Intergenerational Equity: Sup, Inf, Lim Sup, and Lim Inf

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Abstract

We study the problem of intergenerational equity for utility streams and a countable set of agents. A numerical social welfare function is invariant to ordinal transformation, satisfies a weak monotonicity condition, and an invariance with respect to concatenation of utility streams if and only if it is either the sup, inf, lim sup, or lim inf. Keywords: intergenerational equity, supremum, limit superior.

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1. Introduction

This note provides a simple characterization of a family of social choice rules for the problem of intergenerational equity. The goal is to rank utility streams. We imagine that a countably infinite set of agents is given, and we want to aggregate their utilities in a "fair" way. Traditionally, fairness has meant some form of anonymity, so that the names of agents should be irrelevant in making social judgments.

In this work, we discuss a new axiom. It describes what happens when we put together two disjoint societies of agents. Suppose we have two disjoint collections of agents, perhaps agents in different countries or different regions. Suppose we apply our rule to each of these sets of agents, and find out that each of these societies is equally well off according to our rule. What should happen when we treat these two societies as one large society? It seems natural to require that the large society should be just as well-off as either of the individual societies. We refer to this axiom as *reinforcement*.

Usually, the theory of ranking infinite utility streams has taken as primitive a countably infinite set of agents, and has required conditions on functions mapping from utility streams to the reals. Classical results in this theory tell us that such a function cannot be continuous (in the sup-norm topology), anonymous (in almost any sense), and compatible with the strong Pareto relation (for example, see Diamond [8]). The impossibility obtains because we require the existence of a function, as opposed to an ordinal ranking of utility streams. In a sense, the real numbers are not a large enough set to embed all sequences in a strictly monotonic way. If *numerical* measurement is not desired, then possibility results obtain (see Svensson [25]). Many works are devoted to investigating the nature of this impossibility: see, for example, [3, 4, 12, 16, 17, 22, 23, 24, 29]. Other works related to the theory of intergenerational equity include [1, 5, 6, 10]

We bypass these impossibilities by assuming a weakened form of the Pareto principle (which we call *monotonicity*): if everybody in society is made at least as well off under one utility stream than under another, then society as a whole is made at least as well off. We make no statements about strict preference of agents or society in formulating this property.

Our final property formalizes the notion that utility should be interpersonally comparable, yet have no cardinal significance. We refer to it as *ordinal covariance*. A variant of this axiom has been studied in this context before; see Lauwers [15]. It finds a role in social choice theory in general, see for example [2, 11, 13, 20, 21];

where it is often used to axiomatize the *rank-order dictatorships*. In particular, it states that when transforming utility of agents in an arbitrary way, social utility should be transformed in the same way. Our techniques relating to this axiom are adapted from an earlier paper which axiomatizes a broad class of functions as a type of quantile, see Chambers [7].

Our three axioms, reinforcement, monotonicity, and ordinal covariance, characterize a class of exactly four rules: the supremum, the infimum, the limit superior, and the limit inferior.

Section 2 provides the model and axioms. Section 3 discusses the main result. Section 4 studies some alternative axiomatizations. Section 5 is devoted to the independence of the axioms. Finally, Section 6 proposes some open problems.

2. Model and axioms

Let $\mathbb{N}$, the natural numbers, be a countably infinite set of agents. We will work with the set $l^{\infty}(\mathbb{N})$, the set of bounded sequences (we will henceforth call this set $\mathcal{U}$), interpreted as utility streams. A typical element will be specified $u \in \mathcal{U}$, where $u = (u(i))_{i\in\mathbb{N}}$. In particular, note that $u \in \mathcal{U}$, by definition, is formally a function from $\mathbb{N}$ into $\mathbb{R}$. Thus, for example, given a function $\sigma : \mathbb{N} \to \mathbb{N}$, $u \circ \sigma$ is well-defined as $((u \circ \sigma)(i))_{i\in\mathbb{N}} = (u(\sigma(i)))_{i\in\mathbb{N}}$.

Our main interest will be in studying social choice functions $T : \mathcal{U} \to \mathbb{R}$. Note that we are already making a restrictive assumption by requiring that the social ordering of utility streams be representable by a numerical function.

The following axiom is typical for this literature:

Anonymity: For all $u \in \mathcal{U}$ and all bijections from $\sigma : \mathbb{N} \to \mathbb{N}$, $T(u \circ \sigma) = Tu$.

We introduce a reinforcement condition that we believe has not yet been stated in the literature. Consider some finite collection of utility streams. Suppose they are all ranked as equivalent according to the function T. Suppose we form a new utility stream which is composed of the original, finite collection of utility streams, in the sense that every element in every original utility stream is mapped to some element in the new utility stream, and conversely. We require that the new utility stream so constructed is ranked as equivalent to all of the original utility streams. This condition encompasses the strongest notions of anonymity existing in the literature. However, it is a very natural condition if it is believed that the sequencing of when utility values are faced is completely irrelevant. Conditions related to this are pervasive in the social choice literature on variable populations and in fair allocation theory, where the population is typically finite (for example, see Thomson [26] or Young [28]). Such variable population axioms relate the utility or allocation of groups to the utility or allocation of larger groups. Our particular axiom is a weak form of the statement that a large society cannot be strictly better off (or strictly worse off) than each of its subsocieties; applying only when each subsociety has the same social utility.

For $K \in \mathbb{N}$ and a bijection $\sigma : \mathbb{N} \to \mathbb{N} \times \{1, ..., K\}$, we will write for each $i \in \mathbb{N}$, $\sigma(i) = (\sigma_1(i), \sigma_2(i))$, where $\sigma_1 : \mathbb{N} \to \mathbb{N}$ and $\sigma_2 : \mathbb{N} \to \{1, ..., K\}$.

Reinforcement: Let $\{u^j\}_{j=1}^{K} \subset \mathcal{U}$, where $K < +\infty$. Suppose that for all $j, k \in \{1, ..., K\}$, $Tu^j = Tu^k$. Let $\sigma : \mathbb{N} \to \mathbb{N} \times \{1, ..., K\}$ be a bijection. Define $u^{\sigma}(i) \equiv u^{\sigma_2(i)}(\sigma_1(i))$. Then $Tu^{\sigma} = Tu^1$.

Reinforcement is a much stronger axiom than anonymity; it forces the indifference sets of the social welfare function to be large. Anonymity requires indifference sets to be invariant under permutation, whereas reinforcement requires them to be invariant under permutation *and* concatenation. Thus, we rule out many potentially interesting rules by its imposition (see Section 2.2). While mathematically it is quite strong, conceptually it seems natural.

We now discuss our monotonicity axiom.

Monotonicity: Let $u, u' \in \mathcal{U}$ and suppose that for all $i \in \mathbb{N}$, $u(i) \leq u'(i)$. Then $Tu \leq Tu'$.

Ordinal covariance: Let $u \in \mathcal{U}$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing. Then $T(\varphi \circ u) = \varphi(Tu)$.

3. Main result

The following lemma is useful. It illustrates that under ordinal covariance, the social utility must essentially be a utility held by one of the members in the society. In the extreme case in which there are only two levels of utility held by members of society, say, when the utility profile takes the form 1_E for some $E \subset \mathbb{N}$, then $T1_E \in \{0, 1\}$. In particular, this illustrates that a rule which is ordinally covariant rules out tradeoffs or compromises. So, for example, $T1_E \in (0, 1)$ is here ruled out. This is because a compromise in utility space incorporates cardinal information.

Lemma 1: Suppose that T is ordinally covariant. Then for all $u \in \mathcal{U}$, $Tu \in \overline{u(\mathbb{N})}$.[1]

Proof. Suppose that the statement of the Lemma is false. Then there exists $u \in \mathcal{U}$ such that $Tu \in \mathbb{R} \backslash \overline{u(\mathbb{N})}$. In particular, there exists a neighborhood V of Tu so that $V \cap u(\mathbb{N}) = \varnothing$. Without loss of generality, we may assume that V is an open interval, say $(Tu - \varepsilon, Tu + \varepsilon)$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as

$$\varphi(x) \equiv \begin{cases} x \text{ for } x \notin (Tu - \varepsilon, Tu + \varepsilon) \\ \frac{(x-(Tu-\varepsilon))^2}{2\varepsilon} + (Tu - \varepsilon) \text{ for } x \in (Tu - \varepsilon, Tu + \varepsilon) \end{cases}.$$

For all $x \in (Tu - \varepsilon, Tu + \varepsilon)$, $\varphi(x) < x$. In particular, as $(Tu - \varepsilon, Tu + \varepsilon) \cap u(\mathbb{N}) = \varnothing$, $\varphi \circ u = u$. As $Tu \in (Tu - \varepsilon, Tu + \varepsilon)$, $\varphi(Tu) < Tu$. Hence $Tu = T(\varphi \circ u) = \varphi(Tu) < Tu$. Here, the second equality follows from ordinal covariance. The expression $Tu < Tu$ is impossible. ■

Theorem 1: A function T satisfies reinforcement, monotonicity, and ordinal covariance if and only if it is either the supremum, the infimum, the limit superior, or the limit inferior.

Proof. It is easy to check that each of the four functions discussed above satisfies the three axioms. Conversely, suppose that T satisfies the three axioms. By the lemma, for all $E \subset \mathbb{N}$, $T1_E \in \{0, 1\}$; moreover, $T1_{\mathbb{N}} = 1$ and $T0 = 0$. Define $\mathcal{E} \equiv \{E \subset \mathbb{N} : T(1_E) = 0\}$. Note that if $E \in \mathcal{E}$ and if $F \subset E$, then by monotonicity, $0 = T0 \leq T1_F \leq T1_E = 0$, so that $F \in \mathcal{E}$. Moreover, $\varnothing \in \mathcal{E}$ and $\mathbb{N} \notin \mathcal{E}$.

Define $\Sigma = 2^{\mathbb{N}}$. We will establish three facts.

Fact 1: For all $E, F \in \Sigma$ for which $|E| < +\infty$ and $|F| < +\infty$, $E \in \mathcal{E} \Leftrightarrow F \in \mathcal{E}$.

Fact 2: For all $E, F \in \Sigma$ for which $|E| = +\infty$ and $|\mathbb{N} \backslash E| = +\infty$, and $|F| = +\infty$ and $|\mathbb{N} \backslash F| = +\infty$, $E \in \mathcal{E} \Leftrightarrow F \in \mathcal{E}$.

Fact 3: For all $E, F \in \Sigma$ for which $|\mathbb{N} \backslash E| < +\infty$ and $|\mathbb{N} \backslash F| < +\infty$, $E \in \mathcal{E} \Leftrightarrow F \in \mathcal{E}$.

To establish fact 1, we show that $T1_{\{1\}} = T1_{\{1,\ldots,K\}}$ for all $K < +\infty$. Let $u^j = 1_{\{1\}}$ for $j = 1, \ldots, K$. For all $k \leq K$, let $\sigma(mK + k) = (m + 1, k)$. Then $u^\sigma = 1_{\{1,\ldots,K\}}$. By reinforcement, $T1_{\{1,\ldots,K\}} = T1_{\{1\}}$. For any finite set $E \in \Sigma$,

[1] Let $u \in \mathcal{U}$; as is standard we define $u(\mathbb{N}) = \{u(i) : i \in \mathbb{N}\}$. For a set $A \subset \mathbb{R}$, $\overline{A}$ denotes its closure in the Euclidean topology.

let σ be a bijection such that $\sigma(\{1,...,|E|\}) = E$. Then $T1_{\{1,...,|E|\}} = T1_E$. Fact 1 follows.

To establish fact 2, let $E, F \in \Sigma$ satisfy $|E| = +\infty$ and $|\mathbb{N}\backslash E| = +\infty$, and $|F| = +\infty$ and $|\mathbb{N}\backslash F| = +\infty$. List the elements of E as $E = \{e_1, e_2, ...\}$ and the elements of F as $F = \{f_1, f_2, ...\}$. List the elements of $\mathbb{N}\backslash E$ as $\{e_1', e_2', ...\}$ and the elements of $\mathbb{N}\backslash F$ as $\{f_1', f_2', ...\}$. For all i, let $\sigma(e_i) = f_i$ and let $\sigma(e_i') = f_i'$. Then $1_E = (1_F)^{\sigma}$. Hence, by reinforcement, $T1_E = T1_F$. The second fact follows.

Fact 3 follows similarly to fact 1. Therefore, all three facts are true.

Let us now consider several various cases. We will calculate, in each case, an explicit expression for

$$\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : u(i) \geq x\} \in \mathcal{E}\}.$$

A later step of the proof will establish that in general,

$$Tu = \inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : u(i) \geq x\} \in \mathcal{E}\}.$$

Case 1: $\mathcal{E}$ contains only the empty set.

In this case, we claim that for all $u \in \mathcal{U}$,

$$\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : u(i) \geq x\} \in \mathcal{E}\} = \sup_{i\in\mathbb{N}} u(i).$$

Let $z = \inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : u(i) \geq x\} \in \mathcal{E}\}$. Then in particular, for all $\varepsilon > 0$, $\{i \in \mathbb{N} : u(i) \geq z - \varepsilon\} \neq \varnothing$; so there exist $i \in \mathbb{N}$ for which $u(i) \geq z - \varepsilon$. Consequently, as ε is arbitrary, $\sup_{i\in\mathbb{N}} u(i) \geq z$. Moreover, for all $\varepsilon > 0$, $\{i \in \mathbb{N} : u(i) \geq z + \varepsilon\} = \varnothing$; consequently, $z + \varepsilon > \sup_{i\in\mathbb{N}} u(i)$ for all ε. Again as ε is arbitrary, $z \geq \sup_{i\in\mathbb{N}} u(i)$. Conclude $z = \sup_{i\in\mathbb{N}} u(i)$.

Case 2: $\mathcal{E}$ contains a finite nonempty set, but no infinite sets.

By fact 1, $\mathcal{E}$ contains all finite nonempty sets. In this case we verify that

$$\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : u(i) \geq x\} \in \mathcal{E}\} = \limsup_{i\in\mathbb{N}} u(i).$$

Let $z = \inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : u(i) \geq x\} \in \mathcal{E}\}$. Let $\varepsilon > 0$; then by definition $|\{i \in \mathbb{N} : u(i) \geq z - \varepsilon\}|$ is infinite. Consequently, by definition of lim sup, $\limsup_{i\in\mathbb{N}} u(i) \geq z - \varepsilon$, and as ε is arbitrary, $\limsup_{i\in\mathbb{N}} u(i) \geq z$. Let $\varepsilon > 0$; then by definition $|\{i \in \mathbb{N} : u(i) \geq z + \varepsilon\}|$ is finite. Consequently, by definition of lim sup, $\limsup_{i\in\mathbb{N}} u(i) < z + \varepsilon$, and as ε is arbitrary, $\limsup_{i\in\mathbb{N}} u(i) \leq z$. Conclude $z = \limsup_{i\in\mathbb{N}} u(i)$.

Case 3: $\mathcal{E}$ contains an infinite set, but no cofinite set.

By fact 2, $\mathcal{E}$ contains all infinite sets whose complements are infinite and all finite sets. In this case, it can be verified that for all $u \in \mathcal{U}$,

$$\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : u(i) \geq x\} \in \mathcal{E}\} = \liminf_{i \in \mathbb{N}} u(i).$$

This can be verified similarly to case 2.

Case 4: $\mathcal{E}$ contains a cofinite set.

In this case, it contains all cofinite sets by fact 3 and hence contains all sets but $\mathbb{N}$. In this case, it can be verified that

$$\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : u(i) \geq x\} \in \mathcal{E}\} = \inf_{i \in \mathbb{N}} u(i).$$

This can be verified similarly to case 1.

Thus, the theorem will follow if in each case, for all $u \in \mathcal{U}$, $Tu = \inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : u(i) \geq x\} \in \mathcal{E}\}$. We will verify this statement.

As a first step, we show that for all $E \in \Sigma$, $T1_E = \inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : 1_E(i) \geq x\} \in \mathcal{E}\}$. To see this, recall that $T1_E \in \{0, 1\}$ (by Lemma 1). Suppose $T1_E = 0$. Then by definition of $\mathcal{E}$, $E \in \mathcal{E}$. For all $\varepsilon \in (0, 1)$, $\{i \in \mathbb{N} : 1_E(i) \geq \varepsilon\} = E$, so that $\{i \in \mathbb{N} : 1_E(i) \geq \varepsilon\} \in \mathcal{E}$. Hence $\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : 1_E(i) \geq x\} \in \mathcal{E}\} < \varepsilon$. As ε is arbitrary, $\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : 1_E(i) \geq x\} \in \mathcal{E}\} \leq 0$. Now, suppose that $\varepsilon < 0$. Then $\{i \in \mathbb{N} : 1_E(i) \geq \varepsilon\} = \mathbb{N}$ and $\mathbb{N} \notin \mathcal{E}$. Consequently,

$$\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : 1_E(i) \geq x\} \in \mathcal{E}\} = 0.$$

Suppose instead that $T1_E = 1$. Then $E \notin \mathcal{E}$. In particular, for all $\varepsilon > 0$, $\{i \in \mathbb{N} : 1_E(i) \geq 1 + \varepsilon\} = \varnothing$, and $\varnothing \in \mathcal{E}$, so that $\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : 1_E(i) \geq x\} \in \mathcal{E}\} < 1 + \varepsilon$. As ε is arbitrary, conclude $\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : 1_E(i) \geq x\} \in \mathcal{E}\} \leq 1$. Now let $\varepsilon < 0$. Then $E \subset \{i \in \mathbb{N} : 1_E(i) \geq 1 + \varepsilon\}$, so that $\{i \in \mathbb{N} : 1_E(i) \geq 1 + \varepsilon\} \notin \mathcal{E}$; consequently, $\inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : 1_E(i) \geq x\} \in \mathcal{E}\} = 1$. Thus, we obtain $T1_E = \inf\{x \in \mathbb{R} : \{i \in \mathbb{N} : 1_E(i) \geq x\} \in \mathcal{E}\}$.

It remains to extend this representation to the arbitrary utility profiles. This part of the proof mimics [7] (Theorem 1, p. 420). Let $u \in \mathcal{U}$ and let $x^*(u) = \inf\{x : \{i : u(i) \geq x\} \in \mathcal{E}\}$. Let $\varepsilon > 0$ be arbitrary. Then $\{i : u(i) \geq x^*(u) + \varepsilon\} \in \mathcal{E}$ by definition of $x^*(u)$. Let $g^\varepsilon \in \mathcal{U}$ be defined as

$$g^\varepsilon \equiv \begin{cases} \sup u \text{ for } i : u(i) \geq x^*(u) + \varepsilon \\ x^*(u) + \varepsilon \text{ otherwise} \end{cases}.$$

Then $u \leq g^{\varepsilon}$, so that by monotonicity, $Tu \leq Tg^{\varepsilon}$. Note that $\{i : u(i) \geq x^*(u) + \varepsilon\} \in \mathcal{E}$, so that $\{i : g^{\varepsilon}(i) \geq \sup u\} \in \mathcal{E}$. As g^{ε} is an ordinal transformation of the indicator function of $\{i : u(i) \geq x^*(u) + \varepsilon\}$, we may conclude $Tg^{\varepsilon} = x^*(u) + \varepsilon$. As ε is arbitrary, $Tu \leq x^*(u)$.

Let $\varepsilon > 0$ be arbitrary. Let $h^{\varepsilon} \in \mathcal{U}$ be defined as

$$h^{\varepsilon} \equiv \begin{cases} \inf u \text{ for } i : u(i) < x^*(u) - \varepsilon \\ x^*(u) - \varepsilon \text{ otherwise} \end{cases}.$$

Then $u \geq h^{\varepsilon}$. Moreover, $\{i : u(i) \geq x^*(u) - \varepsilon\} \notin \mathcal{E}$. But $u(i) \geq x^*(u) - \varepsilon$ if and only if $h^{\varepsilon}(i) \geq x^*(u) - \varepsilon$. Therefore, $\{i : h^{\varepsilon}(i) \geq x^*(u) - \varepsilon\} \notin \mathcal{E}$. As h^{ε} is an ordinal transformation of the indicator function of $\{i : u(i) \geq x^*(u) - \varepsilon\}$, we may conclude $Th^{\varepsilon} = x^*(u) - \varepsilon$. By monotonicity, $Tu \geq x^*(u) - \varepsilon$. As ε is arbitrary, $Tu \geq x^*(u)$.

Conclude $Tu = x^*(u)$, so that the theorem is true. ■

4. Alternative axiomatizations

Lauwers [15] axiomatizes the infimum rule using axioms very closely related to ours. He works in a completely ordinal framework; however. He also imposes variants of monotonicity and ordinal covariance. His version of monotonicity is slightly stronger; while his version of ordinal covariance is slightly weaker. Moreover, he also requires invariance with respect to arbitrary permutations. In addition to this, however, he requires continuity in the sup-norm topology and a condition that he calls the "repetition-approximation" principle. Very roughly, the repetition approximation principle requires that the value of a function applied to a sequence can be approximated arbitrarily closely by the "value" of finite subsequences of the sequence. He rules out the supremum function by the use of a mild equity axiom.

We formally state two of Lauwers' axioms specialized to our environment. Topologize $\mathcal{U}$ with the sup-norm topology. This topology is defined by the metric $d : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ given by

$$d(u, v) = \sup_{i \in \mathbb{N}} |u(i) - v(i)|.$$

Ordinal level comparability: For all $u, v \in \mathcal{U}$ and all $\varphi : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing, $Tu \geq Tv \Longleftrightarrow T(\varphi \circ u) \geq T(\varphi \circ v)$.

Continuity: T is continuous in the sup-norm topology.

Proposition 1: Suppose that T is monotonic and continuous. If T satisfies ordinal level comparability, then either T is constant, or there is a strictly increasing and continuous function $\rho : T(\mathcal{U}) \to \mathbb{R}$ such that $\rho \circ T$ is ordinally covariant.

Proof. Suppose that T is not constant. Consequently, there exist $u, v \in \mathcal{U}$ for which $Tu > Tv$. There exist $x, y \in \mathbb{R}$ such that $x1_{\mathbb{N}} \geq u$ and $v \geq y1_{\mathbb{N}}$ (as u and v are bounded), so that by monotonicity, $T(x1_{\mathbb{N}}) > T(y1_{\mathbb{N}})$. By monotonicity, it follows that $x > y$. Now, suppose that $w, z \in \mathbb{R}$ and that $w > z$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any strictly increasing and continuous function for which $\varphi(x) = w$ and $\varphi(y) = z$. Note that as $T(x1_{\mathbb{N}}) > T(y1_{\mathbb{N}})$, ordinal level comparability implies that $T(w1_{\mathbb{N}}) > T(z1_{\mathbb{N}})$. In particular, by continuity and monotonicity, this implies that $T(\mathcal{U}) = \{T(r1_{\mathbb{N}}) : r \in \mathbb{R}\}$. To see this, $\{T(r1_{\mathbb{N}}) : r \in \mathbb{R}\} \subset T(\mathcal{U})$, so suppose $u \in \mathcal{U}$. Then by monotonicity and continuity, using the fact that $\{r1_{\mathbb{N}} : r \in \mathbb{R}\}$ is sup norm connected, we establish the existence of $x_u \in \mathbb{R}$ for which $T(u) = T(x_u 1_{\mathbb{N}})$, so that $T(\mathcal{U}) \subset \{T(r1_{\mathbb{N}}) : r \in \mathbb{R}\}$. In particular, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = T(x1_{\mathbb{N}})$ is strictly monotonic and can easily be verified to be continuous; it hence has a continuous and strictly monotonic inverse f^{-1} defined on $T(\mathcal{U})$. Let $\rho = f^{-1}$. We want to show that $\rho \circ T$ is ordinally covariant. In particular, note that for all $x \in \mathbb{R}$, $(\rho \circ T)(x1_{\mathbb{N}}) = x$. So let $u \in \mathcal{U}$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be strictly increasing and continuous. Then $Tu = T(x_u 1_{\mathbb{N}})$ so that by ordinal level comparability, $T(\varphi \circ u) = T(\varphi \circ (x_u 1_{\mathbb{N}})) = T(\varphi(x_u) 1_{\mathbb{N}})$. Moreover, $(\rho \circ T)(u) = (\rho \circ T)(x_u 1_{\mathbb{N}}) = x_u$ and $(\rho \circ T)(\varphi \circ u) = (\rho \circ T)(\varphi(x_u) 1_{\mathbb{N}}) = \varphi(x_u)$, establishing the claim. ■

The preceding result shows that, instead of assuming ordinal covariance, we may have instead assumed sup-norm continuity and ordinal unit comparability and obtained (ordinally) the same results. The only additional rules which would emerge are the constant rules.

The following axiom is also somewhat related to the current study. It appears in this form in Efimov and Koshevoy [9], but the conceptual idea dates back at least to Kolmogorov [14] and Nagumo [18].[2]

ω-Decomposability: Let $N \subset \mathbb{N}$ be a countably infinite set. Then $T\left(T(u_N)^N, u_{-N}\right) = Tu$.[3]

[2] We thank an anonymous referee for suggesting a study of this axiom.

[3] By u_N, we mean the projection of u onto the utility subspace spanned by the utilities of the

The following result establishes that ω-decomposability implies reinforcement under the axioms of ordinal covariance and anonymity.

Proposition 2: Suppose T satisfies ordinal covariance, ω-decomposability, and anonymity. Then it satisfies reinforcement.

Proof. Without loss of generality, suppose $K = 2$, and suppose that $Tu^1 = Tu^2$. The claim for general K follows by induction. Let $\sigma : \mathbb{N} \to \mathbb{N} \times \{1, 2\}$ be a bijection, and consider u^σ. We claim that $Tu^\sigma = Tu^1$. To see this, let $N = \{i : \sigma_2(i) = 1\}$, and note that by decomposability, $Tu^\sigma = T\left((Tu_N)^N, u_{-N}\right)$. By anonymity, $T\left((Tu_N)^N, u_{-N}\right) = T\left((Tu^1)^N, u_{-N}\right)$. By decomposability, $T\left((Tu^1)^N, u_{-N}\right) = T\left((Tu^1)^N, (Tu_{-N})^{-N}\right)$. By anonymity, $T\left((Tu^1)^N, (Tu_{-N})^{-N}\right) = T\left((Tu^1)^N, (Tu^2)^{-N}\right)$. By Lemma 1, as $Tu^1 = Tu^2$, $T\left((Tu^1)^N, (Tu^2)^{-N}\right) = Tu^1$. ■

The preceding result illustrates that we could have replaced our reinforcement axiom with ω-decomposability, characterizing the same family.

5. Independence of axioms

We here demonstrate the independence of the axioms. For each of the three axioms, a rule satisfying the remaining two but not the axiom itself is presented. Verification that the rules satisfy the remaining axioms is left to the reader.

Reinforcement: We here define an analogue of the rank order dictatorships which is meaningful for infinite sets of agents. In general, there may not exist an agent with a "k-th highest utility;" so the standard definition is meaningless here. Instead, let $k \in \mathbb{N}$, and define the k-th order dictatorship as

$$T^k u = \inf \{x \in \mathbb{R} : |\{i \in \mathbb{N} : u(i) \geq x\}| \leq k - 1\}.$$

In particular, for $k = 1$, $T^k u = \sup_i u(i)$. Suppose instead k is arbitrary; we demonstrate its value for a few sequences. Suppose $u^* \in \mathcal{U}$ is defined as $u^*(i) = -1/i$. Then $T^k u^* = 0$ (regardless of k). However, suppose $u' \in \mathcal{U}$ is defined as $u'(i) = 1/i$. Then $T^k u' = 1/k$. In addition to being ordinally

agents in N. By $T(u_N)^N$, we mean the the constant vector in $\mathbb{R}^N$ in which each agent has a utility of $T(u_N)$.

covariant and monotonic, the rank order dictatorships are also anonymous. In fact, together with the rules for which

$$T^k u = \sup \{x \in \mathbb{R} : |\{i \in \mathbb{N} : u(i) \leq x\}| \leq k-1\},$$

and the rules characterized in Theorem 1, these rules exhaust the family of rules satisfying anonymity, monotonicity, and ordinal covariance (the proof is similar to that of the main theorem). Non-anonymous rules satisfying monotonicity and ordinal covariance are easy to construct; for example, consider a dictatorship:

$$Tu = u(1).$$

These rules generally are also not ω-decomposable.

Monotonicity: The rule discussed here is somewhat strange; and we do not recommend it as a normatively appealing rule; it is merely exhibited to demonstrate independence of the axioms. Now define

$$Tu = \begin{cases} \inf_i u(i) \text{ if } |u(\mathbb{N})| < +\infty \\ \liminf_i u(i) \text{ otherwise} \end{cases}.$$

Thus, if there are only a finite number of utility values that agents have in a society, the rule recommends the infimal such value. Otherwise, it recommends the limit inferior.

Ordinal covariance: In this case, let T be a constant rule; for example, let $Tu = 0$ for all $u \in \mathcal{U}$. For a rule which also violates ordinal unit comparability, the function

$$Tu = \min \left\{ \inf_{i \in \mathbb{N}} u(i), 0 \right\}$$

provides an example.

6. Open problems

An interesting fact is that these results do not extend to continuum of agents (or higher cardinalities of agents) models. For example, consider the rule, which is a kind of "countable lim sup," which specifies that the utility of a society is the smallest value for which an at most countable number of agents receive at least that value. This rule satisfies all of the axioms we have posited, yet it is not strictly speaking a inf, sup, lim inf, or lim sup. An interesting question is to study these generalized limit concepts for arbitrary cardinalities of agents.

References

[1] G.B. Asheim and B. Tungodden, Resolving distributional conflicts between generations, *Econ. Theory* **24** (2004), 221-230.

[2] C. D'Aspremont, L. Gevers, Equity and the informational basis of collective choice, *Rev. Econ. Stud.* **44** (1977), 199-209.

[3] K. Banerjee and T. Mitra, On the continuity of ethical social welfare orders on infinite utility streams, *Soc. Choice and Welfare* **30** (2008), 1-12.

[4] K. Basu and T. Mitra, Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian, *Econometrica* **71** (2003), 1557-1563.

[5] K. Basu and T. Mitra, Utilitarianism for infinite utility streams: a new welfare criterion and its axiomatic characterization, *J. Econ. Theory* **133** (2007), 350-373.

[6] W. Bossert, Y. Sprumont, and K. Suzumura, Ordering infinite utility streams, *J. Econ. Theory* **135** (2007), 579-589.

[7] C.P. Chambers, Ordinal aggregation and quantiles, *J. Econ. Theory* **137** (2007), 416-431.

[8] P.A. Diamond, The evaluation of infinite utility streams, *Econometrica* **33** (1965), 170-177.

[9] B.A. Efimov and G.A. Koshevoy, A topological approach to social choice with infinite populations, *Math. Soc. Sci.* **27** (1994), 145-157.

[10] M. Fleurbaey and P. Michel, Intertemporal equity and the extension of the Ramsey criterion, *J. Math. Econ.* **39** (2003), 777-802.

[11] L. Gevers, On interpersonal comparability and social welfare orderings, *Econometrica* **47** (1979), 75-89.

[12] C. Hara, T. Shinotsuka, K. Suzumura, and Y. Xu, Continuity and egalitarianism in the evaluation of infinite utility streams, *Soc. Choice and Welfare*, forthcoming.

[13] P. Hammond, Equity, Arrow's conditions, and Rawls' difference principle, *Econometrica* **44** (1976), 793-804.

[14] A. Kolmogorov, (1930), Sur la notion de la moyenne, *Rendiconti Accademia dei Lincei (6)* **12** (1930), 388-391.

[15] L. Lauwers, Rawlsian equity and generalised utilitarianism with an infinite population, *Econ. Theory* **9** (1997), 143-150.

[16] L. Lauwers, Continuity and equity with infinite horizons, *Soc. Choice and Welfare* **14** (1997), 345-356.

[17] L. Lauwers, Ordering infinite utility streams comes at the cost of a non-Ramsey set, manuscript (2007).

[18] M. Nagumo, Uber eine Klasse der Mittelwerte, *Japan Journal of Mathematics* **7** (1930), 71-79.

[19] J. Rawls, "A Theory of Justice," Harvard University Press, 1971.

[20] K. Roberts, Possibility theorems with interpersonally comparable welfare levels, *Rev. Econ. Stud.* **47** (1980), 409-420.

[21] K. Roberts, Interpersonal comparability and social choice theory, *Rev. Econ. Stud.* **47** (1980), 421-439.

[22] T. Sakai, An axiomatic approach to intergenerational equity, *Soc. Choice and Welfare* **20** (2003), 167-176.

[23] T. Sakai, Equitable intergenerational preferences on restricted domains, *Soc. Choice and Welfare* **27** (2006), 41-54.

[24] T. Shinotsuka, Equity, continuity, and myopia: a generalization of Diamond's impossibility theorem, *Soc. Choice and Welfare* **15** (1998), 21-30.

[25] L.-G. Svensson, Equity among generations, *Econometrica* **48** (1980), 1251-1256.

[26] W. Thomson, "Consistent Allocation Rules," manuscript, 2003.

[27] J. von Neumann, O. Morgenstern, "Theory of Games and Economic Behavior," Princeton University Press, 1944.

[28] H.P. Young, Social choice scoring functions, *SIAM J. Appl. Math.* **28** (1975), 824-838.

[29] W. Zame, Can intergenerational equity be operationalized?, *Theoret. Econ.* **2** (2007), 187-202.