The Structure of Household Preference

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Abstract

In an environment of Samuelsonian aggregation, we characterize those social welfare functions which (*i*) map concave utility profiles to concave household utility functions and (*ii*) map quasiconcave utility profiles to quasiconcave household utility functions. Case (*i*) holds for any concave social welfare function, while case (*ii*) holds only for *generalized maxmin* social welfare functions. Lastly, we establish a simple duality result for computing the household utility under a maxmin social welfare function: the household expenditure function is the sum of the individual expenditure functions.

1 Introduction

The purpose of this paper is to establish a few simple facts on the structure of preferences of representative consumers. Following Samuelson (1956), we assume that a household of individuals allocates aggregate consumption in order to maximize social welfare. The resulting induced preference over aggregate consumption is termed the *household preference*. In this paper, we study the structure of the social welfare function and the implied structure of household utility.

We make one basic assumption: the social welfare function is of the Bergson Samuelson form, and operates directly on utility levels.[1] The results are straight-

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[1]See Burk (1938) for the introduction of the concept.

forward: our first result establishes that a social welfare function is concave if and only if, for any household of individuals with concave utility functions, the resulting household utility is concave. The second result establishes that a social welfare function generates quasiconcave household utility for all households of individuals with quasiconcave preferences if and only if it is "generalized maxmin." By this, we simply mean that the social welfare function takes the form

$$W(u_1, ..., u_n) = \min_i \{\varphi_1(u_1), ..., \varphi_n(u_n)\}$$

for some increasing functions φ_i.

Quasiconcave preferences are relevant as they are the most general class of preferences allowing classical results such as the existence of equilibrium to hold.[2] Moreover, they are (together with nondecreasingness and continuity requirements) the most general class which allow the direct utility function to be completely recoverable from the indirect utility function.[3] As the main result in Samuelson (1956) is to establish that the map which associates indirect utility to direct utility commutes with respect to the social welfare function, we view this kind of result as useful.[4] Hence, only under the case of quasiconcave household preference could we first obtain indirect household utility and then recover direct household utility.

The household preference model is a particularly useful model in applied economics settings; and much has been done to understand its implications in terms of behavior. See Cherchye et al. (2010, 2012, 2011).

Lastly, we establish a simple duality result allowing us to explicitly compute household utility for the generalized maxmin social welfare functions. This result states that for the maxmin social welfare function, the expenditure function of the household is the sum of expenditure functions of each individual in the household. For a quasiconcave and upper semicontinuous preference, the utility function can be recovered from the expenditure function.

[2] As attributed to McKenzie (1954) and Arrow and Debreu (1954).

[3] If $v(p, w)$ is the indirect utility, the direct utility can be recovered via the formula $u(x) = \inf_{p: p\cdot x \leq 1} v(p, 1)$. See Martinez-Legaz (1991) for a very general result; and Lau (1969) for an early such result.

[4] Chipman and Moore (1979) produces the first fully general version of this result; Samuelson assumed quasiconcavity of preference, which is unnecessary. See Dow and Sonnenschein (1986) for an exposition.

2 The model

Let $N = \{1, ..., n\}$ be a set of agents. For m-dimensional Euclidean space $\mathbb{R}^m$, if $x, y \in \mathbb{R}^m$, we write $x \ll y$ if $x_i < y_i$ for all i, and $x \leq y$ if $x_i \leq y_i$ for all i. Say a real-valued function f on a subset of Euclidean space is *increasing* if $x \ll y$ implies $f(x) < f(y)$. It is *strictly increasing* if $x \leq y$ and $x \neq y$ implies $f(x) < f(y)$.

A social welfare function is a continuous and increasing $W : \mathbb{R}^N \to \mathbb{R}$.

A *utility profile* is a tuple consisting of a pair $m \in \mathbb{N}$, and a collection of functions $(U_1, ..., U_n)$, where each $U_i : \mathbb{R}^m_+ \to \mathbb{R}$ is continuous and increasing.

Given a utility profile $(m, U_1, ..., U_n)$, we can define an induced household utility function $V : \mathbb{R}^m_+ \to \mathbb{R}$ given by

$$V(x) = \sup_{\sum_i x_i = x} W(U_1(x_1), ..., U_n(x_n)).$$

This is called *the household utility induced* by $(m, U_1, ..., U_n)$.

The question addressed here relates the structure of V to the structure of the functions U_i.

Proposition 1. *The household utility induced by $(m, U_1, ..., U_n)$ is concave for all concave utility profiles if and only if W is concave.*

Proof. Suppose W is concave. Let $(m, U_1, ..., U_n)$ be a utility profile. Let $\alpha \in [0, 1], x, y \in \mathbb{R}^m_+$. Let $(x_1, ..., x_n)$ satisfy

$$V(x) = W(U_1(x_1), ..., U_n(x_n))$$

and let $(y_1, ..., y_n)$ satisfy

$$V(y) = W(U_1(y_1), ..., U_n(y_n)).$$

Then in particular

$$\begin{aligned}
&\alpha V(x) + (1 - \alpha)V(y) \\
&= \alpha W(U_1(x_1), ..., U_n(x_n)) + (1 - \alpha)W(U_1(y_1), ..., U_n(y_n)) \\
&\leq W(\alpha U_1(x_1) + (1 - \alpha)U_1(y_1), ..., \alpha U_n(x_n) + (1 - \alpha)U_n(y_n)) \\
&\leq W(U_1(\alpha x_1 + (1 - \alpha)y_1), ..., U_n(\alpha x_n + (1 - \alpha)y_n)) \\
&\leq V(\alpha x + (1 - \alpha)y).
\end{aligned}$$

Consequently, V is concave.

Conversely, suppose V concave. Let $m = N$. Then, for all $i \in N$, let $U_i : \mathbb{R}^N_+ \to \mathbb{R}$ be defined as $U_i(x) = x_i$. By definition, $V = W$. Consequently, W is concave. ∎

Observe that the proof of Proposition 1, establishing that only concave W are admissible utilizes preferences which are not strictly monotonic. We can actually show that Proposition 1 holds on the domain of strictly monotonic preferences as well.

Proposition 2. *The household utility induced by $(m, U_1, \ldots, U_n)$ is concave for all strictly monotone and concave utility profiles if and only if W is concave.*

Proof. The first part of the proof is identical to that of the proof of Proposition 1.

Conversely, consider $m = n$ (as in the preceding proof), and for each $\lambda \geq 0$ and $i \in N$, let $U_i^\lambda(x) = x_i + \lambda \sum_{j \neq i} x_j$. For each λ, let $\mathcal{U}(\lambda, x) \equiv \{u \in \mathbb{R}^N : u = U_i^\lambda(x_i), \sum_i x_i = x\}$ be the *utility possibility set* for x.

It is straightforward to observe that, viewed as a correspondence in $\lambda, \mathcal{U}(\lambda, x)$ is continuous and compact-valued.

Let $x, y \in \mathbb{R}^m_+$ and $\alpha \in [0, 1]$. Let V^λ be the household utility induced by $(m, U_1^\lambda, \ldots, U_n^\lambda)$. By monotonicity of W, $W(x) = V^0(x)$, $W(y) = V^0(y)$ (observe that U_i^0 is not strictly monotone; but this holds true in any case).

By the Maximum Theorem (see Berge (1963)), for any $\epsilon > 0$, there therefore exists λ^* for which $|V^{\lambda^*}(x) - V^0(x)| < \epsilon$, $|V^{\lambda^*}(y) - V^0(y)| < \epsilon$, and $|V^{\lambda^*}(\alpha x + (1 - \alpha y)) - V^0(\alpha x + (1 - \alpha)y)| < \epsilon$.

Observe that by hypothesis of the Proposition, $V^{\lambda^*}(\alpha x + (1 - \alpha y) \geq \alpha V^{\lambda^*}(x) + (1 - \alpha)V^{\lambda^*}(y)$.

$$\begin{aligned}
&V^0(\alpha x + (1 - \alpha)y) + \epsilon \\
&\geq V^{\lambda^*}(\alpha x + (1 - \alpha)y) \\
&\geq \alpha V^{\lambda^*}(x) + (1 - \alpha)V^{\lambda^*}(y) \\
&\geq \alpha V^0(x) + (1 - \alpha)V^0(y) - \epsilon.
\end{aligned}$$

The result follows as $\epsilon > 0$ was arbitrary, and as $W(x) = V^0(x)$, $W(y) = V^0(y)$ and $W(\alpha x + (1 - \alpha)y) = V^0(\alpha x + (1 - \alpha)y)$. ∎

3 Quasiconcave households

Say that W is *generalized maxmin* if there is $M \subseteq N$, $M \neq \varnothing$, and for each $i \in M$, there exists $\varphi_i : \mathbb{R} \to \mathbb{R}$ continuous and increasing for which $W(x_1, \ldots, x_n) = \min_{i \in M} \varphi_i(x_i)$.

Proposition 3. *The household utility induced by $(m, U_1, ..., U_n)$ is quasiconcave for all quasiconcave utility profiles if and only if W is generalized maxmin.*

We will use the following lemma, which is both geometrically obvious and almost certainly not novel. It is closely related to work in Hougaard and Keiding (1998); Christensen et al. (1999); Chambers and Miller (2014a,b); Chambers et al. (2014), but does not follow directly from these results. As is standard, $\wedge$ denotes the *meet*, or componentwise minimum of two vectors.

Lemma 3.1. *An increasing and continuous function $W : \mathbb{R}^N \to \mathbb{R}$ is generalized maxmin if and only if for each $x, y \in \mathbb{R}^N$, we have $W(x \wedge y) \geq \min\{W(x), W(y)\}$ (equivalently $W(x \wedge y) = \min\{W(x), W(y)\}$).*

Proof. First, we show that generalized maxmin functions have this property. To this end, observe that, by quasiconcavity,

$$\begin{aligned}
&\min_{i \in M} \varphi_i(\min\{x_i, y_i\}) \geq \min_{i \in M} \min_{z \in \{x,y\}} \varphi_i(z_i) \\
&= \min_{z \in \{x,y\}} \min_{i \in M} \varphi_i(z_i) \\
&= \min\{W(x), W(y)\}
\end{aligned}$$

Now, we establish the converse. Consider $W(\mathbb{R}^N) \subseteq \mathbb{R}$. We claim that for any $\alpha \in W(\mathbb{R}^N)$, there is $x(\alpha) \in \mathbb{R}^N$ such that $W(x) \geq \alpha$ if and only if $x \geq x(\alpha)$.

To see this, first observe that by the fact that W increasing and continuous, $W(\mathbb{R}^N)$ is an open interval in $\mathbb{R}$. Now, let $\alpha \in W(\mathbb{R}^N)$. Now, for each $i \in N$, either there is $x^* \in \mathbb{R}^N$ for which $W(x^*) = \alpha$ and $W(x_i^* - \epsilon, x_{-i}) < \alpha$ for all $\epsilon > 0$, or there is not. Let $N(\alpha) \subseteq N$ constitute the set of $i \in N$ for which such an x^* exists.

First, we claim that $N(\alpha) \neq \varnothing$. Suppose false. Then fix any x' for which $W(x') = \alpha$. Let $\epsilon_1 > 0$ such that $W(x'_1 - \epsilon_1, x_{-1}) = \alpha$ (by monotonicity, continuity, and the fact that $N(\alpha) = \varnothing$, such ϵ_1 must exist). Inductively, let $\epsilon_i > 0$ such that $W(x'_1 - \epsilon_1, \ldots, x'_i - \epsilon_i, x'_{i+1}, \ldots, x_n) = \alpha$. Eventually, we conclude that $W(x'_1 - \epsilon_1, \ldots, x_n - \epsilon_n) = \alpha$, contradicting monotonicity.

Now, if $i \in N(\alpha)$, then if $x^*, y^* \in \mathbb{R}^N$ with $W(x^*) = W(y^*) = \alpha$, and such that there is no $\epsilon > 0$ as described previously, then we claim $x^*_i = y^*_i$. Suppose false, and without loss that $x^*_i < y^*_i$. Then in particular $\alpha = W(x^* \wedge y^*) \leq W(x^*_i, y^*_{-i}) < W(y^*) = \alpha$, a contradiction. Let us label this value as $x_i(\alpha)$.

We now construct a vector $x(\alpha) \in \mathbb{R}^{N(\alpha)}$ by taking, for each $i \in N(\alpha)$, the value x^*_i constructed in the previous paragraph.

We claim that for any $z \in \mathbb{R}^N$, $W(z) \geq \alpha$ if and only if $z_{N(\alpha)} \geq x(\alpha)$.

To this end, suppose that $W(z) \geq \alpha$, and let $i \in N(\alpha)$. We want to show that $z_i \geq x_i(\alpha)$. Suppose by means of contradiction that $z_i < x_i(\alpha)$. By definition of $x_i(\alpha)$, there is $x^* \in \mathbb{R}^N$ such that $W(x^*) = \alpha$, and $x^*_i = x_i(\alpha)$, where if $\epsilon > 0$, then $W(x^*_i - \epsilon, x^*_{-i}) < \alpha$. So, $W(z \wedge x^*) = \alpha$. Choose $\epsilon > 0$ small so that $x_i(\alpha) - \epsilon > z_i$. Then $z \wedge x^* = z \wedge (x^*_i - \epsilon, x^*_{-i})$, so $W(z \wedge (x^*_i - \epsilon, x^*_{-i})) = \alpha$. But also, $W(x^*_i - \epsilon, x^*_i) < \alpha$, so $W(z \wedge (x^*_i - \epsilon, x^*_{-i})) < \alpha$, a contradiction. Conclude that $z_i \geq x_i(\alpha)$ and so $z_{N(\alpha)} \geq x(\alpha)$.

Suppose instead that $z_{N(\alpha)} \geq x(\alpha)$, and by means of contradiction that $W(z) < \alpha$. First, it is clear by construction of $x_i(\alpha)$ for each $i \in N(\alpha)$ and by the meet homomorphism property that there exists $x^* \in \mathbb{R}^N$ such that $W(x^*) = \alpha$ and $x^*_{N(\alpha)} = x(\alpha)$. Now, $W(z \wedge x^*) < \alpha$, and observe that $(z \wedge x^*)_{N(\alpha)} = x(\alpha)$ by assumption. Of course, W retains continuity and monotonicity on the interval $[(z \wedge x^*), x^*]$, so there exists some $y^* \in [(z \wedge x^*), x^*]$ and $i \notin N(\alpha)$ such that $W(y^*) = \alpha$ and for every $\epsilon > 0$, $W(y^*_i - \epsilon, y_{-i}) < \alpha$. This contradicts the fact that $i \notin N(\alpha)$, which is a contradiction. Conclude that $W(z) \geq \alpha$.

Further, observe that for any $x \in \mathbb{R}^N$, if $x_{N(\alpha)} = x(\alpha)$, then $W(x) = \alpha$.

Next, we claim that if $\alpha \leq \beta$, then $N(\alpha) \subseteq N(\beta)$. To this end, suppose by means of contradiction that there is $i \in N(\alpha) \setminus N(\beta)$. By letting $x \in \mathbb{R}^N$ such that $x_{N(\beta)} \geq x(\beta)$ but $x_i < x_i(\alpha)$, we get both $\beta \leq W(x) < \alpha$, a contradiction.

Further, if $\alpha < \beta$, then $x(\alpha) \lll x(\beta)_{N(\alpha)}$. That $x(\alpha) \leq x(\beta)$ is obvious. To see the strict inequality, suppose by means of contradiction that there is $i \in N(\alpha)$ for which $x_i(\beta) = x_i(\alpha)$. Let $x^* \in \mathbb{R}^N$ such that $x^*_{N(\beta)} = x(\beta)$; then $W(x^*) = \beta$, and for any $\epsilon > 0$, $W(x_i^* - \epsilon, x_{-i}^*) < \alpha$, which is a contradiction to continuity of W.

Let $X(i) \equiv \{\alpha : i \in N(\alpha)\}$. We claim that $x_i(\alpha)$ is continuous in α on $X(i)$. That it is continuous from below follows; let $\alpha^k \to \alpha^-$. Observe that for any x, $W(x) \geq \alpha$ if and only if $W(x) \geq \alpha^k$ for all k. This reads: $x_{N(\alpha)} \geq x(\alpha)$ if and only if $x_{N(\alpha^k)} \geq x(\alpha^k)$ for all α^k. It follows that there must exist k sufficiently large so that $N(\alpha^k) = N(\alpha)$, and further that we then have $x(\alpha^k) \to x(\alpha)$. Observe that the same argument demonstrates that $X(i)$ is open from below (it is an interval by the preceding arguments).

To see that x_i is continuous from above, suppose not and suppose that $x_i(\alpha^*) < \inf_{\alpha > \alpha^*} x_i(\alpha)$ for some α^*. Let $x^* \in \mathbb{R}^N$ so that $x^*_{N(\alpha^*)} = x(\alpha^*)$. Observe that for $\epsilon > 0$ small, we have $x_i(\alpha^*) + \epsilon < \inf_{\alpha > \alpha^*} x_i(\alpha)$, so that for all $\alpha > \alpha^*$, $(x^* + \epsilon\mathbb{1})_{N(\alpha)} \geq x(\alpha)$ is false. Hence $W(x^*) = \alpha$ and $W(x^* + \epsilon\mathbb{1}) = \alpha$, contradicting increasingness of W (as $x^* + \epsilon\mathbb{1}) \gg x^*$).

Finally, we show that across all α for which $i \in N(\alpha)$, $\inf x_i(\alpha) = -\infty$. Let us let $\alpha^* = \inf\{\alpha : i \in N(\alpha)\}$. Clearly, by finiteness of N, there is some $\epsilon > 0$ such that for all $\alpha \in (\alpha^*, \alpha^* + \epsilon)$, it follows that $N(\alpha)$ is constant set, say T (which includes i). So, let $y = \inf\{x(\alpha) \in \mathbb{R}^T : \alpha \in (\alpha^*, \alpha^* + \epsilon)\}$. Let $y^* \in \mathbb{R}^N$ such that $y^*_T = y$. Observe that, by continuity, $W(y^*) = \alpha^*$. But, by assumption, $i \notin N(\alpha^*)$. Consequently, $W(y^* - 2_i) = \alpha^*$. But observe that for any $\alpha \in (\alpha^*, \alpha^* + \epsilon)$, $(y^* - 2(1_i) + \mathbf{1})_{N(\alpha)} \geq x(\alpha)$ is false, so in particular, $W(y^* - 2(1_i) + \mathbf{1}) = \alpha^*$, contradicting monotonicity as $y^* - 2(1_i) + \mathbf{1}) \gg y^* - 2(1_i)$.

Finally, let us conclude by defining $M \equiv \{i \in N : \exists \alpha \text{ for which } i \in N(\alpha)\}$. Then, whenever there is α for which $i \in N(\alpha)$, define $\varphi_i(x_i(\alpha)) = \alpha$. Observe that φ_i is increasing and continuous on its domain, which by the preceding is unbounded below. Extend it arbitrarily outside of its domain to remain increasing and continuous.

Finally, we observe that $W(x) = \min_{i \in M} \varphi_i(x_i)$. To this end, observe that $x_{N(W(x))} \geq x(W(x))$, but that for any $\alpha > W(x)$, there is $i \in N(\alpha)$ for which $x_i < x(\alpha)_i$. Conclude that $W(x) = \min_{i \in M} \varphi_i(x_i)$. ∎

of Proposition 3. First, let us suppose that W is generalized maxmin, and let $(m, U_1, ..., U_n)$ be a quasiconcave utility profile. Let $x, y \in \mathbb{R}^n$, and suppose that $V(x) \geq V(y)$. Let $\alpha \in [0,1]$. Let $(x_1, \ldots, x_n) \in (\mathbb{R}^m_+)^N$

such that $W(U_1(x_1),\dots,U_n(x_n)) = V(x)$ and $(y_1,\dots,y_n) \in (\mathbb{R}^m_+)^N$ such that $W(U_1(y_1),\dots,U_n(y_n)) = V(y)$. Then in particular, $V(x) = \min_{i\in N}\varphi_i(U_i(x_i))$ and $V(y) = \min_{i\in N}\varphi_i(U_i(y_i))$. Clearly, by definition, for all $i \in N$, we have $\varphi_i(U_i(x_i)) \geq V(x) \geq V(y)$ and $\varphi_i(U_i(y_i)) \geq V(y)$. Consequently, for all $i \in N$, by quasiconcavity of U_i and increasingness of φ_i, we have $\varphi_i(U_i(\alpha x_i+(1-\alpha)y_i)) \geq V(y)$. Thus, as $\sum_{i\in N}\alpha x_i + (1-\alpha)y_i = \alpha x + (1-\alpha)y$, we conclude that

$$
\begin{aligned}
V(\alpha x + (1-\alpha)y) = &\sup_{\{z:\sum_i z_i=\alpha x+(1-\alpha)y\}} \min_{i\in N}\varphi_i(U_i(z_i)) \\
\geq &\min_{i\in N}\varphi_i(U_i(\alpha x_i + (1-\alpha)y_i)) \\
\geq &V(y)
\end{aligned}
$$

Conversely, let us establish that if the household utility is always quasiconcave, then W is generalized maxmin.

To establish this, suppose false, and by Lemma 3.1 that there are $x, y \in \mathbb{R}^N$ for which $W(x\wedge y) < W(x), W(y)$. In particular, by continuity and monotonicity, there exists $\alpha, \beta \in \mathbb{R}^N$ for which $W(\alpha\wedge\beta) < W(\alpha) = W(\beta)$.[5]

Let us label $c \equiv W(x)$.

Obviously, by monotonicity, for some i, j (say, $1, 2$, $\alpha_1 < \beta_1$ and $\beta_2 < \alpha_2$).

Let $m = N$, and a utility profile whereby for each $i \in N$, we have $U_i(x) = u_i(x_i)$ (*i.e.* U_i depends only on agent i's own consumption, as in the proof of Proposition 1). Our aim will be to construct u_i in order to derive a contradiction. So, each individual is endowed with a dimension (again, this requires a commodity space whose dimension is as high as the space of individuals). So each individual i has $U_i(x) = u_i(x_i)$. Then $V(x) = W(u_1(x_1), ..., u_n(x_n))$.

We pick $x, y \in \mathbb{R}^m_+$ arbitrarily, so that $x_i > y_i$ if and only if $\alpha_i > \beta_i$ and $y_i > x_i$ if and only if $\beta_i > \alpha_i$.

We use x, y to define u_i. First, for all $i \in N$, set $u_i(x_i) = \alpha_i$ and $u_i(y_i) = \beta_i$. Now, for each $i \in N$, choose $u_i(\frac{1}{2}(x_i + y_i))$ low enough so that $V(x) = W(\big(u_i(\frac{1}{2}(x_i + y_i))\big)_{i\in N}) < c$.[6] Extend u_i in a piecewise linear fashion.

[5] Suppose without loss that $W(x\wedge y) < W(x) \leq W(y)$. Obviously, there is $\beta \in [x\wedge y, y]$ for which $W(\beta) = W(x)$, and further, we may conclude that $W(x\wedge\beta) \leq W(x\wedge y)$ by monotonicity, so that letting $\alpha = x$ we obtain the desired claim.

[6] This can be done by using the continuity of W, observing that by choosing $u_i(\frac{1}{2}(x_i+y_i))$ close

Conclude that $V(\frac{1}{2}(x+y)) < \min\{V(x), V(y)\}$, contradicting quasiconcavity. ∎

As in Proposition 1, Proposition 3 also holds when preferences are required to be strictly monotonic. We sketch the argument here.

Proposition 4. *The household utility induced by $(m, U_1, \dots, U_n)$ is concave for all strictly monotone and concave utility profiles if and only if W is concave.*

Proof. Again, one direction of the proposition follows directly from Proposition 3. For the other direction, observe that all of the arguments remain valid in the proof; but we have not established a contradiction because the utility profile chosen there was not strictly monotonic. So, there are x, y for which $W(\big(u_i(\frac{x_i+y_i}{2}\big)_{i\in N}) < \min\{W((u_i(x_i)))_{i\in N}, W((u_i(y_i))_{i\in N})\}$.

To remedy this, let the utility profile $U_i^\lambda(x) = u_i(x_i + \lambda \sum_{j\neq i} x_j)$, and that by the same type of Maximum Theorem argument described in the proof of Proposition 2, for any $\epsilon > 0$, we demonstrate the existence of λ^* for which $V^{\lambda^*}(\frac{x+y}{2}) < W(\big(u_i(\frac{x_i+y_i}{2})\big)_{i\in N}) + \epsilon$ and $V^{\lambda^*}(x) > W((u_i(x_i))_{i\in N}) - \epsilon$ and $V^{\lambda^*}(y) > W((u_i(y_i))_{i\in N}) - \epsilon$; by choosing ϵ small enough we obtain $V^{\lambda^*}(\frac{x+y}{2}) < \min\{V^{\lambda^*}(x), V^{\lambda^*}(y)\}$. ∎

Remark. The intuition behind Proposition 3 is straightforward: the maxmin functions are the "most" quasiconcave amongst all increasing functions (they possess L-shaped level curves). In the case when individual utility functions are quasiconcave, but not very concave, much quasiconcavity is needed in the aggregator to ensure quasiconcavity of household preference. Consequently, it would be inappropriate to read any "ethical" interpretation into Proposition 3. Indeed, dictatorial aggregators are an example of generalized maxmin.

4 Generalized maxmin and expenditure functions: A duality result

In general, it may be feared that maxmin or general maxmin social welfare functions are intractable. Here, we illustrate a simple duality result which allows us to

enough to $\min_i\{\alpha_i, \beta_i\}$, we can guarantee the inequality.

simply describe the expenditure functions for household utility induced by maxmin social welfare functions.[7] As the expenditure function is an equivalent description of utility under minimal conditions, the result establishes that in fact maxmin welfare functions are tractable.

Define, for $U : \mathbb{R}^m_+ \to \mathbb{R}_+$, the *expenditure function* $e(U) : \mathbb{R}^m_+ \times \mathbb{R}_+ \to \mathbb{R}$ by

$$e(U)(p, v) = \inf\{p \cdot x : u(x) \geq v\}.$$

The expenditure function is a standard concept, and measures the expenditure necessary to bring an individual to a given utility level.

Proposition 5. *Suppose that $W(y_1, ..., y_n) = \min_{i \in N} y_i$, and let $V : \mathbb{R}^m_+ \to \mathbb{R}$ be the household utility induced by $(m, U_1, \ldots, U_n)$ for W. Then $e(V) = \sum_{i=1}^n e(U_i)$.*

Proof. To see this, observe that $V(x) \geq v$ if and only if there exists some allocation $(x_1, \ldots, x_n) \in \left(\mathbb{R}^m_+\right)^n$ of x for which $U_i(x_i) \geq v$ for all $i \in N$.

Consequently, $e(V)(p, v) = \inf\{p \cdot x : V(x) \geq v)\} = \inf\{p \cdot (\sum_i x_i) : U_i(x) \geq v\}$. The latter is then $\sum_i \inf\{p \cdot x_i : u_i(x_i) \geq v\} = \sum_i e(U_i)(p, v)$. ∎

Let us demonstrate a simple example to illustrate the result. Suppose individuals $i \in N$ have utility function $U_i(x) = \min_j\{\alpha^i_j x_j\}$ where $\alpha^i_j > 0$ for all $i \in N$ and j, in other words, they are *generalized Leontief.*

Here, it is a standard exercise to demonstrate that $e(U_i)(p, u) \equiv \sum_j u \left(\frac{p_j}{\alpha^i_j}\right)$; consequently, the household expenditure is given by

$$e(V)(p, u) = \sum_{i \in N} \sum_j u \left(\frac{p_j}{\alpha^i_j}\right).$$

A little algebra demonstrates that in fact,

$$e(V)(p, u) = u \sum_j \frac{p_j}{\left(\sum_{i \in N} \left(\alpha^i_j\right)^{-1}\right)^{-1}},$$

evidently also a generalized Leontief utility, given by

$$V(x) \equiv \min_j \left\{ \left(\sum_{i \in N} \left(\alpha^i_j\right)^{-1} \right)^{-1} x_j \right\}.$$

[7]For general maxmin, we would simply need to replace U_i with $\varphi_i \circ U_i$.

5 Conclusion

In this paper, we established several results on the functional forms of household aggregators. Of course, this work is not exhaustive. Many other questions can easily be addressed. For example, one might ask when homogeneous utility functions are aggregated into a homogeneous household utility; or when additive individual utility functions are aggregated into an additive household utility. We leave these questions to future research.

Of further interest is the idea that our aggregator functions are intended to work for a commodity space of any dimension. If the commodity space is known and restricted, however, the results of this paper break down. For example, with a single commodity, *any* increasing W results in a quasiconcave household preference. Understanding the form of permissible social welfare functions when commodity space is fixed is an obvious direction for future research.

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