General Revealed Preference Theory

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A contribution to positive theory.

Axioms here are *descriptive*. (SARP)

A basic decision theory exercise

Axiomatize $\succeq$ for which:

$\exists u$ (satisfying some properties) such that

$$\forall x \forall y, x \succeq y \leftrightarrow u(x) \geq u(y)$$

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Axiomatize R, P (revealed preference relations) for which:

$\exists \succeq$ (satisfying some properties) such that

$\forall x \forall y, x\,R\,y \rightarrow x \succeq y$ and $x\,P\,y \rightarrow x \succ y$

A basic choice theory exercise

Axiomatize R, P (revealed preference relations) for which:

$\exists \succeq$ (satisfying some properties) such that

$\forall x \forall y, x\,R\,y \to x \succeq y$ and $x\,P\,y \to x \succ y$

In both cases, there are theoretical (unobserved) constructs: u in one $\succeq$ in the other

A *positive axiomatization* should be a description of possible data. Should not refer to theoretical objects, but only to observables

Example: Classical revealed preference theory

Observe data R, P

Theorize:
$\exists \succeq, \succ$ such that

- $\forall x \forall y, x\, R\, y \rightarrow x \succeq y$
- $\forall x \forall y, x\, P\, y \rightarrow x \succ y$
- $\forall x \forall y, (x \succeq y) \vee (y \succeq x)$
- $\forall x \forall y \forall z, ((x \succeq y) \wedge (y \succeq z)) \rightarrow x \succeq z$
- $\forall x \forall y, (x \succ y) \leftrightarrow (x \succeq y) \wedge \neg(y \succeq x)$

Equivalent to axiomatization:

$$\forall x_1 \forall x_2 \ldots \forall x_K, \neg \bigwedge_{i=1}^{K} (x_i \, Q_i \, x_{i+1}),$$

where all $Q_i \in \{R, P\}$ and at least one $Q_i = P$.

Absence of cycles–variant of SARP

Universal axiomatization in terms of R, P (observables). Does not refer to theoretical $\succeq$. Effective procedure for falsifying model.

Falsification

Axiomatization should provide *falsifiable* theory (or refutable by observable data)

Data should be able to refute axioms.

The nature of falsifiable theories

Popper's theories:

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All swans are white:
$\forall s W(s)$

There exists a black swan:
$\exists s B(s)$

The nature of falsifiable theories

Popper's theories:

All swans are white:
$\forall s W(s)$

Falsifiable

There exists a black swan:
$\exists s B(s)$

Not falsifiable

A basic question

Does the idea behind SARP generalize?

Under what conditions can we find universal positive description of a given economic theory, which leads to effective falsification procedure?

This paper

For any "open formulas" $\varphi_1, \varphi_2, \dots$ in language with predicates $R_1, \dots, R_n, Q_1, \dots Q_k$, we would like to find "open formulas" $\psi_1, \psi_2, \dots$

$$\exists Q_1 \dots \exists Q_k \text{ such that}$$
$$\forall x_1 \dots \forall x_{n_1} \varphi_1(R_1, \dots, R_n, Q_1, \dots, Q_k, x_1, \dots, x_{n_1})$$
$$\forall x_1 \dots \forall x_{n_2} \varphi_2(R_1, \dots, R_n, Q_1, \dots, Q_k, x_1, \dots, x_{n_2})$$
$$\vdots$$

if and only if

$$\forall x_1 \dots \forall x_{m_1} \psi_1(R_1, \dots, R_n, x_1, \dots, x_{m_1})$$
$$\forall x_1 \dots \forall x_{m_2} \psi_2(R_1, \dots, R_m, x_1, \dots, x_{m_2})$$
$$\vdots$$

Language

Two *relational* languages, $\mathcal{L}$ and $\mathcal{F}$, where $\mathcal{F} = (R_1, ..., R_N)$ and $\mathcal{L} = (R_1, ..., R_N, Q_1, ..., Q_K)$.

Symbols R_i or Q_i used to talk about k-ary relations (for us usually binary relations).

Choice example, $\mathcal{F} = (R, P)$ and $\mathcal{L} = (R, P, \succeq, \succ)$. Language $\mathcal{F}$: observables, language $\mathcal{L}$: (observables and unobservables).

Theories and structures

Theories specify what we believe is "true;" a class of "possible worlds"

A *structure* is a possible world for a language.

For language $\mathcal{F} = (R_1, ..., R_N)$, an $\mathcal{F}$-structure is a set X and a collection of relations $R_1^X, ..., R_N^X$.

Example

$\mathcal{F} = (R, P)$, two binary relation symbols.

$X = \{x, y, z, w\}$
$R^X = \{(x, y), (w, w), (x, z)\}$
$P^X = \{(x, z), (z, x)\}$

$X = \{1, 2, 3, ...\}$
$R^X = >$
$P^X = \leq$

are examples of $\mathcal{F}$-structures.

A *theory* for language $\mathcal{F}$ is a class of $\mathcal{F}$-structures.

Theories specify possible worlds.

Example

The class of all structures (X, R^X, P^X) where R^X is a weak order and P^X is its strict part is a theory.

Class of all structures (X, R^X, P^X) for which there exists $\succeq^X$, $\succ^X$ for which $\succeq^X$ is a weak order, $\succ^X$ is its strict part, and $R^X \subseteq \succeq^X$, $P^X \subseteq \succ^X$ is a theory.

Recall $\mathcal{F} \subseteq \mathcal{L}$. Let T be an $\mathcal{L}$-theory.

Define $F(T)$ to be the class of all $\mathcal{F}$-structures $(X, R_1^X, \ldots, R_N^X)$ for which there exist $Q_1^X, \ldots, Q_K^X$ such that

$$(X, R_1^X, \ldots, R_N^X, Q_1^X, \ldots, Q_K^X) \in T.$$

$F(T)$ is a projection of $\mathcal{L}$-theory T onto language $\mathcal{F}$.

Example

Let $\mathcal{F} = (R, P)$, and let $\mathcal{L} = (R, P, \succ, \succeq)$.

T: $\mathcal{L}$-theory of structures $(X, R^X, Q^X, \succ^X, \succeq^X)$ for which

1. $\succeq^X$ is a weak order
2. $\succ^X$ is its strict part
3. $R^X \subseteq \succeq^X$
4. $P^X \subseteq \succ^X$.

$F(T)$ is the $\mathcal{F}$-theory of all structures (X, R^X, Q^X) for which there exists $\succ^X$, $\succeq^X$ for which 1-4 is satisfied.

Main point: If T has a universal axiomatization in language $\mathcal{L}$, then $F(T)$ has a universal axiomatization in language $\mathcal{F}$.

To formalize this, we need to talk about axiomatizations.

Axioms and axiomatizations

- An axiom is a mathematical statement which formalizes some real-world notion.
- The way to study axioms is called model theory: A branch of logic relating axioms to “concrete” environments

Axioms

Examples of (first order) axioms using symbols $P, R, \succ, \succeq, O$

- Completeness: $\forall x \forall y (x \succeq y) \vee (y \succeq x)$
- Transitivity: $\forall x \forall y \forall y (x \succeq y) \wedge (y \succeq z) \rightarrow (x \succeq z)$
- Nonsatiation: $\forall x \exists y (y \succ x)$
- Subrelation: $\forall x \forall y (x \, R \, y) \rightarrow (x \succeq y)$
- Optimality: $\forall x O(x) \leftrightarrow \forall y (x \, R \, y)$

We can say when a structure *satisfies* an axiom (or collection of axioms).

The theory axiomatized by a collection of axioms is the class of all structures satisfying those axioms.

Quantification operates on variables (first order).

$F(T)$ is implicitly an “existential second order” theory (hypothesizing existence of relations in $\mathcal{L} \backslash \mathcal{F}$).

Any first order axiom can be put in *prenex normal form* with all quantification in front.

Universal axioms are those with universal ($\forall$) quantification, coming at the beginning of the sentence:

$\forall x \forall y (x \succeq y) \vee (y \succeq x)$ is universal.
$\forall x \exists y (x \succ y)$ is not.
$\forall x \forall y O(x) \leftrightarrow \forall y (x \, R \, y)$ is not.

Main result

For languages $\mathcal{F} \subseteq \mathcal{L}$, and $\mathcal{L}$-axioms Σ, the set of $\mathcal{F}$-consequences of Σ is the collection of all logical consequences of Σ involving only symbols from $\mathcal{F}$.

Theorem

Suppose that T is a universally axiomatizable $\mathcal{L}$-theory, and that $\mathcal{F} \subseteq \mathcal{L}$. Then $F(T)$ is a universally axiomatizable $\mathcal{F}$-theory, and is axiomatized by the set of all universal $\mathcal{F}$-consequences of T.

There is an (impractical) algorithm for uncovering this axiomatization.

Recursive enumerability

A set of axioms is **recursively enumerable** if there is an algorithm for listing them out, one by one.

Corollary

If T is universally and recursively enumerably axiomatizable, then so is $F(T)$.

Gives an effective procedure for falsifying a theory: check axioms one by one. If theory is false, its falsehood will be ascertained.

R.e. means that we can check *whether* a given statement is an axiom, not *whether or not*. Latter property is called *recursive*.

Craig's result: any theory with a r.e. axiomatization has a recursive axiomatization. Does not imply that any theory with a universal r.e. axiomatization has a universal recursive axiomatization.

Interpretation

For any open formulas $\varphi_1, \varphi_2, \dots$ in language $\mathcal{L}$, there are open formulas $\psi_1, \psi_2, \dots$ in language $\mathcal{F}$ for which the following is true for $\mathcal{F}$-structures:

$$\begin{gathered}\exists Q_1 \dots \exists Q_k \text{ such that} \\ \forall x_1 \dots \forall x_{n_1} \varphi_1(R_1, \dots, R_n, Q_1, \dots, Q_k, x_1, \dots, x_{n_1}) \\ \forall x_1 \dots \forall x_{n_2} \varphi_2(R_1, \dots, R_n, Q_1, \dots, Q_k, x_1, \dots, x_{n_2}) \\ \vdots\end{gathered}$$

if and only if

$$\begin{gathered}\forall x_1 \dots \forall x_{m_1} \psi_1(R_1, \dots, R_n, x_1, \dots, x_{m_1}) \\ \forall x_1 \dots \forall x_{m_2} \psi_2(R_1, \dots, R_m, x_1, \dots, x_{m_2}) \\ \vdots\end{gathered}$$

Applications

Any application where there are unobserved theoretical relations...

- Multiple selves
- Revealed game theory
- Group preferences (Pareto relation, majority rule, etc)
- Choice theory
- *Potentially* decision theory

Multiple selves, or group preferences

Observe relation R. Hypothesize that R is generated by given finite set of agents N and given social choice rule f (satisfying neutrality and IIA).

Agents hypothesized to have "rational" preferences.

This theory is universally and r.e. axiomatizable.

Special case: Pareto extension relation on N agents

Observe R.

Hypothesize: $\exists P \exists R_i \exists P_i$ such that:

- $\forall x \forall y, x\, P\, y \leftrightarrow \bigwedge_{i \in N} x\, P_i\, y$
- $\forall x \forall y, x\, R_i\, y \vee y\, R_i\, x$
- $\forall y \forall y \forall z, x\, R_i\, y \wedge y\, R_i\, z \rightarrow x\, R_i\, z$
- $\forall x \forall y, x\, P\, y \leftrightarrow x\, R\, y \wedge \neg y\, R\, x$
- $\forall x \forall y, x\, P_i\, y \leftrightarrow x\, R_i\, y \wedge \neg y\, R_i\, x$

Special case: Pareto extension relation on N agents

Observe R.

Hypothesize: $\exists P \exists R_i \exists P_i$ such that:

- $\forall x \forall y, x \, P \, y \leftrightarrow \bigwedge_{i \in N} x \, P_i \, y$
- $\forall x \forall y, x \, R_i \, y \vee y \, R_i \, x$
- $\forall y \forall y \forall z, x \, R_i \, y \wedge y \, R_i \, z \rightarrow x \, R_i \, z$
- $\forall x \forall y, x \, P \, y \leftrightarrow x \, R \, y \wedge \neg y \, R \, x$
- $\forall x \forall y, x \, P_i \, y \leftrightarrow x \, R_i \, y \wedge \neg y \, R_i \, x$

Mathematicians have worked out the axioms for the case $|N| = 2$.

Why proof is non-trivial

Clearly: any $\mathcal{F}$-consequence of T is satisfied by $F(T)$, and if an $\mathcal{F}$ axiom is true for $F(T)$, it is true for T (and hence an $\mathcal{F}$ consequence).

Main difficulty is in establishing that $F(T)$ is axiomatizable. Need not necessarily be true.

Proof of main result relies on a result of Tarski, which characterizes universally axiomatizable theories. Also relies on some form of choice (Szpilrajn's theorem is a corollary of our result).

Important: In general, $F(T)$ need not even be axiomatizable, even if T is. Universality of T is critical.

Complexity

Fagin's theorem guarantees that, for finite worlds, deciding whether or not a structure is an element of $F(T)$ is a problem in NP when T is finitely axiomatizable.

Conclusion

There was once a debate on the falsifiability of economic theories whose specification involves theoretical terms:

> *Given the premise–"All consumers maximize something"–the critic can claim he has found a consumer who is not maximizing anything. The person who assumed the premise is true can respond: "You claim you have found a consumer who is not a maximizer but how do you know there is not something which he is maximizing?" In other words, the verification of the counterexample requires the refutation of a strictly existential statement; and as stated above, we all agree that one cannot refute existential statements.*
>
> *L. Boland*

Others have recognized that existential theoretical terms do not render a theory non-falsifiable:

> *Although existential quantification of an observable is fatal to the falsifiability of a theory, the same is not true when the existentially quantified term is a theoretical one.*
>
> *H. Simon*

> *All of the counterexamples thus far known involve second-order quantification... the counterexamples turn out to be innocuous first-order statements of the usual, refutable type.*
>
> *P. Mongin*

Open questions/Work in progress

There is an *algorithm* for uncovering universal implications on observables. But practically speaking it is not useful.
So far we have no axiomatizations which have not already been uncovered. A basic goal is to use our methodology to uncover this.

- Given some d.t. model, what is its empirical content?
- Would require a result involving functions
- For example, what is the empirical content of SEU? Multiple priors? More general models?
  - Note that the Ellsberg paradox of SEU comes in a form specifically to violate an axiom (STP)
  - But given general data, it is hard to tell if it falsifies SEU
  - There are *some* results in this direction