ON LEXICOGRAPHIC CHOICE†

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ABSTRACT. We show that every lexicographic choice function is acceptant and path independent. Furthermore, there exists an acceptant and path independent choice function that is not lexicographic.

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1. Introduction

In this paper, we investigate a class of choice functions that we call *lexicographic* choice functions. A choice function of this type consists of a q-vector of preference relations over the alternatives; for any group of alternatives, the choice function first picks the highest-ranked alternative with respect to the first preference relation, then the highest-ranked remaining alternative with respect to the second preference relation, and so forth.

A choice function is *path independent* if the choice over a set of alternatives remains the same when the set is segmented arbitrarily, the choice applied to each subset, and finally the choice applied again to all chosen alternatives from the subsets (Plott, 1973). A choice function is $q-$*acceptant* if it always selects q alternatives when at least q alternatives are available, and otherwise selects all of them. A choice rule is *acceptant* if it is q-acceptant for some q. We show that every lexicographic choice function is path independent and acceptant, but that the converse is not true (Theorem 1). That is, there exist acceptant and path-independent choice functions not of this form.

Lexicographic choice functions are a subset of choice functions studied in Kominers and Sönmez (2016). They study a matching environment in which a firm has different slots and for each slot a preference relation over contracts that workers can sign. The firm chooses contracts for slots in a sequence, which is dubbed the *precedence order*. If a worker has a chosen contract for a slot, then her remaining contracts are not considered for the rest of the slots. To apply our model to such a setting, we would assume that each alternative is associated with a different worker, so there is no need to use contracts in our special case.

This paper is a part of the growing literature in the intersection of choice theory and matching theory. Using choice functions as a primitive of a matching model has many benefits. As a result, recent work has focused on choice functions that arise in matching. See for example, Chambers and Yenmez (2018) and Doğan et al. (2017b) for a characterization of *responsive* choice rules which are a subclass of lexicographic choice rules when all preference relations are the same. In a subsequent work to the current one, Doğan et al. (2017a) study lexicographic choice rules when the capacity q is variable. Choice functions have also been used to study diversity (Westkamp (2013), Hafalir et al. (2013), Ehlers et al. (2014), Echenique and Yenmez (2015), Aygün and Bó (2016), and Aygün and Turhan (2016)) and externalities (Pycia and Yenmez, 2014).

2. Model

Suppose $\mathcal{X}$ is the set of alternatives and $\mathcal{P}(\mathcal{X}) = 2^{\mathcal{X}}$ is the powerset of $\mathcal{X}$. There exists an agent with a ***choice function*** $C : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ such that

- for every $X \subseteq \mathcal{X}$, $C(X) \subseteq X$ and
- for every non-empty $X \subseteq \mathcal{X}$, $C(X) \neq \emptyset$.

The interpretation is if X is the set of available alternatives to the agent, then $C(X)$ is the set of chosen ones. We require the chosen set to be non-empty if there is at least one alternative available.

A ***preference relation*** $\succeq$ on $\mathcal{X}$ is a binary relation on $\mathcal{X}$ that is complete, transitive, and antisymmetric.[1] Choice function C is ***lexicographic*** if there exists a sequence of preference relations on $\mathcal{X}$, $\{\succeq_i\}_{i\in I}$, such that for any $X \in \mathcal{X}$

$$C(X) = \bigcup_{i\in I}\{x_i^*\},$$

where x_i^* is defined inductively as $x_1^* = \max\limits_{X} \succeq_1$ and, for $i \geq 2$, $x_i^* = \max\limits_{X\setminus\{x_1^*,\ldots,x_{i-1}^*\}} \succeq_i$. Kominers and Sönmez (2016) study a more general class of choice functions.

Lexicographic choice functions can be used to promote diversity. For example, minorities can be given higher priorities for some of the seats in a public school. See Hafalir et al. (2013), Kominers and Sönmez (2016), and the subsequent literature.

Next we consider two properties of a choice function.

Definition 1. *Choice function C is* ***q-acceptant*** *if for every X,* $|C(X)| = \min\{q, |X|\}$. *Choice function C is* ***acceptant*** *if it is q-acceptant for some q.*

Kojima and Manea (2010) use q-acceptant choice functions to characterize DA. In an earlier work, Alkan (2001) show the lattice structure of stable matchings under acceptant and path independent choice functions.

Definition 2. *Choice function C is* ***path independent*** *if for every X and Y,* $C(X \cup Y) = C(C(X) \cup C(Y))$.

Path independence guarantees the existence of *stable* matchings (Chambers and Yenmez, 2017).

3. Result

Theorem 1. *A lexicographic choice function is acceptant and path independent. There exists an acceptant and path-independent choice function that is not lexicographic.*

Proof. Before we proceed with the proof we introduce the following choice function properties.

Definition 3. *Choice rule C is* ***substitutable*** *if for every* $x \in X \subseteq Y$, $x \in C(Y)$ *implies* $x \in C(X)$.

[1]Complete: For all $x, y \in \mathcal{X}$, $x \succeq y$ or $y \succeq x$. Antisymmetric: For all $x, y \in \mathcal{X}$, $x \succeq y$ and $y \succeq x$ implies $x = y$. Transitive: For all $x, y, z \in \mathcal{X}$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Substitutability requires that if an alternative is chosen from a set, then it must also be chosen from a subset containing it. Kelso and Crawford (1982) were the first to study substitutability in a matching context. However, it was studied in the choice literature earlier and known as Sen's α or Chernoff's axiom (Moulin, 1985).

Definition 4. *Choice rule C satisfies **consistency** if $C(Y) \subseteq X \subseteq Y$ then $C(X) = C(Y)$.*

See a discussion of consistency in Aygün and Sönmez (2013, 2012) who call it the irrelevance of rejected contracts. Path independence is equivalent to the conjunction of consistency and substitutability (Aizerman and Malishevski, 1981).

First, we show that if C is lexicographic with $\{\succeq_i\}_{i\in I}$, then it is acceptant and path independent. It is easy to see that C is $|I|$-acceptant and that it satisfies consistency. We note that the completeness of the preference relations is crucial for acceptance. Next we show that C is also substitutable.[2]

Suppose that $x \in X \subseteq Y$ and $x \in C(Y)$. We show that $x \in C(X)$. Let x_i^* and y_i^* be as in the definition of $C(X)$ and $C(Y)$. We show by induction that $X \setminus \{x_1^*, \ldots, x_k^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_k^*\}$. The claim is true for $k = 0$ because $X \subseteq Y$.[3] Suppose that the claim holds for $k = 0, \ldots, n-1$, we prove it for n where $n \leq |I|$. Recall $y_n^* = \max_{Y \setminus \{y_1^*, \ldots, y_{n-1}^*\}} \succeq_n$ and $x_n^* = \max_{X \setminus \{x_1^*, \ldots, x_{n-1}^*\}} \succeq_n$. There are two cases that we consider. Fist, $y_n^* = x_n^*$. In this case, $X \setminus \{x_1^*, \ldots, x_{n-1}^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_{n-1}^*\}$ implies $X \setminus \{x_1^*, \ldots, x_n^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_n^*\}$. Second, $y_n^* \neq x_n^*$. This implies that $y_n^* \notin X \setminus \{x_1^*, \ldots, x_{n-1}^*\}$, so $X \setminus \{x_1^*, \ldots, x_{n-1}^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_n^*\}$ and $X \setminus \{x_1^*, \ldots, x_n^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_n^*\}$. Thus the claim is true.

Since $x \in C(Y)$, there exists k such that $x = y_{k+1}^*$. By the claim above, $X \setminus \{x_1^*, \ldots, x_k^*\} \subseteq Y \setminus \{y_1^*, \ldots, y_k^*\}$. Therefore, if $x \in X \setminus \{x_1^*, \ldots, x_k^*\}$, then $x = x_{k+1}^*$ and so $x \in C(X)$. Otherwise, if $x \notin X \setminus \{x_1^*, \ldots, x_k^*\}$, then $x = x_i^*$ for some $i \leq k$ and so $x \in C(X)$. Thus C is substitutable.

We prove that not every acceptant path-independent choice function is lexicographic by providing an example. Consider a path-independent choice function C on $\{1,2,3,4,5,6\}$ with $C(1,2,3,4,5,6) = \{1,2\}$, $C(3,4,5,6) = \{3,4\}$, $C(3,5,6) = \{3,5\}$, $C(4,5,6) = \{4,5\}$, $C(1,5,6) = \{1,6\}$, and $C(2,5,6) = \{2,6\}$. This choice function is not completely defined yet since there is only partial information. We complete it using path independence to get the choice function in Figure 1, which shows the lattice representation of the choice function (Johnson and Dean, 2001).

Choice function C is 2-acceptant and path independent. We show that C is not lexicographic. In fact, we only use partial knowledge on C above to get a contradiction.

Consider any choice function C with the stated values above. Since 1 and 2 are symmetric for this choice function and that $C(1,2,3,4,5,6) = \{1,2\}$ we can assume without loss of

[2] An alternative proof can be found in Hatfield and Kominers (2015).

[3] With an abuse of notation let $\{x_1^*, \ldots, x_k^*\} = \{y_1^*, \ldots, y_k^*\} = \emptyset$ for $k = 0$.

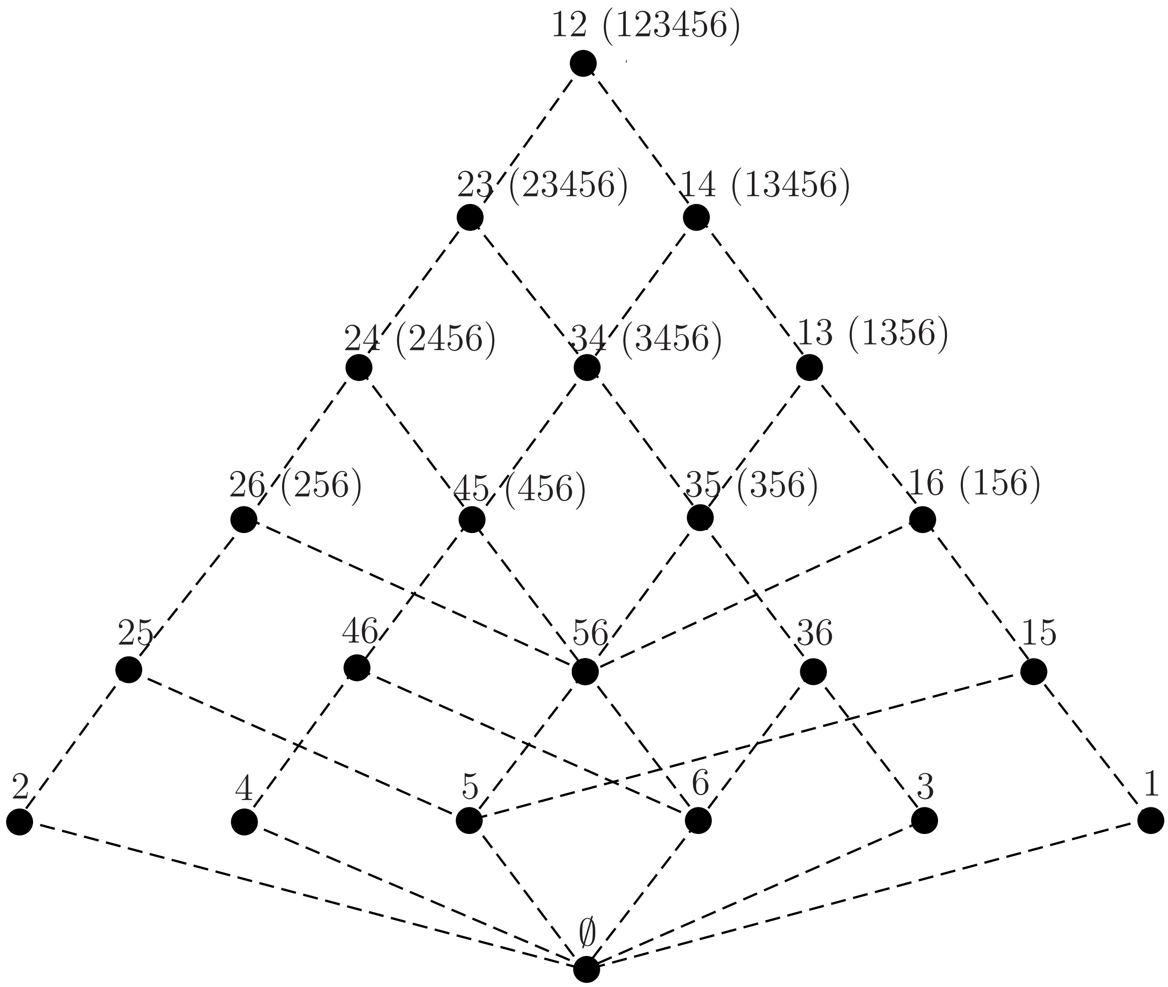

FIGURE 1. Lattice representation of the choice function in the proof of Theorem 1

generality that the highest ranked element in $\succ_1$ is 1 and the highest ranked element in $\succ_2$ is 2. Since $C(3,4,5,6) = \{3,4\}$ either $3 \succ_1 4,5,6$; $4 \succ_2 5,6$ or $4 \succ_1 3,5,6$; $3 \succ_2 5,6$.

Case 1 ($3 \succ_1 4,5,6$; $4 \succ_2 5,6$): Since $C(3,5,6) = \{3,5\}$ and $3 \succ_1 5,6$, we get $5 \succ_2 6$. But $C(1,5,6) = \{1,6\}$ and $1 \succ_1 5,6$, so we have $6 \succ_2 5$, which is a contradiction.

Case 2 ($4 \succ_1 3,5,6$; $3 \succ_2 5,6$): Since $C(4,5,6) = \{4,5\}$ and $4 \succ_1 5,6$, we get $5 \succ_2 6$. But $C(1,5,6) = \{1,6\}$ and $1 \succ_1 5,6$, so we have $6 \succ_2 5$, which is a contradiction. □

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