Allocation Rules for Land Division

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Abstract

This paper studies the classical land division problem formalized by Steinhaus [27] in a multi-profile context. We propose a notion of an allocation rule for this setting. We discuss several examples of rules and properties they may satisfy. Central among these properties is *division independence*: a parcel may be partitioned into smaller parcels, these smaller parcels allocated according to the rule, leaving a recommended allocation for the original parcel. In conjunction with two other normative properties, *division independence* is shown to imply the principle of utilitarianism. *Journal of Economic Literature* Classification Numbers: C71, D63, D71

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1. Introduction

The purpose of this paper is to study a theory of allocation rules in problems of land division. Imagine a group of agents with cardinally comparable utility functions over some universal parcel of land. Suppose they need to allocate a subparcel of this universal parcel. If one subparcel contains another, then all agents prefer the larger one. Thus, a conflict of interests arises. Taking the position of an outside observer, we ask: which allocation of the parcel should be chosen?

We assume that agents' utility functions are "additive"[1] over parcels.[2] An "economy" consists of a parcel to allocate, and a utility function over the universal parcel for each agent.

We study allocation rules. A rule associates a set of allocations with every possible economy. The focal point of this paper is a simple "divide and conquer" procedure, which is easily described using rules.

Consider some rule. Given an economy, one way of reducing the allocation problem is to first partition the parcel into subparcels. Each of these subparcels can then be allocated according to the rule. The totality of what each agent is recommended from the subparcels forms an allocation of the initial parcel. If the resulting allocation of the initial parcel is itself recommended by the rule for the initial economy, we say the rule is *division independent*.

We can thus state our main finding: any rule that recommends only efficient allocations, only depends on agent's utility functions over the parcel to be divided (we call this property *independence of infeasible land*), and that satisfies *division independence* is a subrule of a weighted utilitarian rule. That is, for each agent, there exists a scalar weight such that the rule only selects allocations that maximize the weighted sum of utilities.

This paper combines two separate literatures: the land division literature and the social choice literature. We comment on how it fits in with previous work in the two areas.

In the land division problem formalized by Steinhaus [27], it is well-known that allocations exist which maximize the sum of agents' utilities (Dubins and Spanier [15]). A natural question is what other types of allocations exist. *Efficient* and *envy-free* allocations always exist (Weller, [29]). When agents have endowments,

[1] For any two non-overlapping parcels, the utility of the union of the two parcels is the sum of the utilities of each of the individual parcels.

[2] This assumption is the norm in the literature, with a few exceptions, particularly [9, 10, 21].

equilibria, as defined by Berliant [8], exist (see also Berliant and Dunz [9]). Many other types of allocations are shown to exist in a very general framework, in Berliant et al. [10]. Procedures and algorithms for constructing these allocations are discussed at length in Brams and Taylor [11].

It turns out that in this model, an allocation is efficient if and only if it maximizes a weighted sum of utilities (Barbanel and Zwicker [6]). This characterization amounts to restating the convexity of the utility possibility set (a fact known since Dvoretzky, Wald, and Wolfovitz [16] and Dubins and Spanier [15]). However, for any efficient allocation, there may be *several* supporting weight vectors (Barbanel [5]).[3] Our theorem differs from this characterization in that there is a unique supporting vector which is independent of the economy.

Utilitarianism is an important concept in the literature on social choice. Using conditions relating to the informational content of utility functions, several authors (for example, D'Aspremont and Gevers [3] and Maskin [22]) characterize utilitarian social welfare functions. In a single-profile exercise, Harsanyi [17] assumes social alternatives are lotteries, using the added structure to obtain a single-profile characterization of social welfare functions which mimic utilitarianism. The main distinction between our work and those cited above is that our result relies on a procedural property, *division independence*, instead of statements about the interpersonal comparability of utility or statements about the risk-preference of society.

Section 2 discusses the formal model. Section 3 is devoted to introducing axioms for rules. In Section 4, we discuss our main results. Finally, Section 5 concludes.

2. The model

We set up a formal framework to study methods of dividing land among a set of agents with utility functions over land. Let (X, Σ, μ) be a measure space, where Σ are the μ-measurable sets, and μ is a non-atomic, non-degenerate, countably additive, and finite measure.[4] Let $\Sigma_\mu \subset \Sigma$ be the sets of μ-positive measure. The set X is interpreted as the largest possible parcel of land that may be divided. Elements of Σ_μ are smaller parcels that may have to be divided. All mathematical

[3]Thus, in general, the utility possibilities set may have "kinks" in its boundary.

[4]A measure μ is non-atomic if for all $A \in \Sigma$ with $\mu(A) > 0$, there exists some $B \subset A$, $B \in \Sigma$ such that $0 < \mu(B) < \mu(A)$.

statements that we make are assumed to hold up to μ-measure zero, without further mention.

Let $N \equiv \{1, ..., n\}$ be a set of agents, each of whom possesses a utility function over Σ_μ. We take $\mathcal{M}(X, \Sigma, \mu)$, the set of finite measures mutually absolutely continuous with respect to μ, as the domain of possible utility functions for all agents.[5] Such utility functions can be viewed as having a "constant marginal utility" property.[6] For all $i \in N$, the generic notation for agent i's utility function is ν_i. A **utility profile** is a list $\nu = (\nu_i)_{i \in N} \in \mathcal{M}(X, \Sigma, \mu)^N$.[7] An **economy** is a pair $(E, \nu) \in \Sigma_\mu \times \mathcal{M}(X, \Sigma, \mu)^N$. The set E is interpreted as the parcel to be divided, and ν is interpreted as the agents' utility functions over all of Σ. By $\mathcal{E} \equiv \Sigma_\mu \times \mathcal{M}(X, \Sigma, \mu)^N$, we mean the set of all economies. For all $E \in \Sigma_\mu$, let $\Sigma_E \equiv \{A \cap E : A \in \Sigma\}$. For all $E \in \Sigma_\mu$, an **allocation of E** is a function $P : N \to \Sigma_E$ satisfying $\bigcup_{i \in N} P_i = E$, and for all $i, j \in N$ such that $i \neq j$, $P_i \cap P_j = \varnothing$.[8, 9] This notion of an allocation reflects the requirements that a) the parcel must be completely allocated and b) no two agents' subparcels can overlap. Let $\mathcal{P}(E)$ denote the set of all allocations of E and $\mathcal{P} \equiv \bigcup_{E \in \Sigma_\mu} \mathcal{P}(E)$ the set of all allocations. We are interested in studying how allocations should be chosen across economies. A **rule** is a nonempty-valued correspondence $\psi : \mathcal{E} \rightrightarrows \mathcal{P}$ such that for all $(E, \nu) \in \mathcal{E}$, $\psi(E, \nu) \subset \mathcal{P}(E)$. A rule should be understood as a deterministic method of recommending allocations for all economies.

A simple example of a rule follows. Let $E \in \Sigma_\mu$. For all $P \in \mathcal{P}(E)$, and all $F \in \Sigma_\mu$ such that $E \cap F \neq \varnothing$, let $P \cap F \in \mathcal{P}(E \cap F)$ be the allocation such that for all $i \in N, (P \cap F)_i = P_i \cap F$. Let $P \in \mathcal{P}(X)$. Then for all $(E, \nu) \in \mathcal{E}$, the **P-constant rule** recommends $P \cap E$.

Another simple example of a rule is the rule which selects all of the allocations that maximize a sum of agents' utilities. Formally, define the **utilitarian rule**

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[5] Two measures μ and ν are **mutually absolutely continuous** if for all $E \in \Sigma$, $\mu(E) = 0$ if and only if $\nu(E) = 0$.

[6] Specifically, let $\nu \in \mathcal{M}(X, \Sigma, \mu)$. Let $E, F \in \Sigma_\mu$ such that $E \cap F = \emptyset$. Then the "marginal utility" of E to an agent who consumes F is given by $\nu(E \cup F) - \nu(F) = \nu(E)$.

[7] For arbitrary sets L and M, M^L denotes the set of functions from L to M.

[8] An allocation is also known as an ordered partition. We use the term allocation as it is consistent with the economics literature.

[9] Some authors consider a more general notion of an allocation; defining it to be a *partition of unity*, or a list of positive Σ_μ-measurable functions which sum to one. This more general notion provides more structure, so that the set of allocations becomes a convex set. For example, see [1, 12, 13].

U by

$$U(E,\nu) \equiv \arg \max_{P \in \mathcal{P}(E)} \sum_N \nu_i(P_i),$$

where arg max refers to the set of maximizers of the corresponding optimization problem. The allocations arising from the utilitarian rule are introduced and shown to exist by Dubins and Spanier [15].

3. Properties of rules

This section is devoted to introducing properties of rules. We use ψ as our generic notation for a rule.

The first property needs no explanation.

Let $(E,\nu) \in \mathcal{E}$. An allocation $P \in \mathcal{P}(E)$ is **efficient for** $(\mathbf{E}, \boldsymbol{\nu})$ if for all $P' \in \mathcal{P}(E)$ for which there exists $i \in N$ such that $\nu_i(P_i') > \nu_i(P_i)$, then there exists $j \in N$ for which $\nu_j(P_j) > \nu_j\left(P_j'\right)$.

Efficiency: For all $(E,\nu) \in \mathcal{E}$, if $P \in \psi(E,\nu)$, P is efficient for (E,ν).

Properties similar to the next one are implicit in many axiomatic studies. The property states that allocations recommended by a rule for a specific economy should be independent of agents' utility functions outside of the parcel to be divided.

For all $E \in \Sigma_\mu$ and all $\nu \in \mathcal{M}(X,\Sigma,\mu)$, define $\nu_E : \Sigma_E \to \mathbb{R}$ as follows: for all $B \in \Sigma_E$,

$$\nu_E(B) = \nu(B).$$

Independence of infeasible land: For all $(E,\nu), (E',\nu') \in \mathcal{E}$ such that $E = E'$, if $\nu_E = \nu'_E$, then $\psi(E,\nu) = \psi(E',\nu')$.[10]

One method of reducing an economy to "smaller" economies is to deal with the economy "piece by piece". The parcel can be divided into subparcels. Each agent receives the union of what he receives according to the rule for each of the subparcels. The next axiom states that an allocation rule should be invariant under such divisions.

[10] This axiom bears resemblance to the notion of *independence of infeasible alternatives*, prevalent in the social choice literature.

Let $E \in \Sigma_\mu$ and let $\{E^m\}_{m=1}^M$ be a finite partition of E such that for all $m = 1, ..., M$, $E^m \in \Sigma_\mu$. If for all $m = 1, ..., M$, $P^m \in \mathcal{P}(E^m)$, let

$$\bigcup_{m=1}^{M} P^m \equiv \left\{ P \in \mathcal{P}(E) : \text{for all } i \in N,\ P_i = \bigcup_{m=1}^{M} P_i^m \right\}.$$

Division independence: For all $(E, \nu) \in \mathcal{E}$ and all finite partitions $\{E^m\}_{m=1}^M \subset \Sigma_\mu$ of E,

$$\psi(E, \nu) = \left\{ \bigcup_{m=1}^{M} P^m : \text{for all } m = 1, ..., M,\ P^m \in \psi(E^m, \nu) \right\}.$$

Division independence makes sense when many parcels may become available to divide in the future, or when parcels may be taken away. It is equivalent to saying that the order in which parcels become available or are taken away is irrelevant. The final set of recommended allocations is always the same.

It is clear that not many rules are *division independent*. The constant rules, introduced at the end of the last section, are *division independent*, as is the utilitarian rule. *Division independence* should itself be viewed as an additivity or separability condition, applied to rules instead of preferences. Theorem 2 in the next section establishes just how strong a condition *division independence* is, by establishing a strong implication of *efficiency*, *independence of infeasible land*, and *division independence*.

For the case of single-valued rules, *division independence* is equivalent to the following weaker property:

Weak division independence: For all $(E, \nu), (E', \nu') \in \mathcal{E}$ such that $\nu = \nu'$ and $E \cap E' = \varnothing$, if $P \in \psi(E, \nu)$ and $P' \in \psi(E', \nu)$, then $P \cup P' \in \psi(E \cup E', \nu)$.

A common requirement in the fair division literature is that agents with similar characteristics be treated similarly. In our context, agents are parametrized solely by indices and utility functions. The next property requires only that if *all* agents have identical utility functions, then the rule recommends some allocation that each agent prefers to receiving nothing. It is impossible to give identical agents identical subparcels. However, it is not impossible to give identical agents identical *utility values*; much of the research found in Hill [19, 20] studies these types of allocations.

Positive treatment of equals: For all $(E, \nu) \in \mathcal{E}$ such that for all $i, j \in N$, $\nu_i = \nu_j$, there exists $P \in \psi(E, \nu)$ such that for all i, $\nu_i(P_i) > 0$.

One might wonder why we do not use a stronger equity requirement. The following is a stronger version of *positive treatment of equals*. It differs in the fact that if all agents have identical utility functions, then the rule must only recommend allocations which each agent prefers to receiving nothing. We introduce this notion because it still appears very weak, but there is a sense in which it is much stronger than *positive treatment of equals*.

Strong positive treatment of equals: For all $(E, \nu) \in \mathcal{E}$ such that for all $i, j \in N$, $\nu_i = \nu_j$, for all $P \in \psi(E, \nu)$, for all i, $\nu_i(P_i) > 0$.

Strong positive treatment of equals is incompatible with our main requirement of *division independence*, as is shown in Proposition 1. Note that Proposition 1 implies that there is no single-valued rule satisfying *weak division independence* and *positive treatment of equals*.

Proposition 1: There is no rule satisfying *division independence* and *strong positive treatment of equals*.

Proof: Suppose the statement of the proposition is false, and let ψ be a rule satisfying the stated properties. Let $(E, \nu) \in \mathcal{E}$ satisfy for all $i, j \in N$, $\nu_i = \nu_j$. Let $P \in \psi(E, \nu)$. Let $i \in N$. By *strong positive treatment of equals*, $\mu(P_i) > 0$. By *division independence*, there exists $P' \in \psi(P_i, \nu)$ and $P'' \in \psi(E \backslash P_i, \nu)$ such that for all $j \in N$, $P_j = P'_j \cup P''_j$. In particular, this implies that $P'_i = P_i$. But $P' \in \mathcal{P}(P_i)$, so that for all $j \neq i$, $\mu\left(P'_j\right) = 0$. Hence, *strong positive treatment of equals* is violated.■

When the values that utility functions take do not represent objective or verifiable quantities, it is natural to require that a rule be independent of the scale of utility. The following axiom formalizes this notion. [11,12]

Scale-invariance: For all $(E, \nu) \in \mathcal{E}$ and all $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^N_{++}$,

$$\psi(E, (\alpha_1 \nu_1, ..., \alpha_n \nu_n)) = \psi(E, \nu).$$

[11] This axiom has been discussed extensively in the theory of bargaining, beginning with Nash [24]. See Thomson [28].

[12] Here, $\mathbb{R}^N_+ \equiv \left\{x \in \mathbb{R}^N : x_i \geq 0 \text{ for all } i \in N\right\}$, $\mathbb{R}^N_{++} \equiv \left\{x \in \mathbb{R}^N : x_i > 0 \text{ for all } i \in N\right\}$, and $\mathbb{R}^N_+ \backslash \{0\} \equiv \left\{x \in \mathbb{R}^N : x \neq 0 \text{ and } x_i \geq 0 \text{ for all } i \in N\right\}$.

To give an example of a rule and investigate which axioms it satisfies, we define the **efficient and envy-free rule** F so that for all $(E,\nu) \in \mathcal{E}$, $F(E,\nu) \equiv \{P \in \mathcal{P}(E) : P \text{ is efficient for } (E,\nu) \text{ and for all } i,j \in N,\ \nu_i(P_i) \geq \nu_i(P_j)\}$. It is well-known that F is well-defined and nonempty (Weller [29]). Clearly, F is *efficient* and satisfies *independence of infeasible land*. However; F violates *weak division independence.* This can be most easily seen through the use of a simple example. Specifically, let $X = [0,1]$, let Σ be the Borel sets of $[0,1]$, and let μ be Lebesgue measure. Let $N = \{1,2\}$. Let $E \equiv [0,1/2]$ and $E' \equiv (1/2,1]$. Suppose ν_1 is also Lebesgue measure, and for all $A \in \Sigma$, $\nu_2(A) \equiv \mu(A \cap E) + 2\mu(A \cap E')$. Then both (E,ν) and (E',ν) are well-defined economies. Note that $P \equiv \{([0,1/4],(1/4,1/2])\} \in F(E,\nu)$ and $P' \equiv \{((1/2,3/4],(3/4,1])\} \in F(E',\nu)$. If F were *weakly division independent*, $P \cup P' = \{([0,1/4] \cup (1/2,3/4], (1/4,1/2] \cup (3/4,1])\} \in F(X,\nu)$. However, $P \cup P'$ is not efficient for (X,ν). The allocation $P^* \equiv \{([0,1/2],(1/2,1])\}$ Pareto dominates $P \cup P'$. Clearly, as F violates *weak division independence*, it also violates *division independence.* It is clear to see that F satisfies *positive treatment of equals* and *scale-invariance.*

If we are willing to sacrifice *efficiency* in the preceding example, then we can obtain *weak division independence.* Define the **envy-free rule** F^* so that for all $(E,\nu) \in \mathcal{E}$, $F^*(E,\nu) \equiv \{P \in \mathcal{P}(E) : \text{for all } i,j \in N,\ \nu_i(P_i) \geq \nu_i(P_j)\}$. Then F^* can be shown to satisfy *weak division independence.* Unfortunately, F^* is not *efficient*; in fact, it also does not satisfy *division independence.*

To close the section, we introduce an important class of rules which will be discussed in the next section. These rules generalize the utilitarian rule, defined in the previous section.

Let $\lambda \in \mathbb{R}^N_+ \setminus \{0\}$. Define the **$\boldsymbol{\lambda}$-utilitarian rule** U^λ by

$$U^\lambda(E,\nu) \equiv \arg \max_{P \in \mathcal{P}(E)} \sum_N \lambda_i \nu_i(P_i).$$

Call the class of such rules the **weighted utilitarian rules**. The utilitarian rule is a weighted utilitarian rule for which $\lambda = (1,...,1)$.

4. Results

We begin with the following characterization of efficient allocations (Barbanel and Zwicker [6], see also Akin [1] and Weller [29]).

Theorem 1: Let $(E, \nu) \in \mathcal{E}$. An allocation $P \in \mathcal{P}(E)$ is efficient for (E, ν) if and only if there exists $\lambda \in \mathbb{R}^N_+ \setminus \{0\}$ such that

$$P \in \arg \max_{P' \in \mathcal{P}(E)} \sum_N \lambda_i \nu_i (P'_i).$$

The mathematical tool used in proving Theorem 1 is a generalization of Lyapunov's Convexity Theorem ([2], p.444) due to Dvoretzky, Wald, and Wolfovitz [16].

The following is our main result. For two rules ψ and ψ', $\psi \subset \psi'$ means that for all $(E, \nu) \in \mathcal{E}$, $\psi(E, \nu) \subset \psi'(E, \nu)$.[13]

Theorem 2: Let ψ be a rule satisfying *efficiency*, *independence of infeasible land*, and *division independence*. Then there exists $\lambda \in \mathbb{R}^N_+ \setminus \{0\}$ such that $\psi \subset U^\lambda$. For all $\lambda \in \mathbb{R}^N_+ \setminus \{0\}$, U^λ satisfies the above axioms.

The proof of Theorem 2 (specifically, Lemma 6) establishes that the λ discussed in the statement of the theorem is unique up to scalar transformations. Therefore, a consequence of Theorem 2 is that the weighted utilitarian rules are the maximal (with respect to set inclusion) rules satisfying *efficiency*, *independence of infeasible land*, and *division independence*. By this, we mean that for any rule ψ satisfying the three axioms, there exists a weight vector λ such that $\psi \subset U^\lambda$, and for all λ, if there exists a ψ satisfying the three axioms such that $U^\lambda \subset \psi$, then $\psi = U^\lambda$. This statement follows because there exists some λ' such that $\psi \subset U^{\lambda'}$ (by Theorem 2) and the vector λ delivered by Theorem 2 is unique up to scalar transformations, so that $U^\lambda \subset \psi \subset U^{\lambda'}$ (and hence $U^\lambda \subset U^{\lambda'}$). But as $U^\lambda \subset U^\lambda$, λ' is a scalar transformation of λ (as U^λ satisfies all of our axioms) Hence, $U^\lambda = U^{\lambda'}$, so that $\psi = U^\lambda$.

The intuition behind Theorem 2 is as follows. By Theorem 1, for all economies and all efficient allocations for a given economy, there exists a list of weights such that the allocation maximizes the sum of weighted utilities. We wish to show that all economies and recommended allocations have the same list of weights. Suppose, for example, that there are two economies whose parcels are disjoint and for which the recommended allocations do not have the same list of weights. Consider the larger economy which results by piecing together these two parcels and a utility function that agrees with the original utility functions in each of

[13] The notation '$\subset$' refers to weak set inclusion.

the original economies. *Division independence*, together with *independence of infeasible land*, implies that the allocation which arises by piecing together the original recommended allocations should be recommended by the rule. However, this allocation is not efficient, as there is not a vector of weights so that this allocation maximizes the weighted sum of agents' utilities. Most of the work of the proof of Theorem 2 is in establishing that if a rule is not a subrule of a weighted utilitarian rule, then there exist economies whose parcels are disjoint and recommended allocations whose weight vectors are different.

Theorem 2 does not provide a characterization of the weighted utilitarian rules, although we do provide such a characterization later. The **lexicographic utilitarian rule**, U^L, is defined as follows. Let $>_L$ be an order over $\mathbb{R}^N$ given by $(x_1, ..., x_n) >_L (y_1, ..., y_n)$ if *a)* $\sum_N x_i > \sum_N y_i$ or *b)* $\sum_N x_i = \sum_N y_i$, and for some i, $(x_1, ..., x_i) = (y_1, ..., y_i)$ and $x_{i+1} > y_{i+1}$. Then for all $(E, \nu) \in \mathcal{E}$,

$$U^L(E, \nu) \equiv \arg\max_{>_L} \left\{ \left(\nu_i (P_i)_{i \in N} \right) : P \in \mathcal{P}(E) \right\}.$$

It can be shown that U^L satisfies all of the axioms of Theorem 2; however, it is evident that it is not a weighted utilitarian rule. To see why it is not a weighted utilitarian rule, first observe that by (a), $U^L \subset U$. Therefore, as the weight delivered by Theorem 2 is unique up to scalar transformations, if U^L were a weighted utilitarian rule, then $U^L = U$. Let $\nu \in \mathcal{M}(X, \Sigma, \mu)^N$ be a profile such that for all $i, j \in N$, $\nu_i = \nu_j$. Then $U^L(X, \nu) = \{P\}$, where P is such that $P_1 = X$, and for all $i \in N \setminus \{1\}$, $P_i = \varnothing$. But if $U^L = U$, then $U^L(x, \nu)$ recommends every possible allocation, which is a contradiction.

The next corollary discusses the additional imposition of *positive treatment of equals* to our other axioms; it provides a foundation for classical utilitarianism in our model.

Corollary 1: Let ψ be a rule satisfying *efficiency*, *independence of infeasible land*, *division independence*, and *positive treatment of equals*. Then $\psi \subset U$. Moreover, U satisfies the above axioms.

Proof: It is trivial to verify that U satisfies the axioms.

Let ψ be a rule satisfying the axioms. By Theorem 2, there exists $\lambda \in \mathbb{R}^N_+ \setminus \{0\}$ such that $\psi \subset U^\lambda$. We will show that $\lambda = (1, ..., 1)$. Let $\nu \in \mathcal{M}(X, \Sigma, \mu)^N$ be such that for all $i, j \in N$, $\nu_i = \nu_j = \nu'$ and $\nu_i(X) = 1$. Then

$$\{(\nu_1(P_1), ..., \nu_n(P_n)) : P \in \mathcal{P}(X)\} = \Delta(N),$$

which follows as $\sum_{i\in N}\nu_i(P_i)=\sum_{i\in N}\nu'(P_i)=1$.[14] Consider the economy (X,ν). *Positive treatment of equals* implies that there exists $P\in\psi(X,\nu)$ such that for all $i\in N$, $\nu_i(P_i)>0$. We know that $P\in\arg\max_{P'\in\mathcal{P}(X)}\sum_N\lambda_i\nu_i(P'_i)$. Let $(x_i)_{i\in N}\equiv(\nu_i(P_i))_{i\in N}$. Then $x\in\arg\max_{y\in\Delta(N)}\sum_N\lambda_i y_i$. However, as $x\in\operatorname{int}\Delta(N)$, for all $i,j\in N$, $\lambda_i=\lambda_j$. ■

Informally speaking, weighted utilitarian rules are not "fair." For *all* weighted utilitarian rules, there exist economies in which some agents receive nothing at all, and there exist economies in which some agent receives everything. Further, dictatorial rules are weighted utilitarian rules. The following characterization establishes that upon adding *scale-invariance* to our other axioms, only dictatorial rules are possible. This corollary also shows that any rule satisfying our axioms that only depends on *ordinal* information is dictatorial.[15]

Corollary 2: Let ψ be a rule satisfying *efficiency*, *independence of infeasible land*, *division independence*, and *scale-invariance*. Then there exists $i\in N$ such that for all $(E,\nu)\in\mathcal{E}$, and all $P\in\psi(E,\nu)$, $P_i=E$. Any such rule satisfies the four axioms.

Proof: It is trivial to verify that these rules satisfy the axioms.

Let ψ be a rule satisfying the axioms. By Theorem 2, there exists $\lambda\in\mathbb{R}^N_+\setminus\{0\}$ such that $\psi(E,\nu)\subset U^\lambda(E,\nu)$. We show, by contradiction, that for all but one $i\in N$, $\lambda_i=0$. Thus, assume that there exist i, $j\in N$ such that $\lambda_i,\lambda_j>0$. Without loss of generality, suppose $(i,j)=(1,2)$ and that $\lambda_1,\lambda_2\geq\max_{i\in N\setminus\{1,2\}}\lambda_i$. Let $\beta>\lambda_2/\lambda_1$. Let $(E,\nu)\in\mathcal{E}$, where ν is such that $\nu_1=\beta\nu_2$ and for all $i\in N\setminus\{1,2\}$, $\nu_i=\nu_2$. For all $i\in N\setminus\{1\}$, $\lambda_1\nu_1>\lambda_i\nu_i$. Thus, if $P\in\psi(E,\nu)$, then $P_1=E$.

Let $\alpha>\beta\lambda_1/\lambda_2$. For all $i\in N\setminus\{2\}$, $\lambda_2\alpha\nu_2>\lambda_i\nu_i$. Thus, if $P\in\psi(E,(\nu_1,\alpha\nu_2,\nu_3,...,\nu_n))$, $P_2=E$. The rule ψ violates *scale-invariance*, a contradiction. ■

The preceding theorem is simple to interpret; unless agents can agree that cardinal comparisons of utility are meaningful, it is impossible to require *division independence* and non-dictatorship if other minimal properties are required.

Although the primary purpose of our study is to understand the implications of *division independence*, Theorem 2 establishes that *division independence* is inti-

[14] Let $\Delta(N)\equiv\left\{y\in\mathbb{R}^N_+:\sum_N y_i=1\right\}$.

[15] For example, if utility functions are not observable, the implementation question becomes relevant. A consequence of Corollary 2 is that the only weighted utilitarian rules which are implementable in Nash or dominant strategy equilibrium are the dictatorial rules.

mately related to weighted utilitarianism. It is therefore natural to ask whether or not the weighted utilitarian rules possess any distinguishing characteristics among the class of rules discussed in Theorem 2. It turns out that the weighted utilitarian rules are the only such rules which are *continuous*, where continuity is defined in a very weak sense.

To make the idea precise, we specify a sequential notion of convergence on the domain of utility functions. Topologize the set of utility functions so that $\left\{\nu^k\right\}_{k=1}^{\infty} \subset \mathcal{M}(X, \Sigma, \mu)$ converges to $\nu \in \mathcal{M}(X, \Sigma, \mu)$ if for all $E \in \Sigma_\mu$, $\nu^k(E) \to \nu(E)$. We say a rule ψ is **continuous** if for all $E \in \Sigma_\mu$, for all $\left\{\left(\nu_i^k\right)_{i \in N}\right\}_{k=1}^{\infty} \subset \mathcal{M}(X, \Sigma, \mu)^N$ such that for all $i \in N$ $\nu_i^k \to \nu_i \in \mathcal{M}(X, \Sigma, \mu)$, if $P \in \psi\left(E, \left(\nu_i^k\right)_{i \in N}\right)$ for all k, then $P \in \psi(E, \nu)$. *Continuity* in this context says nothing more than that for a fixed parcel, the set of utility profiles which map to a specific allocation is closed under pointwise convergence.

Continuity: The rule ψ is continuous.

Corollary 3: A rule satisfies *efficiency*, *independence of infeasible land*, *division independence*, and *continuity* if and only if it is a weighted utilitarian rule.

The proof of Corollary 3 is relegated to Appendix A, as it relies on techniques introduced in proving Theorem 2.

5. Conclusion

Our paper works with a very restrictive utility domain–the domain of additive utility functions. Suppose that we choose to work with a larger domain, so that utility functions are no longer required to be additive. In this scenario, *division independence* loses its normative appeal. The reason is that when given a parcel of land, an additive utility function induces a natural unique additive utility function over subparcels of this parcel. When utility functions are not additive, this is no longer true.

To illustrate, suppose we have given an arbitrary set-function ν over (X, Σ_μ), interpreted as a utility function for some agent. Let $E, F \in \Sigma_\mu$ be such that $E \cap F = \varnothing$. What is the induced utility function over Σ_E? This naturally depends on which element of Σ_F the agent consumes. A natural way of defining the induced utility function over Σ_E is simply by using its marginal utility function. If the agent consumes $F' \in \Sigma_F$, define $\nu_{F'}^* : \Sigma_E \to \mathbb{R}$ as $\nu_{F'}^*(E^*) \equiv \nu(E^* \cup F') -$

$\nu(F')$. If ν is additive, $\nu^*_{F'}$ is exactly ν_E, which is independent of F'. *Division independence* states that Σ_E and Σ_F should be allocated independently of each other. In the case of nonadditive utility, the marginal utility functions over Σ_E depend on consumption in Σ_F. Thus, it is not natural to allocate Σ_F and Σ_E independently; indeed, it is not even clear as to which marginal utility function over Σ_E should be used as the induced utility function.

Working with more general domains of utility functions should be a motivating goal in this model. But our claim is that *division independence* is sensible if and only if utility functions are additive. Nevertheless, we may work with a larger domain of utility functions. A natural definition of *division independence* in this extended framework would hold only between economies which feature additive utility functions. Our theorems would then discuss the restrictions of such rules to the class of economies featuring additive utility functions.

What would real life examples of economies with additive utility functions look like? A canonical example is of a collection of farmers, each of whom produces a different crop. Different parcels are more amenable to producing different crops. A farmer's utility function over land would simply be his production function, whose value is the output produced given a parcel. Additivity in this context simply means the total crop yield from two separate parcels is the sum of the yields of each of the individual parcels.

We illustrate that the commodity space considered in this paper is more general than the commodity spaces considered in standard exchange economy models. In an exchange economy, before speaking of agents' preferences over differing quantities of the commodity, we implicitly assume that all agents are indifferent between any two distinct units of the commodity. However, one may imagine an agent who distinguishes between two units of the commodity in terms of preference. Thus, any time we have a complete description of all relevant commodity types, we may conceive of an agent who distinguishes between two units of the same type. To quote Debreu, ([14], p. 30)

> "...a *commodity* is therefore defined by a specification of all its physical characteristics, of its availability date, and of its availability location. As soon as one of these three factors changes, a *different* commodity results."

Viewing additivity from this angle, the additive feature of utility, while questionable when discussing such concepts as preference for well-shaped parcels (but see [18]), is not an unreasonable assumption. In fact, this additivity is implicit in the context of single commodity exchange economies, where preferences are defined over *quantities* of a commodity. Sprumont [26] discusses this issue at length, taking a preference based theory. If all agents have additive preferences over parcels of land, and these additive preferences coincide, then land forms what Sprumont calls a *cardinal commodity*. This simply means that we can assign some measure to parcels of land so that any two parcels of land with the same measure are indifferent according to all agents' preferences. The measure can be interpreted as measuring the quantity of a heterogeneous commodity, so that we can speak of preferences over quantities–exactly what is assumed from the outright in an exchange economy.

6. Appendix A: Proofs

We here discuss two further properties of rules which will be useful in the proof of Theorem 2.

The first is a monotonicity property. It says that given two economies in which parcels are nested and utilities are the same, allocations should also be "nested," in a formal sense.

Allocation independence for nested parcels: For all $(E,\nu),(E',\nu') \in \mathcal{E}$ such that $E \subset E'$ and $\nu = \nu'$, $\psi(E,\nu) = \{P \cap E : P \in \psi(E',\nu')\}$.

The following condition was introduced by Barbanel [5].

Let $E \in \Sigma_\mu$. A set $\Pi \subset \mathcal{P}(E)$ is **convex** if for all $P^1, P^2, ..., P^M \in \Pi$, if $P \in \mathcal{P}(E)$ satisfies for all $i \in N$, $P_i \subset \bigcup_{j=1}^{M} P_i^j$, then $P \in \Pi$. Proposition 2 states that *convexity* is equivalent to the apparently weaker condition that if for all $P, P' \in \Pi$ and all $P'' \in \mathcal{P}(E)$ if for all $i \in N$, $P_i'' \subset P_i \cup P_i'$, $P'' \in \Pi$. This version of convexity is useful for verifying that a collection of allocations is convex.

Proposition 2: Let $\Pi \subset \mathcal{P}(E)$ be a set of allocations such that for all $P, P' \in \Pi$ and all $P'' \in \mathcal{P}(E)$, if for all $i \in N$ $P_i'' \subset P_i \cup P_i'$, $P'' \in \Pi$. Then Π is convex.

Proof: Let $\Pi \subset \mathcal{P}(E)$ satisfy the hypothesis of the proposition. Let $\{P^k\}_{k=1}^M \subset \Pi$ and let $P^* \in \mathcal{P}(E)$ satisfy for all $i \in N$, $P_i^* \subset \bigcup_{k=1}^M P_i^k$. We claim that $P^* \in \Pi$.

The proof proceeds by induction. We construct a sequence $\left\{P^{*k}\right\}_{k=1}^{M} \subset \Pi$ for which $P^{*1} = P^1$ and $P^{*M} = P^*$, such that for all $k = 1, ..., M$, $P^{*k} \in \Pi$.

Thus, define $P^{*1} \equiv P^1$.

Suppose that for all $k = 1, ..., m-1$, P^{*k} is defined and an element of Π. For all $i \in N$, $P_i^m \cap P_i^* \subset P_i^m$, and $P_i^{(m-1)*} \backslash \bigcup_{l \in N} \left(P_l^m \cap P_l^*\right) \subset P_l^{(m-1)*}$. Define $P_i^{m*} \equiv \left(P_i^m \cap P_i^*\right) \cup \left(P_i^{(m-1)*} \backslash \bigcup_{l \in N} \left(P_l^m \cap P_l^*\right)\right)$. We claim that $P^{m*} \in \mathcal{P}(E)$. To see this, let $x \in E$. If there exists $i \in N$ such that $x \in P_i^m \cap P_i^*$, then $x \in P_i^{m*}$. Otherwise, there exists $i \in N$ such that $x \in P_i^{(m-1)*}$. In this case, $x \in P_i^{(m-1)*} \backslash \bigcup_{l \in N} \left(P_l^m \cap P_l^*\right)$, so that $x \in P_i^{m*}$. Suppose by means of contradiction that $x \in P_i^{m*} \cap P_j^{m*}$ and that $i \neq j$. Suppose that $x \in P_i^m \cap P_i^*$. As $P^* \in \mathcal{P}(E)$, $x \notin P_i^* \cap P_j^*$, implying $x \notin P_j^m \cap P_j^*$, so that $x \in P_j^{(m-1)*} \backslash \bigcup_{l \in N} \left(P_l^m \cap P_l^*\right)$, contradicting the fact that $x \in P_i^m \cap P_i^*$. Next, suppose that $x \in P_i^{(m-1)*} \backslash \bigcup_{l \in N} \left(P_l^m \cap P_l^*\right)$. As $P^{(m-1)*} \in \mathcal{P}(E)$, $x \notin P_i^{(m-1)*} \cap P_j^{(m-1)*}$, so that $x \notin P_j^{(m-1)*} \backslash \bigcup_{l \in N} \left(P_l^m \cap P_l^*\right)$, implying $x \in P_j^m \cap P_j^*$, contradicting the fact that $x \notin P_j^m \cap P_j^*$.

By the hypothesis of the proposition, we may conclude that $P^{m*} \in \Pi$.

Finally, we establish that $P^{M*} = P^*$. Let $x \in P_i^*$; we claim that $x \in P_i^{M*}$. In particular, $x \in P_i^k$ for some $k = 1, ..., M$ (as $P_i^* \subset \bigcup_{k=1}^{M} P_i^k$). Let K be the maximal such k. Then $x \in P_i^K \cap P_i^*$, so that $x \in P_i^{K*}$. We proceed by induction. Suppose that for $m \geq K+1$, $x \in P_i^{(m-1)*}$. We will show that $x \in P_i^{m*}$. By definition of K, $x \notin \left(P_i^{m*} \cap P_i^*\right)$. Moreover, for all $j \neq i$, $x \notin \left(P_j^{m*} \cap P_j^*\right)$ (because $x \in P_i^*$). Conclude that $x \in P_i^{(m-1)*} \backslash \bigcup_{l \in N} \left(P_l^m \cap P_l^*\right)$, so that $x \in P_i^{m*}$. This induction argument establishes that $x \in P_i^{M*}$. This shows that $P^{M*} = P^*$. ■

Convexity: For all $(E, \nu) \in \mathcal{E}$, $\psi(E, \nu)$ is convex.

We note that *convexity* is automatically satisfied by any single-valued rule.

The following theorem details an important connection between our axioms.

Theorem 3: A rule satisfies *division independence* if and only if it satisfies both *allocation independence for nested parcels* and *convexity*.

Proof: Step 1: Division independence implies both allocation independence for nested parcels and convexity.

Let ψ be a rule satisfying *division independence*. We first show that ψ satisfies *allocation independence for nested parcels*. Let $(E', \nu) \in \mathcal{E}$. Let $P' \in \psi(E', \nu)$, and let $E \subset E'$, $E \in \Sigma_\mu$. By *division independence*, there exist $P \in \psi(E, \nu)$ and $P'' \in \psi(E' \backslash E, \nu)$ such that $P' = P \cup P''$. Thus $P = P' \cap E \in \psi(E, \nu)$.

Next, let $(E, \nu) \in \mathcal{E}$. Suppose that $P \in \psi(E, \nu)$ and $E \subset E'$, $E' \in \Sigma_\mu$. Let $P'' \in \psi(E' \backslash E, \nu)$. By *division independence*, there exists $P' \in \psi(E', \nu)$ such that $P' = P \cup P''$. Therefore, $P' \cap E = P$, establishing *allocation independence for nested parcels*.

We now show that ψ satisfies *convexity*. Let $(E, \nu) \in \mathcal{E}$. Let $P', P'' \in \psi(E, \nu)$ and let P be an allocation of E such that for all $i \in N$, $P_i \subset P_i' \cup P_i''$. Let $B \equiv \bigcup_i (P_i \cap P_i')$, and let $A \equiv E \backslash B$. By *allocation independence for nested parcels*, proved in the previous paragraph, there exist $P^* \in \psi(B, \nu)$ and $P^{**} \in \psi(A, \nu)$ such that for all $i \in N$, $P_i^* = P_i \cap P_i'$ and $P_i^{**} = (P_i \cap P_i'') \backslash P_i'$. *Division independence* then implies $P = P^* \cup P^{**} \in \psi(E, \nu)$. Appeal to Proposition 2 to obtain the general statement.

Step 2: Allocation independence for nested parcels and convexity imply division independence.

Conversely, let ψ be a rule satisfying *allocation independence for nested parcels* and *convexity*. We show that ψ satisfies *division independence*. Let $(E, \nu) \in \mathcal{E}$ and let $\left\{E^k\right\}_{k=1}^M \subset \Sigma_\mu$ be a finite partition of E. Let $P \in \psi(E, \nu)$. By *allocation independence for nested parcels*, for all k, $P \cap E^k \in \psi\left(E^k, \nu\right)$. Thus $P = \bigcup_{k=1}^M P^k$, where $P^k = P \cap E^k \in \psi\left(E^k, \nu\right)$.

Lastly, let $(E, \nu) \in \mathcal{E}$ and let $\left\{E^1, ..., E^M\right\}$ be a finite partition of E. Suppose that for all $k = 1, ..., M$, $P^k \in \psi\left(E^k, \nu\right)$. By *allocation independence for nested parcels*, for all k, there exists $P^{k\prime} \in \psi(E, \nu)$ such that $P^{k\prime} \cap E^k = P^k$. By *convexity*, as for all $i \in N$, $\left(\bigcup_{k=1}^M P^k\right)_i \subset \bigcup_{k=1}^M P_i^{k\prime}$, we conclude that $\bigcup_{k=1}^M P^k \in \psi(E, \nu)$. ■

6.1. Proof of Theorem 2

By virtue of Theorem 3, we may instead prove the following:

Theorem 4: Let ψ be a rule satisfying *efficiency*, *independence of infeasible land*, *allocation independence for nested parcels*, and *convexity*. Then there exists $\lambda \in \mathbb{R}_+^N \backslash \{0\}$ such that $\psi \subset U^\lambda$. For all $\lambda \in \mathbb{R}_+^N \backslash \{0\}$, U^λ satisfies the above axioms.

First, to check that for all λ, U^λ satisfies the axioms is straightforward, with the possible exception of *convexity*. However, *convexity* follows directly from ([5], Theorem 1).

We divide the other part of the proof into a series of lemmas.

Lemma 1: Let ψ be a rule satisfying *efficiency*, *independence of infeasible land*, *allocation independence for nested parcels*, and *convexity.* Let $\{E^m\}_{m=1}^{M}$ be a partition of E and let $\nu \in \mathcal{M}(X, \Sigma, \mu)^N$, and for all m, let $\nu^m \in \mathcal{M}(X, \Sigma, \mu)^N$ satisfy $v^m_{E^m} = \nu_{E^m}$. Let $P^m \in \psi(E^m, \nu^m)$. Then:

$$\bigcup_{m=1}^{M} P^m \in \psi(E, \nu).$$

Proof: *Independence of infeasible land* implies that for all $m = 1, ..., M$, $\psi(E^m, \nu) = \psi(E^m, \nu^m)$. Thus, for all $m = 1, ..., M$, $P^m \in \psi(E^m, \nu)$. For all $m = 1, ..., M$, *allocation independence for nested parcels* implies the existence of an allocation $P^{m*} \in \psi(E, \nu)$ such that $P^m = P^{m*} \cap E^m$. Applying *convexity* establishes the conclusion. ■

Lemma 2: Let $(E, \nu) \in \mathcal{E}$ and $P \in \mathcal{P}(E)$. Suppose

$$P \in \arg \max_{P' \in \mathcal{P}(E)} \sum_N \lambda_i \nu_i(P'_i),$$

where $\lambda \in \mathbb{R}^N_+ \setminus \{0\}$. For all $i \in N$, let f_i be the Radon-Nikodym derivative of ν_i with respect to μ. If $x \in P_i$, then

$$i \in \arg \max_{j \in N} \lambda_j f_j(x).$$

Conversely, if for all $i \in N$ and all $x \in E$, $x \in P_i$ implies

$$i \in \arg \max_{j \in N} \lambda_j f_j(x),$$

then

$$P \in \arg \max_{P' \in \mathcal{P}(E)} \sum_N \lambda_i \nu_i(P'_i).$$

Proof: See Dubins and Spanier [15] and Weller [29]. ■

Let $(E, \nu) \in \mathcal{E}$ and $P \in \mathcal{P}(E)$. Let

$$\Lambda(P, (E, \nu)) \equiv \left\{ \lambda \in \mathbb{R}^N_+ \setminus \{0\} : P \in \arg \max_{P' \in \mathcal{P}(E)} \sum_N \lambda_i \nu_i \left(P'_i\right) \right\}.$$

Thus, $\Lambda(P, (E, \nu))$ is the set of weight vectors "supporting"an allocation P. By Theorem 1, P is efficient for (E, ν) if and only if $\Lambda(P, (E, \nu)) \neq \varnothing$.

Lemma 3: Let $(E,\nu) \in \mathcal{E}$ and $P \in \mathcal{P}(E)$. Suppose that P is efficient for (E,ν). Let $F \subset E$. Then $\Lambda(P,(E,\nu)) \subset \Lambda(P \cap F,(F,\nu))$.

Proof: Immediate from Lemma 2. ■

Lemma 4: Suppose that P is efficient for (E,ν). Then $\Lambda(P,(E,\nu)) \cup \{0\}$ is a closed, convex cone.

Proof: Let $(E,\nu) \in \mathcal{E}$ and $P \in \psi(E,\nu)$. We wish to show that $\Lambda(P,(E,\nu)) \cup \{0\}$ is a closed, convex cone. Convexity follows from [5], Theorem 1. To see that it is a cone, if $\lambda \in \Lambda(P,(E,\nu)) \cup \{0\}$, it is clear that $\alpha \geq 0$ implies $\alpha\lambda \in \Lambda(P,(E,\nu))$. To show that it is closed, let $\{\lambda^k\}_{k=1}^{\infty} \subset \Lambda(P,(E,\nu)) \cup \{0\}$ be a sequence converging to some λ. By definition of $\Lambda(P,(E,\nu)) \cup \{0\}$, for all $P' \in \mathcal{P}(E)$,

$$\sum_N \lambda_i^k (\nu_i(P_i) - \nu_i(P_i')) \geq 0.$$

By continuity of the preceding linear form,

$$\sum_N \lambda_i (\nu_i(P_i) - \nu_i(P_i')) \geq 0,$$

or, equivalently, $\lambda \in \Lambda(P,(E,\nu)) \cup \{0\}$. ■

Lemma 4 establishes that each $\Lambda(P,(E,\nu))$ is characterized by its intersection with $\Delta(N)$. To this end, define

$$S(P,(E,\nu)) \equiv \Lambda(P,(E,\nu)) \cap \Delta(N).$$

Let $A \subset \Delta(N)$. Define the closed ε-neighborhood of A, $\mathcal{N}(A,\varepsilon)$, by

$$\mathcal{N}(A,\varepsilon) \equiv \{x \in \Delta(N) : \text{ there exists } y \in A \text{ such that } \|y - x\| \leq \varepsilon\},$$

where $\|\cdot\|$ indicates the Euclidean norm.

The following lemma establishes that if a rule violates the conclusions of our theorem, then there exists a *finite* collection of economies with recommended allocations such that the sets of weight vectors supporting these efficient allocations have an empty intersection. It also establishes that the parcels to be divided in these economies need not exhaust the global parcel X. A similar technique was developed in Myerson [23] in a Nash bargaining framework.

Lemma 5: Let ψ be a rule satisfying *efficiency*, *independence of infeasible land*, *allocation independence for nested parcels*, and *convexity*. Suppose there does not exist $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ such that $\psi \subset U^\lambda$. Then there exists a finite set $\{(E^m, \nu^m)\}_{m=1}^M \subset \mathcal{E}$ and allocations $P^m \in \psi(E^m, \nu^m)$ satisfying $\bigcap_{m=1}^M S(P^m, (E^m, \nu^m)) = \varnothing$ and $\mu\left(X \setminus \bigcup_{m=1}^M E^m\right) > 0$.

Proof: Step 1: There is a finite set of economies and recommended allocations whose sets of weight vectors have an empty intersection.

Let ψ be a rule such that there does not exist $\lambda \in \mathbb{R}_+^N \setminus \{0\}$ such that $\psi \subset U^\lambda$. We show that there exists $\{(E^m, \nu^m)\}_{m=1}^M \subset \mathcal{E}$ and allocations $P^m \in \psi(E^m, \nu^m)$ satisfying $\bigcap_{m=1}^M S(P^m, (E^m, \nu^m)) = \varnothing$.

Suppose that the statement is false. Then, for all finite collections $\{(E^m, \nu^m)\}_{m=1}^M \subset \mathcal{E}$ and allocations $P^m \in \psi(E^m, \nu^m)$,

$$\bigcap_{m=1}^M S(P^m, (E^m, \nu^m)) \neq \varnothing.$$

By the preceding and by Lemma 4, $\{S(P, (E, \nu))\}_{(E,\nu)\in\mathcal{E}, P\in\psi(E,\nu)}$ is a family of closed sets possessing the finite intersection property. Furthermore, $\Delta(N)$ is compact. Conclude that

$$\bigcap_{(E,\nu)\in\mathcal{E}} \bigcap_{P\in\psi(E,\nu)} S(P, (E, \nu)) \neq \varnothing.$$

Let $\lambda \in \bigcap_{(E,\nu)\in\mathcal{E}} \bigcap_{P\in\psi(E,\nu)} S(P, (E, \nu))$. By definition, $\psi \subset U^\lambda$, in contradiction to our assumption.

Step 2: The economies described in Step 1 can be chosen so as not to exhaust the global parcel.

We show that $\{(E^m, \nu^m)\}_{m=1}^M$ can be chosen so that $\mu\left(X \setminus \bigcup_{m=1}^M E^m\right) > 0$.

Suppose we have constructed a family $\{P^m, (E^m, \nu^m)\}_{m=1}^M$ satisfying the property of Step 1. As $\{S(P^m, (E^m, \nu^m))\}_{m=1}^M$ is a collection of closed, convex sets in $\Delta(N)$ having empty intersection, there exists $\varepsilon > 0$ such that $\{\mathcal{N}(S(P^m, (E^m, \nu^m)), \varepsilon)\}_{m=1}^M$ is also a collection of closed, convex sets having empty intersection.

Let $\left\{F^k\right\}_{k=1}^\infty \subset \Sigma_\mu$ be a sequence satisfying for all k, $\mu\left(F^k\right) > 0$, $F^{k+1} \subset F^k$ and $\mu\left(\bigcap_{k=1}^\infty F^k\right) = 0$. (Such a sequence exists as μ is non-atomic). By Lemma 3, for all $m = 1, ..., M$, $S\left(P^m \backslash F^k, \left(E^m \backslash F^k, \nu^m\right)\right)$ is a nested, decreasing sequence of convex, compact sets.

We now establish that for all $m = 1, ..., M$, $\bigcap_{k=1}^{\infty} S\left(P^m \backslash F^k, \left(E^m \backslash F^k, \nu^m\right)\right) = S\left(P^m, \left(E^m, \nu^m\right)\right)$.

Let $\lambda \in S\left(P^m, \left(E^m, \nu^m\right)\right)$. By Lemma 3, for all k, $\lambda \in S\left(P^m \backslash F^k, \left(E^m \backslash F^k, \nu^m\right)\right)$, so that $\lambda \in \bigcap_{k=1}^{\infty} S\left(P^m \backslash F^k, \left(E^m \backslash F^k, \nu^m\right)\right)$.

For the opposite inclusion, let $\lambda \in \bigcap_{k=1}^{\infty} S\left(P^m \backslash F^k, \left(E^m \backslash F^k, \nu^m\right)\right)$. Then for all k, $\lambda \in S\left(P^m \backslash F^k, \left(E^m \backslash F^k\right), \nu^m\right)$. By definition of S, for all $P' \in \mathcal{P}\left(E^m\right)$

$$\sum_N \lambda_i \nu_i^m \left(P_i^m \backslash F^k\right) \geq \sum_N \lambda_i \nu_i^m \left(P_i' \backslash F^k\right). \tag{1}$$

For all $i \in N$ and all $m = 1, ..., M$, mutual absolute continuity of ν_i^m with respect to μ implies that $\nu_i^m \left(\bigcap_k F^k\right) = 0$. Countable additivity of ν_i^m implies that $\nu_i^m \left(F^k\right) \to 0$ as $k \to \infty$, so that for all $G \in \Sigma_\mu$,

$$\lim_{k \to \infty} \nu_i^m \left(G \backslash F^k\right) = \nu_i^m (G).$$

By (1) and the preceding remark,

$$\sum_N \lambda_i \nu_i^m \left(P_i^m\right) \geq \sum_N \lambda_i \nu_i^m \left(P_i'\right).$$

Thus, $\lambda \in S\left(P^m, \left(E^m, \nu^m\right)\right)$.

Therefore,

$$\bigcap_{k=1}^{\infty} S\left(P^m \backslash F^k, \left(E^m \backslash F^k, \nu^m\right)\right) = S\left(P^m, \left(E^m, \nu^m\right)\right).$$

For all $m = 1, ..., M$ and all $\varepsilon > 0$, there exists an integer $\mathcal{L}_m$ such that for all $k \geq \mathcal{L}_m$,

$$S\left(P^m \backslash F^k, \left(E^m \backslash F^k, \nu^m\right)\right) \subset \mathcal{N}\left(S\left(P^m, \left(E^m, \nu^m\right)\right), \varepsilon\right). \tag{2}$$

For example, see Berge [7], (p. 96, Theorem 2). Let $\mathcal{L} \equiv \max_m \mathcal{L}_m$; then (2) holds for all m and all $k \geq \mathcal{L}$. Thus

$$\bigcap_m S\left(P^m \backslash F^{\mathcal{L}}, \left(E^m \backslash F^{\mathcal{L}}, \nu^m\right)\right) = \varnothing$$

and $\mu\left(F^{\mathcal{L}}\right) > 0$. Thus, the economies $\left\{\left(E^m \backslash F^{\mathcal{L}}, \nu^m\right)\right\}_{m=1}^{M}$ with recommended allocations $P^m \backslash F^{\mathcal{L}} \in \psi\left(E^m \backslash F^{\mathcal{L}}, \nu^m\right)$ satisfy the required properties. ■

Suppose we have given an economy and a recommended allocation for this economy. Associated with this recommended allocation is a list of supporting vectors. Suppose we study a parcel of land which is disjoint from the parcel in the economy. We wish to show the existence of a utility profile such that the economy induced by the new parcel and utility profile has an associated recommended allocation whose supporting vectors are also supporting vectors for the original economy. The following lemma establishes that, under certain conditions, this is possible. The proof is divided into steps. We mention that the construction used in Lemma 6 also shows that we can construct an economy so that each efficient allocation for this economy has a unique (up to scalar transformation) supporting weight vector. An immediate consequence of this statement is that for any $\lambda, \lambda' \in \Delta(N)$, there is no inclusion relation between U^{λ} and $U^{\lambda'}$.

Lemma 6: Let ψ be a rule satisfying *efficiency*, *independence of infeasible land*, *allocation independence for nested parcels*, and *convexity*. Let $(E, \nu) \in \mathcal{E}$ and $P \in \psi(E, \nu)$. Suppose $F \subset X \backslash E$, $F \in \Sigma_{\mu}$. Then there exists $\nu' \in \mathcal{M}(X, \Sigma, \mu)^N$, $P' \in \mathcal{P}(F)$ such that $P' \in \psi(F, \nu')$ and

$$\Lambda(P', (F, \nu')) \subset \Lambda(P, (E, \nu)).$$

Proof: Our proof is constructive; Step 1 establishes the preliminaries for our construction. The idea is to construct a new economy whose parcel to divide is $E \cup F$. This new economy, when restricted to E, is the same as the initial economy. When restricted to F, all efficient allocations have a unique supporting weight vector (up to normalization). Hence, the boundary of the utility possibility set for this problem is "smooth." These two facts taken together will imply our result.

Let ψ be a rule satisfying the axioms, and let $(E, \nu) \in \mathcal{E}$, $P \in \psi(E, \nu)$ and $F \subset X \backslash E$, $F \in \Sigma_{\mu}$ be given. We construct the corresponding economy and allocation. Throughout the proof, f_i denotes the Radon-Nikodym derivative of ν_i with respect to μ.

Step 1: Partitioning the parcel in order to construct an expanded economy

Partition F into $n+1$ elements of Σ_{μ}, $\{F_i\}_{i=1}^{n}$ and F'. For all $i \in N$, partition F_i into countably infinite elements of Σ_{μ}, $\left\{F_i^k\right\}_{k=1}^{\infty}$, so that $\mu\left(F_i^k\right) = \frac{\theta_i}{k^3}$, where $\theta_i \equiv \frac{\mu(F_i)}{\sum_{k=1}^{\infty}(1/k^3)}$. Such a partition can be constructed because Lyapunov's convexity theorem implies that as μ is nonatomic, it is convex-ranged. For all

i and all k, partition F_i^k into countably infinite elements of Σ_μ, say, $\left\{F_i^{k,l}\right\}_{l=1}^{\infty}$. Again, this can be done by the nonatomicity of μ. Next, for all k, let $\left\{q^{k,l}\right\}_{l=1}^{\infty}$ be an ordering of $\mathbb{Q} \cap (k-1,k]$.[16]

Step 2 constructs a new utility profile which is measurable with respect to the sets described in Step 1. The new utility profile is constructed in such a way so that for each agent and each rational number, there is a parcel of land where agent i's Radon-Nikodym derivative takes that rational value. For agent i, this parcel is a subset of F_i. On $F \backslash F_i$, agent i's Radon-Nikodym derivative takes the constant value of one–so that on this parcel, his utility function coincides with μ. The density of the rational numbers will ensure that for any recommended allocation, the set of supporting vectors is unique.

Step 2: Constructing the utility profile on the expanded economy

We construct a utility profile ν^* which satisfies the following properties. First, for all $i \in N$, $\nu_E^* = \nu_E$. On F_i, all agents' except for agent i's Radon-Nikodym derivatives agree and are constant, and agent i's Radon-Nikodym derivative takes every possible value in the rational numbers. Any recommended allocation will then have a unique supporting weight vector.

Thus, for all $i \in N$, define

$$f_i^* \equiv \begin{cases} q^{k,l} \text{ on } F_i^{k,l} \\ 1 \text{ on } F \backslash F_i \\ f_i \text{ otherwise} \end{cases}.$$

We claim that $\{f_i^*\}_{i \in N}$ induces a well-defined profile of utility functions $\nu^* \in \mathcal{M}(X, \Sigma, \mu)$. To see this, we only need to show that for all $i \in N$, $\nu_i^*(F_i) < \infty$, as $\nu_i^*(F \backslash F_i) = \mu(F \backslash F_i)$ and $\nu_i^*(X \backslash F) = \nu_i(X \backslash F)$. Thus, by definition of ν_i^* and F_i, $\nu_i^*(F_i) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} q^{k,l} \mu\left(F_i^{k,l}\right)$. As for all k and all l, $q^{k,l} \leq k$, $\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} q^{k,l} \mu\left(F_i^{k,l}\right) \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k \mu\left(F_i^{k,l}\right)$. By definition of $F_i^{k,j}$, $\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k \mu\left(F_i^{k,l}\right) = \sum_{k=1}^{\infty} k \mu\left(F_i^k\right)$. By definition of F_i^k, $\sum_{k=1}^{\infty} k \mu\left(F_i^k\right) = \sum_{k=1}^{\infty} \frac{\theta_i}{k^2} < \infty$.

Step 3 verifies that the set of supporting vectors for any recommended allocation is unique (up to scalar transformation).

Step 3: Showing that for any recommended allocation, the set of supporting vectors is a singleton

[16] Here, $\mathbb{Q}$ denotes the rational numbers.

Let $P^* \in \psi(E \cup F, \nu^*)$. By *allocation independence for nested parcels*, $P^* \cap F \in \psi(F, \nu^*)$. We claim that $S(P^* \cap F, (F, \nu^*))$ is a singleton. Let $i \in N$ satisfy $\mu(P_i^* \cap F) = 0$. We claim that for all $\lambda \in S(P^* \cap F, (F, \nu^*))$, $\lambda_i = 0$. To see why, suppose by means of contradiction that there exists $\lambda \in S(P^* \cap F, (F, \nu^*))$ such that $\lambda_i > 0$. Then for some K large, for all $j \in N$, $\lambda_i(K-1) > \lambda_j$. By Lemma 2, this implies that for all $k \geq K$, $F_i^k \subset P_i^* \cap F$. Thus, $\mu(P_i^* \cap F) > 0$, a contradiction. Conversely, it is trivial to see that if $\lambda_i = 0$, then $\mu(P_i^* \cap F) = 0$.

By the preceding paragraph, we may assume without loss of generality that for all $i \in N$ and for all $\lambda \in S(P^* \cap F, (F, \nu^*))$, $\lambda_i > 0$.

Let $\lambda \in S(P^* \cap F, (F, \nu^*))$. Let $i^* \in N$ be an agent such that $\mu(F' \cap P_{i^*}^*) > 0$. By Lemma 2, it follows that $\lambda_{i^*} \geq \lambda_i$ for all $i \in N$ (as the weighted Radon-Nikodym derivatives of i^* and i respectively on F' are the constant functions λ_{i^*} and λ_i). Next, let $i \in N$ so that $i \neq i^*$. We claim that λ_{i^*}/λ_i is uniquely determined. To see this, as $\lambda_i > 0$, there exists some pair k, l such that $\lambda_i q^{k,l} > \lambda_{i^*}$. In particular, Lemma 2 establishes that if $\lambda_i q^{k,l} > \lambda_{i^*}$, then $F_i^{k,l} \subset P_i^*$. Therefore, the set $\left\{q^{k,l} : F_i^{k,l} \cap P_i^* \neq \varnothing\right\}$ is nonempty. Let $\overline{q}_i \equiv \inf\left\{q^{k,l} : F_i^{k,l} \cap P_i^* \neq \varnothing\right\}$. We claim that $\lambda_{i^*}/\lambda_i = \overline{q}_i$. First, we establish that $\overline{q}_i \leq \lambda_{i^*}/\lambda_i$ By definition of $\overline{q}_i$, if $q^{k,l} < \overline{q}_i$, then $F_i^{k,l} \cap P_i^* = \varnothing$. By Lemma 2, this implies that there exists $j \in N$ such that $\lambda_j \geq \lambda_i q^{k,l}$. By definition of i^*, $\lambda_{i^*} \geq \lambda_j \geq \lambda_i q^{k,l}$, so that $q^{k,l} \leq \lambda_{i^*}/\lambda_i$. This inequality holds for all rational $q^{k,l}$ which are less than $\overline{q}_i$; as the rational numbers are dense, it also holds for $\overline{q}_i$. Hence, $\overline{q}_i \leq \lambda_{i^*}/\lambda_i$. Next, we establish that $\overline{q}_i \geq \lambda_{i^*}/\lambda_i$. Thus, suppose that $q^{k,l} \geq \overline{q}_i$. By definition of $\overline{q}_i$, there exists $q^{k',l'} \leq q^{k,l}$ such that $F_i^{k',l'} \cap P_i^* \neq \varnothing$, so that by Lemma 2, $\lambda_i q^{k',l'} \geq \lambda_{i^*}$. Hence $\lambda_i q^{k,l} \geq \lambda_i q^{k',l'} \geq \lambda_{i^*}$, so that $q^{k,l} \geq \lambda_{i^*}/\lambda_i$. But this inequality holds for all rational numbers greater than $\overline{q}_i$, so that by the density of the rational numbers, $\overline{q}_i \geq \lambda_{i^*}/\lambda_i$. Hence, $\overline{q}_i = \lambda_{i^*}/\lambda_i$. We have established that for all $i \in N$, λ_{i^*}/λ_i is uniquely determined; hence, λ is unique. Thus, $S(P^* \cap F, (F, \nu))$ is a singleton.[17]

The uniqueness of the supporting weight vector is useful in Step 4, as it allows us to establish that this unique weight vector is actually a supporting weight vector for the recommended allocation of the initial economy.

Step 4: Verifying that the unique weight vector for the new economy is a weight vector for the old economy

By Lemma 3 and *independence of infeasible land*,

[17]In particular, this argument establishes that if $\psi \subset U^\lambda$, then λ is unique up to scalar transformation. This follows because for the economy just constructed, any efficient allocation has a unique (up to scalar transformation) list of supporting weights.

$$S\left(P^{*},\left(E\cup F,\nu^{*}\right)\right)\subset S\left(P,\left(E,\nu\right)\right).$$

Moreover,

$$S\left(P^{*},\left(E\cup F,\nu^{*}\right)\right)\subset S\left(P^{*}\cap F,\left(F,\nu^{*}\right)\right).$$

But $S\left(P^{*}\cap F,\left(F,\nu^{*}\right)\right)$ is a singleton, and as $S\left(P^{*},\left(E\cup F,\nu^{*}\right)\right)\neq\varnothing$,

$$S\left(P^{*},\left(E\cup F,\nu^{*}\right)\right)=S\left(P^{*}\cap F,\left(F,\nu^{*}\right)\right),$$

and the desired result holds for $P^{*}\cap F$ with utility profile ν^{*}, as

$$S\left(P^{*}\cap F,\left(F,\nu^{*}\right)\right)\subset S\left(P,\left(E,\nu\right)\right).\ \blacksquare$$

To conclude the proof, we suppose that ψ is a rule which satisfies the axioms, but which is not a subrule of a weighted utilitarian rule. We use the preceding lemmata to construct a set of economies and recommended allocations whose parcels to be divided are disjoint and whose corresponding sets of weight vectors for the recommended allocations are also disjoint. This leads to an immediate contradiction.

Let ψ be a rule satisfying the axioms listed in the theorem. Suppose there does not exist $\lambda\in\mathbb{R}_{+}^{N}\backslash\{0\}$ such that $\psi\subset U^{\lambda}$. By Lemma 5, there exists a finite set $\{(E^{m},\nu^{m})\}_{m=1}^{M}\subset\mathcal{E}$ and allocations $P^{m}\in\psi\left(E^{m},\nu^{m}\right)$ such that $\mu\left(X\backslash\bigcup_{m=1}^{M}E^{m}\right)>0$, and

$$\bigcap_{m=1}^{M}\Lambda\left(P^{m},\left(E^{m},\nu^{m}\right)\right)=\varnothing. \tag{3}$$

Partition $X\backslash\bigcup_{m=1}^{M}E^{m}$ into M parcels, say $\{G^{m}\}_{m=1}^{M}\subset\Sigma_{\mu}$. By Lemma 6, for all $m=1,...,M$, there exists $\nu^{*m}\in\mathcal{M}\left(X,\Sigma,\mu\right)$ and $P^{*m}\in\mathcal{P}\left(G^{m}\right)$ such that

$$P^{*m}\in\psi\left(G^{m},\nu^{*m}\right)$$

and

$$\Lambda\left(P^{*m},\left(G^{m},\nu^{*m}\right)\right)\subset\Lambda\left(P^{m},\left(E^{m},\nu^{m}\right)\right). \tag{4}$$

Let $\nu'\in\mathcal{M}\left(X,\Sigma,\mu\right)$ satisfy for all $m=1,...,M$, $\nu'|_{G^{m}}=\nu^{*m}|_{G^{m}}$. By Lemma 1,

$$\bigcup_{m=1}^{M}P^{*m}\in\psi\left(\bigcup_{m=1}^{M}G^{m},\nu'\right)=\psi\left(X\backslash\bigcup_{m=1}^{M}E^{m},\nu'\right).$$

By Lemma 3, $\Lambda\left(\bigcup_{m=1}^{M} P^{*m}, \left(X \setminus \bigcup_{m=1}^{M} E^m, \nu'\right)\right) \subset \bigcap_{m=1}^{M} \Lambda\left(P^{*m}, (G^m, \nu^{*m})\right)$. By (4), $\bigcap_{m=1}^{M} \Lambda\left(P^{*m}, (G^m, \nu^{*m})\right) \subset \bigcap_{m=1}^{M} \Lambda\left(P^{m}, (E^m, \nu^{m})\right)$. By (3), $\bigcap_{m=1}^{M} \Lambda\left(P^{m}, (E^m, \nu^{m})\right) = \varnothing$. Conclude

$$\Lambda\left(\bigcup_{m=1}^{M} P^{*m}, \left(X \setminus \bigcup_{m=1}^{M} E^m, \nu'\right)\right) = \varnothing. \tag{5}$$

As ψ satisfies *efficiency*, $\bigcup_{m=1}^{M} P^{*m}$ is efficient for $\left(X \setminus \bigcup_{m=1}^{M} E^m, \nu'\right)$. Thus, by Theorem 1, $\Lambda\left(\bigcup_{m=1}^{M} P^{*m}, \left(X \setminus \bigcup_{m=1}^{M} E^m, \nu'\right)\right) \neq \varnothing$, in contradiction to (5). ■

We now establish the proof of Corollary 3.

Proof: To see that the weighted utilitarian rules satisfy *continuity*, let U^λ be a weighted utilitarian rule. Let $\left\{\nu^k\right\}_{k=1}^{\infty} \subset \mathcal{M}(X, \Sigma, \mu)^N$ converge to ν. Suppose that $P \in U^\lambda\left(E, \nu^k\right)$ for all k. By definition, this means that for all $P' \in \mathcal{P}(E)$, $\sum_N \lambda_i \nu_i^k(P_i) \geq \sum_N \lambda_i \nu_i^k(P_i')$; in particular, $\sum_N \lambda_i\left(\nu_i^k(P_i) - \nu_i^k(P_i')\right) \geq 0$. As $\nu_i^k(P_i) \to \nu_i(P_i)$ and $\nu_i^k(P_i') \to \nu_i(P_i')$, we conclude that $\sum_N \lambda_i\left(\nu_i(P_i) - \nu_i(P_i')\right) \geq 0$, so that $P \in U^\lambda(E, \nu)$.

Next, suppose that ψ is a rule satisfying the axioms listed in the statement of the corollary. We know by Theorem 2 that there exists $\lambda \in \Delta(N)$ such that $\psi \subset U^\lambda$. We claim that $\psi = U^\lambda$. Thus, let $(E, \nu) \in \mathcal{E}$. Let $P \in U^\lambda(E, \nu)$. We will show that $P \in \psi(E, \nu)$. Thus, let f_i be the profile of Radon-Nikodym derivatives associated with ν. By Lemma 2, for all $i \in N$, $\lambda_i f_i(x) \geq \lambda_j f_j(x)$ on P_i. Construct a sequence f_i^k by defining $f_i^k = f_i + \frac{1}{k} 1_{P_i}$, where 1_{P_i} is the indicator function of P_i. Let ν_i^k be the element of $\mathcal{M}(X, \Sigma, \mu)$ associated with f_i^k, and note that $\nu_i^k \to \nu_i$. Further, note that for all i, $\lambda_i f_i^k(x) > \lambda_j f_j(x)$ on P_i. By Lemma 2, we may conclude that for all k, $U^\lambda\left(E, \nu^k\right)$ is a singleton and is equal to P. Thus, for all k, $\psi\left(E, \nu^k\right)$ is also a singleton and equal to P. Hence, $P \in \psi\left(E, \nu^k\right)$ for all k, and $\nu^k \to \nu$. By *continuity*, we conclude $P \in \psi(E, \nu)$. Thus $U^\lambda(E, \nu) \subset \psi(E, \nu)$, so that $\psi = U^\lambda$. ■

7. Appendix B: Independence of the Axioms

The following examples demonstrate the independence of the axioms used in Theorem 4.

Example 1: Efficiency is not implied by the other axioms

Let $\Pi(X) \subset \mathcal{P}(X)$ be a convex set of allocations, and let $\lambda \in \mathbb{R}^N$. For all $(E,\nu) \in \mathcal{E}$, if $\max_{P\in\Pi(X)} \sum_N \lambda_i \nu_i (P_i \cap E)$ exists, define

$$\psi_{\lambda,\Pi(X)}(E,\nu) \equiv \arg\max_{P\in\Pi(X)} \sum_N \lambda_i \nu_i (P_i \cap E).$$

There are clearly examples of $\Pi(X)$ in which the preceding expression is well-defined. It is trivial to check that ψ is not generally *efficient*, yet satisfies *independence of infeasible land, allocation independence for nested parcels*, and *convexity.* Note that for all $P \in \mathcal{P}(X)$, the P-constant rule is the rule in which $\Pi(X) = \{P\}$. The *reverse weighted utilitarian rules*, which recommend for $(E,\nu) \in \mathcal{E}$ those allocations in $\arg\min_{P'\in\mathcal{P}(E)} \sum_i \lambda_i \nu_i (P'_i)$ are defined by $\psi_{-\lambda,\mathcal{P}(X)}$.

Example 2: Allocation independence for nested parcels is not implied by the other axioms

For all $(E,\nu) \in \mathcal{E}$, let

$$U(E,\nu) \equiv \{(\nu_1(P_1), ..., \nu_n(P_n)) : P \in \mathcal{P}(E)\}.$$

Such a set is called the *utility possibility set* for (E,ν). Define a function φ from compact, convex subsets of $\mathbb{R}^N$ to $\mathbb{R}^N$, satisfying for all compact and convex $C \subset \mathbb{R}^N$, $\varphi(C) \in C$, and there does not exist $y \in C$ such that $y > \varphi(C)$.[18] For all such φ, the rule ψ given by

$$\psi(E,\nu) \equiv \{P \in \mathcal{P}(E) : (\nu_1(P_1), ..., \nu_n(P_n)) = \varphi(U(E,\nu))\}$$

is well-defined as $U(E,\nu)$ is compact and convex. It can be trivially verified that ψ satisfies *independence of infeasible land* (as the allocation rule is based solely on the shape of the utility possibility set). Next, define $(E,\nu),(E',\nu')$ to be *equivalent* if $E = E'$ and $\psi(E,\nu) = \psi(E',\nu')$. By applying the Axiom of Choice to these equivalence classes, we define a single-valued selection ψ' of ψ such that for all $(E,\nu),(E',\nu') \in \mathcal{E}$, $\psi(E,\nu) = \psi(E',\nu')$ implies $\psi'(E,\nu) = \psi'(E',\nu')$. Then ψ' satisfies *efficiency*, *independence of infeasible land*, and *convexity.* However, ψ' typically violates *allocation independence for nested parcels.* This will be the case if $\varphi(C) \equiv \arg\max_{x\in C} \prod_N x_i$ (see Nash, [24]), for example.

Example 3: Independence of infeasible land is not implied by the other axioms

[18] For $x, y \in \mathbb{R}^N$, $x > y$ if for all $i \in N$, $x_i \geq y_i$ and $x \neq y$.

For all $\nu \in \mathcal{M}(X, \Sigma, \mu)$, let $P(\nu) \in \mathcal{P}(X)$ be efficient. Then for all $(E, \nu) \in \mathcal{E}$, let $\psi(E, \nu) \equiv P(\nu) \cap E$. Then, ψ satisfies *efficiency, allocation independence for nested parcels,* and *convexity*, but in general, not *independence of infeasible land.*

Example 4: Convexity is not implied by the other axioms

Let $\Psi \subset \mathbb{R}_{+}^{N} \backslash \{0\}$ contain two linearly independent vectors. Let

$$\psi(E, \nu) \equiv \bigcup_{\lambda \in \Psi} U^{\lambda}(E, \nu).$$

Then ψ satisfies *efficiency*, *independence of infeasible land*, and *allocation independence for nested parcels. Convexity* is violated.

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