for every economy (N, Z). Importantly, as is standard in the implementation literature, we allow φ to be empty-valued.

For $i \in N$ and $z_i, x \in X$, let $UC_i(z_i, x) = \{y \in X \mid \|y - z_i\| \leq \|x - z_i\|\}$ be the upper contour set for the preference relation represented by the point z_i at the point x. This is simply the set of all outcomes agent i weakly prefers to x. We will say that the preference relation represented by a point z_i' is a monotonic transformation of the preference relation represented by z_i at a point x if $UC_i(z_i', x) \subseteq UC_i(z_i, x)$. Let $MT(z_i, x)$ be the set of all monotonic transformations of the preference relation represented by the point z_i at the point x and $MT(Z, x)$ be the set of all monotonic transformations of a preference profile represented by the set of points Z at a point x.

For $x, y \in X$ we will denote the line segment with x and y as endpoints by $\overline{xy} = \{t \in X \mid \|x - t\| + \|t - y\| = \|x - y\|\}$.

For a set of points $(a_1, \ldots, a_n)$ with each $a_i \in X$, we define a geometric median as a solution to the following minimization problem:

$$\min_{x \in X} \sum_{i=1}^{n} \|x - a_i\|. \tag{1}$$

That is, a geometric median minimizes the sum of distances between itself and all of the points.

The geometric median for a finite set of points always exists; further, it is unique if n is odd or if the points $(a_1, \ldots, a_n)$ are not collinear (Haldane (1948)). In the case of collinear points, there could be multiple geometric medians, in particular when the number of distinct points is even. In this case, the geometric median is a set-valued concept.

It can be shown that (1) has a convenient, simplified dual characterization (Güler (2010)). To see this, first note that for any $x \in X$ we have $\|x\| = \max_{\|y\|\leq 1}\langle x, y\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product. This allows us to rewrite (1) as the minimax problem

$$\min_{x\in X} \max_{\|y_i\|\leq 1} \sum_{i=1}^{n} \langle a_i - x, y_i \rangle. \tag{2}$$

The dual to (1) can then be regarded as the maximin problem corresponding to (2). After some simplification, the dual problem becomes

$$\max \left\{ \sum_{i=1}^{n} \langle a_i, y_i \rangle \middle| \|y_i\| \leq 1, 1 \leq i \leq n, \sum_{i=1}^{n} y_i = 0 \right\}. \tag{3}$$

These duality results give us a simple way of checking whether or not a point is a geometric median. For a candidate point, x, first consider all i for which $x \neq a_i$. For these points, $y_i = \frac{a_i - x}{\|a_i - x\|}$ i.e. the unit vector in the direction $a_i - x$. For i with $x = a_i$, y_i can be any point inside the unit ball. If $\sum_{i=1}^{n} y_i = 0$, then x is a geometric median.

Two economies (N, Z) and (N', Z') are *disjoint* if $N \cap N' = \emptyset$. We will denote the concatenation of Z and Z' as (Z, Z'). We say economy (N', Z') is an *m-fold replica* of economy (N, Z) if $m > 0$ is an integer such that $|N'| = m|N|$ and for all $x \in X$ we have $|\{j \in N' : z'_j = x\}| = m|\{i \in N : z_i = x\}|$.

Axiom 2.1. A social choice rule φ satisfies *Maskin monotonicity* if for all economies (N, Z) and all $x \in \varphi(N, Z)$, if $Z' \in MT(Z, x)$ we have $x \in \varphi(N, Z')$.

Axiom 2.2. A social choice rule satisfies *mean equality* if for any economy (N, Z) and $x = \frac{1}{|N|}\sum_{i\in N} z_i$ such that x satisfies:

$$\|x - z_i\| = \|x - z_j\| \text{ for all } i, j \in N$$

we have $x \in \varphi(N, Z)$.

Mean equality asserts that if the coordinate-wise mean makes all agents "equally well off" with a Euclidean utility representation, then it should be chosen as an outcome.

Axiom 2.3. A social choice rule satisfies *reinforcement* if for any disjoint economies (N, Z) and (N', Z'), $\varphi(N, Z) \cap \varphi(N', Z') \subseteq \varphi(N \cup N', (Z, Z'))$.

Reinforcement simply requires any two economies with different agents, if a choice x is recommended for each, then it is recommended for the combined economy as well.

Axiom 2.4. A social choice rule satisfies *replication invariance* if for any economy (N, Z) and m-fold replica (N', Z') of (N, Z) we have $\varphi(N', Z') = \varphi(N, Z)$.

Replication invariance guarantees that the social choice outcomes do not change when replicating economies.

Axiom 2.5. A social choice rule satisfies *continuity* if for any sequence of economies $\{(N, Z^k)\}$ and outcomes $\{x^k\}$ such that $Z^k \to Z \in X^N$, $x^k \to x \in X$, and $x^k \in \varphi(N, Z^k)$ for each k, we have $x \in \varphi(N, Z)$.

In what follows, we will denote $GM(N, Z)$ as the (set of) geometric median(s) for economy (N, Z). The following is our main result.

Theorem 2.1. *If φ satisfies Maskin monotonicity, mean equality, reinforcement, replication invariance, and continuity, then $GM(N, Z) \subseteq \varphi(N, Z)$ for every economy (N, Z). Further, if $\varphi(N, Z) = GM(N, Z)$ for every economy (N, Z) then φ satisfies Maskin monotonicity, mean equality, reinforcement, replication invariance, and continuity.*

Before presenting the proof, we first prove two Lemmas. The first characterizes monotonic transformations and the second establishes that the geometric median satisfies Maskin monotonicity.

Lemma 2.1. *Suppose $z_i, z_i' \in X$ represent two preference relations. For $x \in X$, $z_i' \in MT(z_i, x)$ if and only if $z_i' \in \overline{z_i x}$.*

Proof. Suppose $z_i' \in \overline{z_i x}$. Then, there exists $\lambda \in [0, 1]$ such that $(x - z_i') = \lambda(x - z_i)$. Consequently $(z_i' - z_i) = (z_i' - x) + (x - z_i) = -\lambda(x - z_i) + (x - z_i) = (1 - \lambda)(x - z_i)$.

Suppose $y \in UC_i(z_i', x)$. Then, $\|y - z_i\| = \|y - z_i' + z_i' - z_i\| \leq \|y - z_i'\| + \|z_i' - z_i\| \leq \|x - z_i'\| + (1 - \lambda)\|x - z_i\| = \lambda\|x - z_i\| + (1 - \lambda)\|x - z_i\| = \|x - z_i\|$ where the first inequality follows by the triangle inequality and the second by assumption. It follows that $y \in UC_i(z_i, x)$ and thus $z_i' \in MT(z_i, x)$ by definition.

Conversely, suppose $z_i' \in MT(z_i, x)$. If $z_i' = z_i$ or $x = z_i$ or $z_i' = x$ then the result is trivially true so assume the points are distinct. This is equivalent to x solving the following optimization problem:

$$\max_{y \in X} -(y - z_i') \cdot (y - z_i') \text{ subject to } -(y - z_i) \cdot (y - z_i) \leq -(x - z_i) \cdot (x - z_i)$$

If we define $g(y) = -(y-z_i)\cdot(y-z_i) + (x-z_i)\cdot(x-z_i)$ then our constraint becomes $g(y) \leq 0$. Clearly the Jacobian $\frac{\partial g(y)}{\partial y} \neq 0$ evaluated at the solution x. Thus, the first order conditions at the solution then imply that $(x - z_i') = \lambda(x - z_i)$ for some $\lambda \geq 0$ (see Corollary 9.6 in Güler (2010)). Establishing $\lambda \leq 1$ would give the desired result. Suppose however that $\lambda > 1$. The first order conditions then imply that $z_i' \neq z_i$ and $z_i \in \overline{z_i' x}$. It then follows that there exists a $t \in X$ such that $\|t - z_i\| > \|x - z_i\|$ but $\|t - z_i'\| \leq \|x - z_i'\|$. Thus, $UC_i(z_i', x) \nsubseteq UC_i(z_i, x)$ which contradicts that $z_i' \in MT(z_i, x)$. So we must have $\lambda \leq 1$ and therefore $z_i' \in \overline{z_i x}$. □

Lemma 2.2. *Let $N \in \mathcal{N}$. For $Z \in X^N$, if $x_Z^* \in GM(N, Z)$ and $Z' \in MT(Z, x_Z^*)$, then $x_Z^* \in GM(N, Z')$ is also a geometric median of Z'.*

Proof. If $Z' = Z$ then the result is trivially true, so suppose $Z' \neq Z$. Further, suppose by means of contradiction that x_Z^* is not a geometric median for Z'. Let us let $x_{Z'}^*$ be any geometric median for Z'. It follows that

$$\sum_{i=1}^{n} \|x_Z^* - z_i\| \leq \sum_{i=1}^{n} \|x_{Z'}^* - z_i\| \tag{4}$$

and

$$\sum_{i=1}^{n} \|x_{Z'}^* - z_i'\| < \sum_{i=1}^{n} \|x_Z^* - z_i'\|. \tag{5}$$

Adding (4) and (5) yields

$$\sum_{i=1}^{n} \left(\|x_Z^* - z_i\| + \|x_{Z'}^* - z_i'\|\right) < \sum_{i=1}^{n} \left(\|x_{Z'}^* - z_i\| + \|x_Z^* - z_i'\|\right). \tag{6}$$

It follows by Lemma 2.1 that $\|x_Z^* - z_i\| = \|x_Z^* - z_i'\| + \|z_i' - z_i\|$ for each i. Using this in (6) with some cancellation yields

$$\sum_{i=1}^{n} \left(\|z_i' - z_i\| + \|x_{Z'}^* - z_i'\|\right) < \sum_{i=1}^{n} \|x_{Z'}^* - z_i\|. \tag{7}$$

By the triangle inequality $\|x_{Z'}^* - z_i\| \leq \|x_{Z'}^* - z_i'\| + \|z_i' - z_i\|$ for each i, which by (7) implies that

$$\sum_{i=1}^{n} \|x_{Z'}^* - z_i'\| < \sum_{i=1}^{n} \|x_{Z'}^* - z_i'\| \tag{8}$$

a contradiction. □

Proof. **Step 1: Sufficiency of the axioms**

Suppose φ satisfies Maskin monotonicity, mean equality, reinforcement, replication invariance, and continuity. Let (N, Z) be an economy and let $x \in GM(N, Z)$. Let $M = \{i \in N : z_i = x\}$.

Case 1: For all $i \in M$, there is v_i such that $\|v_i\| = 1$ and $\sum_{i \in N \setminus M} \frac{z_i - x}{\|z_i - x\|} + \sum_{i \in M} v_i = 0$.

Let $\alpha = \max\{\max_{i \in N \setminus M} \|z_i - x\|, 1\}$. Let (N, Z') be an economy for which for all $i \in N \setminus M$, $z'_i = x + \alpha \frac{z_i - x}{\|z_i - x\|}$, and for all $i \in M$, $z'_i = x + \alpha v_i$. Note that by mean equality, $x \in \varphi(N, Z')$. Finally, by Lemma 2.1, $Z \in MT(Z', x)$, so $x \in \varphi(N, Z)$.

Case 2: All remaining cases.

In the remaining cases, by the duality argument presented in equation (3), for each $i \in M$, there is v_i for which $\|v_i\| \leq 1$ and

$$\sum_{i \in N \setminus M} \frac{z_i - x}{\|z_i - x\|} + \sum_{i \in M} v_i = 0. \tag{9}$$

Clearly, it is without loss of generality to assume that for all $i, j \in M$, we have $v_i = v_j$.[4] So equation (9) becomes:

$$\sum_{i \in N \setminus M} \frac{z_i - x}{\|z_i - x\|} + |M| v = 0 \tag{10}$$

for some v for which $\|v\| \leq 1$.

Now, suppose that $\|v\| \in \mathbb{Q}$, so that it may be written as $\|v\| = \frac{p}{q}$, where $p \geq 0$ and $q > 0$ are integers.

Let (N', Z') be an economy with $q|N \setminus M| + p|M|$ agents, consisting of a q-replica of $(N \setminus M, Z_{N \setminus M})$ concatenated with $p|M|$ agents with ideal point x. By equation (10), if $p > 0$, we have

$$\sum_{i \in N \setminus M} \frac{z_i - x}{\|z_i - x\|} + |M| \frac{p}{q} \frac{v}{\|v\|} = 0, \tag{11}$$

and otherwise if $p = 0$, we obviously have $\sum_{i \in N \setminus M} \frac{z_i - x}{\|z_i - x\|} = 0$.

Multiplying equation (11) by q, we get, when $p > 0$:

$$q \sum_{i \in N \setminus M} \frac{z_i - x}{\|z_i - x\|} + |M| p \frac{v}{\|v\|} = 0, \tag{12}$$

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[4]Thus, for each $i \in M$, replace v_i by $\frac{\sum_{j \in M} v_j}{|M|}$.

from which we conclude, using Case 1, that $x \in \varphi(N', Z')$. Likewise, when $p = 0$, we get $q \sum_{i \in N \setminus M} \frac{z_i - x}{\|z_i - x\|} = 0$, from which we conclude, using Case 1, that $x \in \varphi(N', Z')$.

Now, let (N'', Z'') be an economy consisting of $q - p$ agents with ideal point x. We know by mean equality that $x \in \varphi(N'', Z'')$. Hence, by reinforcement, we have $x \in \varphi(N' \cup N'', (Z', Z''))$. But note that $(N' \cup N'', (Z', Z''))$ is by construction a q-replica of (N, Z). Hence, by replication invariance, $x \in \varphi(N, Z)$.

If $\|v\| \notin \mathbb{Q}$, the result follows from a standard application of continuity.

Step 2: Necessity of the axioms for the geometric median

Now suppose $\varphi(N, Z) = GM(N, Z)$ for every economy (N, Z). That φ satisfies Maskin monotonicity follows immediately by Lemma 2.2. That φ satisfies reinforcement and replication invariance is a simple exercise relying on the additive separability of the objective function being minimized.

Let $x = \frac{1}{|N|} \sum_{i \in N} z_i$ and $\|z_i - x\| = \|z_j - x\|$ for all $i, j \in N$. Then it is easy to see by the duality results that $x \in GM(N, Z) = \varphi(N, Z)$. Thus, φ satisfies mean equality.

Finally, continuity can be shown to follow from a version of the Maximum Theorem of Berge (1963). Let $\{(N, Z^k)\}$ be a sequence of economies and $\{x^k\}$ a sequence of outcomes such that $Z^k \to \hat{Z}$, $x^k \to \hat{x}$, and $x^k \in \varphi(N, Z^k)$ for each k. By assumption, $x^k \in GM(N, Z^k)$ for each k. Define $f(Z, x) := \sum_{i \in N} \|z_i - x\|$. Fix any $y \in X$. By definition we have

$$f(Z^k, x^k) \leq f(Z^k, y)$$

for each k. Thus, it follows by continuity of f that

$$f(\hat{Z}, \hat{x}) \leq f(\hat{Z}, y).$$

Thus, $x \in GM(N, Z) = \varphi(N, Z)$ and we can conclude φ satisfies continuity, which completes the proof. □

3. Conclusion

Our main contribution comes from proposing a specific social choice rule that has intuitive appeal in a spatial setting. This social choice rule, the geometric median, has useful incentive properties and also is independent of the choice of coordinates. We have shown that this social choice rule is the smallest rule to satisfy a collection of axioms.

4. Appendix: Independence of the axioms

Each axiom is followed by a rule satisfying all of the remaining axioms. The rules described here are meant to be illustrative of the proof construction.

For each of the properties, we describe the smallest rule (with respect to set inclusion) satisfying the remaining axioms (just as GM constitutes the smallest rule with respect to set inclusion satisfying all of the axioms).

Maskin monotonicity: For any economy (N, Z), if $x = \frac{1}{|N|}\sum_{i\in N} z_i$ satisfies:

$$\|x - z_i\| = \|x - z_j\| \text{ for all } i, j \in N,$$

then $\varphi(N, Z) = \{x\}$; otherwise, $\varphi(N, Z) = \varnothing$.

Mean equality: For all (N, Z), $\varphi(N, Z) = \varnothing$.

For the remaining cases, recall in the proof the following construction: Let (N, Z) be an economy and let $x \in GM(N, Z)$. Let $M = \{i \in N : z_i = x\}$.

Reinforcement: For any (N, Z), if there is $x \in GM(N, Z)$ for which for all $i \in M$, there is v_i such that $\|v_i\| = 1$ and $\sum_{i\in N\setminus M} \frac{z_i - x}{\|z_i - x\|} + \sum_{i\in M} v_i = 0$, then let $\varphi(N, Z) = GM(N, Z)$. If there is a unique geometric median, and for all $i \in N$, z_i is the geometric median, again let $\varphi(N, Z) = GM(N, Z)$. Otherwise $\varphi(N, Z) = \varnothing$.

Replication invariance: For any (N, Z), if there is $x \in GM(N, Z)$ for which for all $i \in M$, there is v_i such that $\|v_i\| = 1$ and $\sum_{i\in N\setminus M} \frac{z_i - x}{\|z_i - x\|} + \sum_{i\in M} v_i = 0$, then let $\varphi(N, Z) = GM(N, Z)$. If there is a unique geometric median, and for all $i \in N$, z_i is the geometric median, again let $\varphi(N, Z) = GM(N, Z)$. If (N, Z) can be written as $N = \bigcup_k N_k$, where N_k partitions N and each of (N_k, Z_{N_k}) is in the preceding class, then let $\varphi(N, Z) = GM(N, Z)$. Otherwise, $\varphi(N, Z) = \varnothing$.

Continuity: For any (N, Z), if there is $x \in GM(N, Z)$ for which for all $i \in M$, there is v_i such that $\|v_i\| \in \mathbb{Q}$ and $\sum_{i\in N\setminus M} \frac{z_i - x}{\|z_i - x\|} + \sum_{i\in M} v_i = 0$, then let $\varphi(N, Z) = GM(N, Z)$. Otherwise, let $\varphi(N, Z) = \varnothing$.

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