An axiomatization of quantiles on the domain of distribution functions

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Abstract

In an environment in which the primitive is the space of distribution functions, we characterize the quantile functions by the axioms ordinal covariance, monotonicity with respect to first order stochastic dominance, and upper semicontinuity. We show how to characterize the VaR in a similar manner. Keywords: risk measure, VaR, quantile, axiom.

1 Introduction

Here, our primary purpose is to understand quantiles from an axiomatic perspective. Given is the space of distribution functions. We understand an α-quantile of a distribution function F to be the smallest value x for which $F(x) > \alpha$. Properties of quantile functions are well-understood. Two particular appealing properties of quantile functions stand out. First, they are covariant with respect to arbitrary ordinal transformations. Thus, transforming the units of measurement of the outcome of the random variable in an arbitrary monotonic way transforms the quantile in the same way. Secondly, quantiles are weakly monotonic with respect to first order stochastic dominance. These properties of quantiles are both well-known; see, for example, Manski [8] or Denneberg [4]. Our primary contribution in this note is to show that these two properties essentially *characterize* the quantiles. Our characterization theorems utilize a simple continuity condition; however, this condition simply allows us to obtain a one-parameter family of quantiles. Without the continuity condition, we would have an axiomatization of two one-parameter families of quantiles (usually referred to in the literature as "upper" and "lower" quantiles).

The related literature is relatively small, given the importance of quantiles. This note is similar to Chambers [3], whereby a generalized notion of quantile

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is introduced. In that paper, the concern is not with the space of distribution functions, but with the space of bounded real-valued functions on some measurable space. A representation theorem for all functions which are invariant under ordinal transformation and monotonic is provided. The results in that paper are related to several results found in the social choice and mathematics literature, namely [2, 5, 6, 9, 10, 12, 13]. Manski [8] suggests using quantiles in a decision-theoretic framework similar to that of Savage [15]. Rostek [14] provides a decision-theoretic analysis of quantiles in terms of order structures. She uses Savage-style axioms to uncover the complete behavioral implications of a decision maker who behaves as if she forms a unique probability measure over the measurable space, as well as a state-independent utility index. The decision maker chooses that random variable which maximizes some α-quantile of her utility index according to the endogenous probability measure. Her axiomatization does not rely on the ordinal invariance concept per se, although one axiom in her analysis is closely related.

The literature on risk measures also contains related work. Our work is distinguished from previous contributions in that our axiomatization is not based on any type of additivity condition. Artzner et al [1] characterize a class of risk measures termed the "coherent" risk measures based on subadditivity. Classes of risk measures more general than the quantiles are characterized by Wang et al [17] and Heyde et al [7]. Both of these works rely on additivity principles which are not present here. Specifically, these authors characterize Choquet-expectation (see Schmeidler [16]) style risk measures based on commonotonic additivity and commonotonic subadditivity conditions. Instead of any additivity condition, we utilize a strong covariance property not found in these works to single out the quantiles.

In the next section, we introduce the model and results. We show how to characterize quantiles on the space of bounded distribution functions, as well as on the space of unbounded distribution functions. We also show how to axiomatize a popular risk measure, the *Value at Risk*. A Value at Risk is simply a negation of a quantile; therefore, simple restatements of our axioms allow us to characterize these as well.

2 The model

We will consider two domains. One is the domain of all bounded random variables, and the other is which returns can be unbounded.

In the bounded case, the domain $\mathcal{F}_{bdd}$ is the set of all functions $F : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

i) $\{0, 1\} \subset F(\mathbb{R}) \subset [0, 1]$

ii) F is nondecreasing

iii) F is right-continuous

The domain $\mathcal{F}$ consists of all functions $F : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

i)* $\{0, 1\} \subset \overline{F(\mathbb{R})} \subset [0, 1]$

ii) F is nondecreasing
iii) F is right-continuous

We study mappings $\rho : \mathcal{F} \to \mathbb{R}$ from an axiomatic perspective. A function $\rho : \mathcal{F}_{bdd} \to \mathbb{R}$ or $\rho : \mathcal{F} \to \mathbb{R}$ is interpreted as a statistic. Our primary purpose is to understand the theoretical properties of quantiles as statistics. A few definitions are necessary.

For a strictly increasing and continuous onto function $\varphi : \mathbb{R} \to \mathbb{R}$, $F \circ \varphi^{-1} : \mathbb{R} \to \mathbb{R}$ is that function which results when the payoffs are altered according to the function φ. Let $F, F' \in \mathcal{F}$ and say that $\mathbf{F'}$ **first order stochastically dominates F** if for all $x \in \mathbb{R}$, $F'(x) \leq F(x)$. In this case, we write $F' \geq_{FOSD} F$. This means for any value x, F guarantees a weakly greater probability of achieving at most x. We will say that a sequence $\{\mathbf{F}_n\} \subset \mathcal{F}$ **converges to F** $\in \mathcal{F}$ **in distribution** if for all $x \in \mathbb{R}$ at which F is continuous, $F_n(x) \to F(x)$. In this case, we will write $F_n \to F$.

2.1 Bounded distribution functions

When random variables are bounded, the following axioms are meaningful.

Monotonicity: If $F' \geq_{FOSD} F$, then $\rho(F') \geq \rho(F)$.

For a random variable X and an arbitrary continuous and strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi \circ X$ is that random variable whose returns are the returns of X composed with φ. Note that if F is the distribution function of X, then $F \circ \varphi^{-1}$ is the distribution function of $\varphi \circ X$.

Ordinal covariance: For all $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\varphi(\mathbb{R}) = \mathbb{R}$, strictly increasing and continuous, $\rho\left(F \circ \varphi^{-1}\right) = \varphi(\rho(F))$.

Our last axiom in this environment is a simple continuity condition, requiring that there are no "downward jumps" in the behavior of a statistic.

Upper semicontinuity: If $\{F_n\} \to F$ and $\rho(F_n) \geq \alpha$ for all n, then $\rho(F) \geq \alpha$.

We could equivalently define a notion of lower semicontinuity, and characterize the resulting functions. However, imposing both upper semicontinuity and lower semicontinuity together with monotonicity and ordinal covariance leads to an impossibility.

Theorem 1: The function $\rho : \mathcal{F}_{bdd} \to \mathbb{R}$ satisfies monotonicity, ordinal covariance, and upper semicontinuity if and only if there exists $\alpha \in [0, 1)$ such that $\rho(F) = \inf\{x \in \mathbb{R} : F(x) > \alpha\}$.[1]

[1] Dropping the upper semicontinuity condition would admit functions of the following form:

$$\rho(F) = \sup\{x \in \mathbb{R} : F(x) < \alpha\},$$

for some $\alpha \in (0, 1]$. These are the only additional functions that satisfy the monotonicity and ordinal covariance conditions.

Proof. Suppose $\rho : \mathcal{F} \to \mathbb{R}$ can be represented as $\rho(F) = \inf\{x \in \mathbb{R} : F(x) > \alpha\}$ for some $\alpha \in [0,1)$. To see that ρ is monotonic, suppose that $F \geq_{FOSD} F'$. Then $\{x : F(x) > \alpha\} \subset \{x : F'(x) > \alpha\}$, so that $\inf\{x : F(x) > \alpha\} \geq \inf\{x : F'(x) > \alpha\}$, or $\rho(F) \geq \rho(F')$. To see that ρ satisfies ordinal covariance, let $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(\mathbb{R}) = \mathbb{R}$, φ strictly increasing and continuous. Then

$$\begin{aligned}
& \rho\left(F \circ \varphi^{-1}\right) \\
= \; & \inf\left\{x : \left(F \circ \varphi^{-1}\right)(x) > \alpha\right\} \\
= \; & \inf\left\{x : F\left(\varphi^{-1}(x)\right) > \alpha\right\} \\
= \; & \inf\{\varphi(x) : F(x) > \alpha\} \\
= \; & \varphi(\inf\{x : F(x) > \alpha\}) \\
= \; & \varphi(\rho(F)).
\end{aligned}$$

Here, the fourth equality follows from continuity of φ. Lastly, to verify upper semicontinuity, let $\{F_n\} \subset \mathcal{F}$ for which $\rho(F_n) \geq \beta$. Suppose that $\{F_n\} \to F$. Therefore, for all n, $\inf\{x : F_n(x) > \alpha\} \geq \beta$. Hence, for all $\beta' < \beta$, $F_n\left(\beta'\right) \leq \alpha$. Let $\{\beta_m\} \to \beta$ be a sequence of continuity points of F for which $\beta_m < \beta$ for all m. As $\{F_n\} \to F$, $F_n(\beta_m) \to F(\beta_m)$ for all m, so that $F(\beta_m) \leq \alpha$ for all β_m. As F is nondecreasing, if $F(x) > \alpha$, then $x \geq \beta$. Conclude that $\inf\{x : F(x) > \alpha\} \geq \beta$.

Conversely, suppose that $\rho : \mathcal{F}_{bdd} \to \mathbb{R}$ satisfies monotonicity, ordinal covariance, and upper semicontinuity.

First, let I be any interval for which $F|_I$ is constant. We claim that if $y \in \operatorname{int} I$, then $\rho(F) \neq y$. To see this, suppose by means of contradiction that the statement is false. Then there exists $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon) \subset \operatorname{int} I$. Define

$$\varphi(x) \equiv \begin{cases} x \text{ for } x \notin (y-\varepsilon, y+\varepsilon) \\ \frac{(x-(y-\varepsilon))^2}{2\varepsilon} + (y-\varepsilon) \text{ for } x \in (y-\varepsilon, y+\varepsilon) \end{cases}.$$

Note that $\varphi^{-1}(x) = x$ for all $x \notin (y-\varepsilon, y+\varepsilon)$, and $\varphi^{-1}(x) \neq x$ for all $x \in (y-\varepsilon, y+\varepsilon)$. Clearly, $F \circ \varphi^{-1} = F$. But $\rho(F) = \rho\left(F \circ \varphi^{-1}\right) = \varphi(\rho(F)) \neq \rho(F)$, a contradiction.

Therefore, if F is of the following form:

$$F(x) = \begin{cases} 0 \text{ if } x < y \\ 1 \text{ if } x \geq y \end{cases},$$

then $\rho(F) = y$. Moreover, if F is of the following form,

$$F(x) = \begin{cases} 0 \text{ if } x < y \\ \alpha \text{ if } y \leq x < z \\ 1 \text{ if } x \geq z \end{cases},$$

then $\rho(F) \in \{y, z\}$.

Now, let $F_\alpha \in \mathcal{F}$ be defined as

$$F_\alpha(x) = \begin{cases} 0 \text{ if } x < 0 \\ \alpha \text{ if } 0 \leq x < 1 \\ 1 \text{ if } x \geq 1 \end{cases}.$$

By the preceding argument, for all $\alpha \in [0,1]$, $\rho(F_\alpha) \in \{0,1\}$. By monotonicity, if $\alpha < \beta$, then $F_\alpha \geq_{FOSD} F_\beta$, so that $\rho(F_\alpha) \geq \rho(F_\beta)$. Therefore, there exists some α^* so that for $\alpha > \alpha^*$, $\rho(F_\alpha) = 0$ and for $\alpha < \alpha^*$, $\rho(F_\alpha) = 1$. If $\alpha^* = 0$, we know that $\rho(F_0) = 1$. Suppose $\alpha^* > 0$. Let $\{\alpha^n\}$ be an increasing sequence approaching α^*. Then $\{F_{\alpha^n}\} \to F_{\alpha^*}$. Moreover, $\rho(F_{\alpha^n}) \geq 1$ for all n, so that by upper semicontinuity, $\rho(F_{\alpha^*}) = 1$. This tells us, in particular, that $\alpha^* < 1$.

It is easily verified that for all $\alpha \in [0,1]$, $\rho(F_\alpha) = \inf\{x : F_\alpha(x) > \alpha^*\}$.

Let $F \in \mathcal{F}_{bdd}$. Define $\alpha^*(F) \equiv \inf\{x \in \mathbb{R} : F(x) > \alpha^*\}$. We establish that $\rho(F) = \alpha^*(F)$.

Let $\varepsilon > 0$. Define

$$F_\varepsilon^+(x) = \begin{cases} 0 \text{ if } x < \inf F^{-1}([\alpha^* + \varepsilon, 1]) \\ \alpha^* + \varepsilon \text{ if } \inf F^{-1}([\alpha^* + \varepsilon, 1]) \leq x < \inf F^{-1}(1) \\ 1 \text{ if } \inf F^{-1}(1) \leq x \end{cases}.$$

Clearly, $F_\varepsilon^+ \geq_{FOSD} F$. Therefore, $\rho(F_\varepsilon^+) \geq \rho(F)$. Of course, if $\inf F^{-1}([\alpha^* + \varepsilon, 1]) = \inf F^{-1}(1)$, then there does not exist x such that $\inf F^{-1}([\alpha^* + \varepsilon, 1]) \leq x < \inf F^{-1}(1)$, from which we conclude that $\rho(F_\varepsilon^+) = \inf F^{-1}([\alpha^* + \varepsilon, 1])$. Otherwise, let $\varphi : \mathbb{R} \to \mathbb{R}$ be any strictly increasing and continuous function for which $\varphi(0) = \inf F^{-1}([\alpha^* + \varepsilon, 1])$ and $\varphi(1) = \inf F^{-1}(1)$, for which $\varphi(\mathbb{R}) = \mathbb{R}$.[2] Note that $F_\varepsilon^+ = F_{\alpha^*+\varepsilon} \circ \varphi^{-1}$. Hence, by ordinal covariance, $\rho(F_\varepsilon^+) = \varphi(\rho(F_{\alpha^*+\varepsilon}))$. But as $\alpha^* + \varepsilon > \alpha^*$, $\rho(F_{\alpha^*+\varepsilon}) = 0$, so that $\rho(F_\varepsilon^+) = \inf F^{-1}([\alpha^* + \varepsilon, 1])$. Therefore, $\rho(F) \leq \inf F^{-1}([\alpha^* + \varepsilon, 1])$, but as ε is arbitrary, we conclude $\rho(F) \leq \inf F^{-1}((\alpha^*, 1]) = \alpha^*(F)$.

Define

$$F^-(x) = \begin{cases} 0 \text{ if } x < \inf F^{-1}((0,1]) \\ \alpha^* \text{ if } \inf F^{-1}((0,1]) \leq x < \inf F^{-1}((\alpha^*, 1]) \\ 1 \text{ if } \inf F^{-1}((\alpha^*, 1]) \leq x \end{cases}.$$

F^- is clearly well-defined for $\alpha^* = 0$. Clearly, $F \geq_{FOSD} F^-$. Therefore, $\rho(F) \geq \rho(F^-)$. Moreover, as in the previous paragraph, one can verify that $\rho(F^-) = \inf F^{-1}((\alpha^*, 1])$. Therefore, $\rho(F) \geq \inf F^{-1}((\alpha^*, 1]) = \alpha^*(F)$.

We have shown that $\alpha^*(F) \leq \rho(F) \leq \alpha^*(F)$. Therefore, $\rho(F) = \inf\{x : F(x) > \alpha^*\}$. ■

[2] For example, let

$$\varphi(x) = \left[\inf F^{-1}(1) - \inf F^{-1}([\alpha^* + \varepsilon, 1])\right] x + \inf F^{-1}([\alpha^* + \varepsilon, 1]).$$

2.2 Unbounded distribution functions

In this section, we discuss the theory of unbounded distribution functions. While we do not wish to allow for infinite payoffs, there is a limiting sense in which unbounded risks are ordinal transformations of bounded risks. Thus, we need to strengthen the notion of ordinal covariance in this section.

Strong ordinal covariance: Let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ be increasing, strictly increasing and continuous on $\varphi^{-1}(\mathbb{R})$, and $\mathbb{R} \subset \varphi(\mathbb{R})$. Let $F \in \mathcal{F}$. Suppose that $F \circ \varphi^{-1} \in \mathcal{F}$. Then $\rho\left(F \circ \varphi^{-1}\right) = \varphi(\rho(F))$.

This is distinct from ordinal covariance in that it allows for transformations that result in unbounded random variables. Note the requirement that $F \circ \varphi^{-1} \in \mathcal{F}$. This requirement is necessary as for general φ, it is possible that $F \circ \varphi^{-1}$ puts an atom at an infinite value (which we do not wish to allow).

Theorem 2: The function $\rho : \mathcal{F} \to \mathbb{R}$ satisfies monotonicity, strong ordinal covariance, and upper semicontinuity if and only if there exists $\alpha \in (0, 1)$ such that $\rho(F) = \inf\{x \in \mathbb{R} : F(x) > \alpha\}$.

Proof. It is straightforward to verify that such functions satisfy monotonicity, strong ordinal covariance, and upper semicontinuity (similarly to Theorem 1).

It remains to establish that if ρ satisfies monotonicity, strong ordinal covariance, and upper semicontinuity, then there exists α for which $\rho(F) = \inf\{x : F(x) > \alpha\}$. The function $\rho|_{\mathcal{F}_{bdd}}$ has such a representation for some $\alpha \in [0, 1)$, as established in the preceding theorem. We first establish that $\alpha > 0$. Suppose, by means of contradiction, that $\alpha = 0$. Let $G \in \mathcal{F}$ such that $1 \in G(\mathbb{R})$, but $0 \notin G(\mathbb{R})$. Let $G_n \in \mathcal{F}$ be defined as $G_n(x) = G(x) 1_{\{x \geq -n\}}$, and note that $G_n \in \mathcal{F}_{bdd}$. Hence $\rho(G_n) = -n$. Moreover, note that $G_n \geq_{FOSD} G$. Consequently, by monotonicity, $\rho(G) \leq \rho(G_n)$ for all n, from which we conclude that $\rho(G)$ is not real-valued, a contradiction. Hence, $\alpha > 0$.

Let $F \in \mathcal{F}$. Consider the function

$$\varphi(x) = \begin{cases} \tan(x) \text{ for } x \in (-\pi/2, \pi/2) \\ +\infty \text{ for } x \geq \pi/2 \\ -\infty \text{ for } x \leq -\pi/2 \end{cases}.$$

Define F' as follows:

$$F'(x) \equiv \begin{cases} F(\tan(x)) \text{ for } x \in (-\pi/2, \pi/2) \\ 1 \text{ for } x \geq \pi/2 \\ 0 \text{ for } x \leq -\pi/2 \end{cases}.$$

Clearly, $F' \in \mathcal{F}_{bdd}$. This follows as tan is strictly increasing and continuous, and $\lim_{x \to -\pi/2} \tan(x) = -\infty$, so that $\lim_{x \to -\pi/2+} F'(x) = 0$, so that right continuity is maintained. Therefore, $\rho(F') = \inf\{x : F'(x) > \alpha\}$. Now, we verify that $F' \circ \varphi^{-1} = F$. Clearly, $\varphi^{-1} : \mathbb{R} \to \mathbb{R}$ is simply the arctan function.

Thus, let $x \in \mathbb{R}$ be arbitrary. Then $F'(\arctan(x)) = F(\tan(\arctan(x))) = F(x)$, where the second equality follows as $\arctan(\mathbb{R}) \subset (-\pi/2, \pi/2)$.

Hence, $F' \circ \varphi^{-1} \in \mathcal{F}$, so by strong ordinal covariance, $\rho(F) = \rho\left(F' \circ \varphi^{-1}\right) = \varphi(\rho(F'))$. Therefore,

$$\begin{aligned} & \varphi(\rho(F')) \\ = & \varphi(\inf\{x : F(\tan(x)) > \alpha\}) \\ = & \varphi(\inf\{\arctan(x) : F(\tan(\arctan(x))) > \alpha\}) \\ = & \inf\{\tan(\arctan(x)) : F(x) > \alpha\} \\ = & \inf\{x : F(x) > \alpha\}. \end{aligned}$$

Therefore, $\rho(F) = \inf\{x \in \mathbb{R} : F(x) > \alpha\}$. ■

2.3 Risk measures and the Value at Risk

The preceding theorems can also be investigated in a financial setting. Formally, a risk measure $\rho : \mathcal{F} \to \mathbb{R}$ is a function which, for any state-dependent monetary outcome, recommends an amount of money that needs to be added to each state to induce an agent to take the risk. The less risky something is, then, the better it is to take it.

By changing our ordinal covariance axiom to the following, we are able to characterize an important class of risk measures, namely, the Value at Risk measures:

Inverse strong ordinal covariance: Let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ be increasing, strictly increasing and continuous on $\varphi^{-1}(\mathbb{R})$, and $\mathbb{R} \subset \varphi(\mathbb{R})$. Let $F \in \mathcal{F}$. Suppose that $F \circ \varphi^{-1} \in \mathcal{F}$. Then $\rho\left(F \circ \varphi^{-1}\right) = -\varphi(-\rho(F))$.

Antimonotonicity: For all $F, F' \in \mathcal{F}$, if $F \leq_{FOSD} F'$, then $\rho(F) \geq \rho(F')$.

These two conditions are natural for a risk measure. Inverse ordinal covariance is the requirement that a risk measure should be invariant under units of measurement; thus, the tax structure should not affect the choices recommended by a particular risk measure.

Define the α-VaR, or $\mathrm{VaR}_\alpha : \mathcal{F} \to \mathbb{R}$ by

$$\mathrm{VaR}_\alpha(F) \equiv -\inf\{x \in \mathbb{R} : F(x) > \alpha\}.$$

This definition coincides with the definition given by Artzner et al [1].

We also need the following:

Lower semicontinuity: If $\{F_n\} \to F$ and $\rho(F_n) \leq \alpha$ for all n, then $\rho(F) \leq \alpha$.

Together with real-valuedness, these axioms are enough to characterize the α-VaR for $\alpha \in (0,1)$.

Theorem 3: A risk measure $\rho : \mathcal{F} \to \mathbb{R}$ satisfies antimonotonicity, inverse strong ordinal covariance, and lower semicontiuity if and only if there exists $\alpha \in (0,1)$ for which $\rho = \mathrm{VaR}_\alpha$.

Proof. Clearly, if there exists such an α, ρ satisfies the corresponding axioms (verified similarly to Theorem 1).

Conversely, suppose that ρ satisfies the axioms. Consider the function $-\rho$. Then it is simple to see that $-\rho$ satisfies monotonicity, ordinal covariance, and upper semicontinuity. Therefore, there exists some $\alpha \in (0,1)$ for which $-\rho(F) \equiv \inf\{x : F(x) > \alpha\}$. Consequently, $\rho(F) = -\inf\{x : F(x) > \alpha\}$. ■

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