Non-Diversified Portfolios with Subjective Expected Utility

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Abstract

Diversification is the typical investment strategy of risk-averse agents. However, non-diversified positions that allocate all resources to a single asset, state of the world or revenue stream are common too. We show that whenever finitely many non-diversified demands under uncertainty are compatible with risk-averse subjective expected utility maximization under strictly positive beliefs, they are also rationalizable under the same beliefs by many qualitatively distinct risk-averse as well as risk-neutral and risk-seeking preferences.

Keywords: Investment under uncertainty; Non-diversification; Subjective expected utility; Demand; Revealed preference.

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"But the wise man saith,
'Put all your eggs in the one basket and
- WATCH THAT BASKET'."
Mark Twain[1]

"Diversification is protection against ignorance,
but if you don't feel ignorant,
the need for it goes down drastically."
Warren Buffett[2]

1 Introduction

Risk-averse decision makers in real-world and experimental markets typically allocate their available funds or revenue streams in ways that exhibit diversification. Yet at the same time it is not uncommon for individuals operating in such environments to choose non-diversified portfolios that allocate all resources to a single asset or state of the world.[3]

Considering the intuitive link between risk aversion and diversification,[4] the first question that arises naturally is whether agents who have been observed to make such *non*-diversified choices can also be portrayed as risk-*averse* subjective expected utility (SEU; Savage, 1954; Echenique and Saito, 2015; Pollison, Quah and Renou, 2020) maximizers under some beliefs and tastes. The answer to this question is positive and implicitly follows from the analysis of Inada (1963). In particular, such behaviour is compatible with a risk-averse SEU agent, but only if her marginal utility is bounded above at all non-negative wealth levels; hence, if the respective Inada (1963) condition on marginal utility is violated.

In light of this fact then, we ask: What is it possible for the analyst to infer from a non-diversifying SEU agent's demand under uncertainty about their possible attitudes toward risk? We answer this question by showing that if a finite dataset consisting of non-diversified positions is compatible with strictly risk-averse SEU under some full-support beliefs, then there is in fact a very general class of preferences over wealth that feature bounded marginal utility and also rationalize this dataset under the *same* beliefs. In particular, we show that such a simultaneous rationalization is achievable with risk-neutral, risk-seeking, constant, as well as increasing absolute risk aversion –although not constant relative risk aversion– preferences. Thus, when non-diversified choice behavior is SEU-rationalizable by some strictly concave utility function and full-support beliefs,

[1]Source: *Pudd'nhead Wilson*, Charles L. Webster & Co, 1894.

[2]Source: *Warren Buffett: The $59 Billion Philanthropist*, Forbes Media, 2018.

[3]From the 207 experimental subjects in Halevy, Persitz and Zrill (2018), for example, 45% made such a non-diversified demand over Arrow-Debreu securities at least once, 11% did so in at least half of their 22 decisions, while the overall rate of such behavior was 16%. For the 93 subjects in Choi, Fisman, Gale and Kariv (2007) these figures were similar at 51%, 8.6% and 11%, respectively, even though these subjects made 50 decisions instead (more details are available in our online supplementary appendix). For a theoretical account on how investing in non-diversified portfolios could be observed in the presence of financial complexity see Galanis (2018).

[4]For example, because firm CEOs are often considered averse to non-diversified revenue streams, some firm boards provide more *"risk-taking incentives"* in the CEOs' compensation packages *"to offset their risk of non-diversified revenue streams, thereby preventing excessive managerial conservatism at the expense of value maximization"* (Chen, Su, Tian and Xu, 2022).

there is a precise sense in which this behavior is largely uninformative about the decision maker's risk attitudes conditional on those beliefs.

Although perhaps seemingly paradoxical, the result is intuitive. The main insight is that rationalization via a strictly concave utility function implies that the chosen asset/state of the world uniquely maximizes the probability/price ratio. This unique maximization in turn implies that the corresponding risk-neutral investor with the same subjective probabilities will have the same demand. Our extension to the other families results from an approximation argument, as any member of this class can approximate a linear preference on a bounded set.

2 Analysis

$\Omega := \{\omega_1, \ldots, \omega_n\}$ is a finite set of states, with generic element $\omega \in \Omega$, and π is a probability measure over Ω. $\mathcal{D} = \{(p^i, x^i)\}_{i=1}^k$ is a finite dataset of prices and asset demands, where $p^i, x^i \in \mathbb{R}^n_+$ and $p^i \gg \mathbf{0}$ for all $i \leq k$.[5] We will refer to any x^i in $\mathcal{D}$ as a *non-diversified demand* if there exists $\omega \in \Omega$ for which $x^i_\omega > 0$ and $x^i_{\omega'} = 0$ for all $\omega' \neq \omega$.

Definition 1. *A dataset $\mathcal{D} = \{(p^i, x^i)\}_{i=1}^k$ is rationalizable by subjective expected utility* (SEU-rationalizable) *if there is a probability measure π over Ω and an increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $i \leq k$,*

$$\mathbb{E}_\pi u(x^i_\omega) \geq \mathbb{E}_\pi u(x_\omega) \text{ for all } x \in \mathbb{R}^n_+ \text{ that satisfies } p^i \cdot x \leq p^i \cdot x^i$$

Remark 1. *If an SEU-rationalizable dataset contains a non-diversified demand, then it is not rationalizable by a Constant Relative Risk Aversion (CRRA) utility function defined by $u(x) := x^\alpha$ for some $\alpha \in (0, 1)$.*

This claim implicitly follows from Inada (1963) and generalizes to any class of utility functions with unbounded marginal utility at zero wealth.

We now recall that for any function $u : \mathbb{R} \to \mathbb{R}$, a *supergradient* at a point $\overline{x}$ is an element $y \in \mathbb{R}$ for which

$$u(x) \leq u(\overline{x}) + y(x - \overline{x})$$

holds for all $x \in \mathbb{R}$. If a point has a single supergradient, then that supergradient is its derivative. The *superdifferential* at $\overline{x}$ is denoted by $\partial u(\overline{x})$ and consists of all supergradients at $\overline{x}$.

The next definition introduces the class of models that we take interest in. We envision a model as a class of utility functions which is "closed" under certain operations. Importantly, this class need not be globally increasing in wealth: our first requirement is only that there exists a function in this class which is strictly increasing in a neighborhood of zero (the relevance of this will be shown below). Our second requirement is that this neighborhood can be made arbitrarily large.

Definition 2. *A collection $\mathcal{U}$ of concave and continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ is* scalable *if it has the following properties:*

[5] $x \gg y$ means that $x_\omega > y_\omega$ for all $\omega \in \Omega$.

1. *There is $u \in \mathcal{U}$ for which there is a supergradient $u'(0)$ at 0, so that for all $x \in \mathbb{R}$, $u(x) \leq u(0) + u'(0)x$; further, all supergradients of u at 0 are strictly positive.*

2. *For all $\kappa \in (0,1]$ and all $v \in \mathcal{U}$, v_κ defined as $v_\kappa(x) := v(\kappa x)$ for all $x \in \mathbb{R}$ satisfies $v_\kappa \in \mathcal{U}$.*

Remark 2. *As supergradients are real-valued, the first part of Definition 2 implies that the relevant $u \in \mathcal{U}$ must satisfy $u'(0) < \infty$, thereby satisfying the premise of Remark 1.*

Proposition 1. *Suppose that a dataset $\mathcal{D} = \{(x^i, p^i)\}_{i=1}^k$ is SEU-rationalizable by a strictly concave, strictly increasing utility index and a full-support probability measure π, and that each x^i is a non-diversified demand. Then, for any scalable family of concave and continuous utility functions $\mathcal{U}$ there is $u \in \mathcal{U}$ such that u is an SEU rationalization of the dataset under π.*

Proof. Let v be the utility index rationalizing the data. Without loss, we may assume that $v(0) = 0$. Let x^i be such that coordinate $x^i_{\omega^i} > 0$, and all remaining coordinates are zero. Slater's condition is satisfied here, so by Theorems 28.2 and 28.3 of Rockafellar (1970), this implies that there is a supergradient y^i_ω of v for each $\omega \neq \omega^i$ at 0 and a supergradient $y^i_{\omega^i}$ at $x^i_{\omega^i}$, and a multiplier $\lambda^i > 0$ for which

$$\pi(\omega)y^i_\omega - \lambda^i p^i(\omega) \leq \pi(\omega^i)y^i_{\omega^i} - \lambda^i p^i(\omega^i) = 0.$$

[The fact that supergradients are additive follows from Theorem 23.8 in Rockafellar (1970).] Each $y^i_\omega > 0$ as v is strictly increasing. We may conclude then that for all $\omega \neq \omega^i$, $\lambda^i p^i(\omega) \geq \pi(\omega)y^i_\omega$ and that $\lambda^i p^i(\omega^i) = \pi(\omega^i)y^i_{\omega^i}$. So

$$\frac{p^i(\omega^i)}{p^i(\omega)} \leq \frac{y^i_{\omega^i}\pi(\omega^i)}{y^i_\omega \pi(\omega)} < \frac{\pi(\omega^i)}{\pi(\omega)}, \tag{1}$$

where the strict inequality follows from strict concavity of v and the fact that the superdifferential is strictly decreasing.

Now, fix any $u \in \mathcal{U}$ with finite supergradient at 0, whose supergradients are all strictly positive there. Without loss, suppose that $u(0) = 0$. Let $u'(0)$ denote the minimal such supergradient (the one with the smallest value); the set of supergradients (the superdifferential) is well-known to be closed (see p. 215 in Rockafellar, 1970), so such an element exists. Without loss assume $u'(0) = 1$ (this is possible because $\mathcal{U}$ is scalable). If the superdifferential correspondence is constant and equal to $u'(0)$, no more work is needed (this means that u is a linear function). Otherwise, we claim that for any $\epsilon > 0$, there exists $x^* > 0$ with a supergradient bounded below by $1 - \epsilon$. To see why, observe that if $x_n \to 0$ strictly monotonically and $y_n \in \partial u(x_n)$, then y_n is weakly increasing and thus has a limit; the limit must be a member of $\partial u(0)$ by Theorem 24.4 of Rockafellar (1970), and hence must be at least as large as 1 (as 1 was the minimal element of $\partial u(0)$). Consequently there is $x_n > 0$ small so that y_n is a supergradient of u at x_n and $y_n \geq 1 - \epsilon$, which is what we wanted to show. Obviously, $y_n \leq 1$.

Now choose $\epsilon > 0$ small so that, for all i, we have $\frac{(1-\epsilon)\pi(\omega^i)}{\pi(\omega)} > \frac{p^i(\omega^i)}{p^i(\omega)}$; this can be done by finiteness of the set of observations. Let $\overline{x} = \max_i x^i_{\omega^i}$ be the maximal nonzero consumed commodity; obviously $\overline{x} > 0$. Let $0 < x_\epsilon < \overline{x}$ have a supergradient of at least $1 - \epsilon$. Let $\alpha = \frac{x_\epsilon}{\overline{x}} < 1$, so that for all i, $\alpha x^i_{\omega^i} < x_\epsilon$. Observe that $u_\alpha(\overline{x}) = u(x_\epsilon)$.

Now, define $\overline{u}(x) = \frac{u_\alpha(x)}{\alpha}$. By assumption, $u_\alpha \in \mathcal{U}$, and since $\overline{u}$ is cardinally equivalent to u_α, they have the same optimizers in any constrained optimization problem.

Observe that $1 \in \partial\overline{u}(0)$ and that there is a supergradient of $\overline{u}$ at x_ϵ at least as large as $1-\epsilon$. Therefore for each i, $\overline{u}$ has a supergradient at $x^i_{\omega^i}$ at least as large as $1-\epsilon$, as $\alpha x^i_{\omega^i} < x_\epsilon$. Consequently, by letting $z^i_{\omega^i}$ be any member of the supergradient of $x^i_{\omega^i}$ at least as large as $1-\epsilon$, we have

$$\frac{p^i(\omega^i)}{p^i(\omega)} < \frac{z^i_{\omega^i}\pi(\omega^i)}{\pi(\omega)}.$$

Set $\lambda^i = \frac{\pi(\omega^i)z^i_{\omega^i}}{p^i(\omega^i)}$ and observe that we then have $\pi(\omega^i)z^i_{\omega^i} - \lambda^i p^i(\omega^i) = 0$ and $\pi(\omega) - \lambda^i p^i(\omega) < 0$. Conclude again by Theorem 28.3 of Rockafellar (1970), using the fact that 1 is a supergradient of $\overline{u}$ at 0. □

Proposition 1 can be extended to the risk-seeking case by adapting Definition 2.

Definition 3. *A class $\mathcal{U}^*$ of continuous, increasing and convex functions from $\mathbb{R}_+$ to $\mathbb{R}$ is* scalable *if it has the following properties:*

1. *There is $v \in \mathcal{U}^*$ with a positive subgradient at 0.*
2. *For all $\alpha \in (0,1]$ and all $v \in \mathcal{U}^*$, v_α defined as $v_\alpha(x) := v(\alpha x)$ for all $x \in \mathbb{R}$ satisfies $v_\alpha \in \mathcal{U}^*$.*

Proposition 2. *Suppose that a dataset $\{(x^i, p^i)\}_{i=1}^k$ is SEU-rationalizable by a strictly concave utility function and a full-support probability measure π, and that each x^i is a non-diversified demand. Then, for any scalable family of increasing, convex, and continuous utility functions $\mathcal{U}^*$ there is some $u \in \mathcal{U}^*$ such that u is an SEU rationalization of the dataset under π.*

Proof. Observe that (1) implies $\frac{\pi(\omega^i)}{p^i(\omega^i)} > \frac{\pi(\omega)}{p^i(\omega)}$ for any $\omega \neq \omega^i$. Consequently, the linear utility given by $v(x) = \sum_\omega \pi(\omega)x(\omega)$ is maximized uniquely at x^i on the budget $\{x : p^i \cdot x \leq p^i \cdot x^i\}$.

The argument is roughly the same as the preceding one, so we only sketch the remaining. Choose $\alpha > 0$ so that (a suitably normalized) u has a subgradient of 1 at 0, and so that for a given $\epsilon > 0$, there is a subgradient of $1 + \frac{\epsilon}{\overline{x}}$, where again $\overline{x}$ is the maximal observed consumption bundle. Now choose $\epsilon > 0$ so that

$$\begin{aligned}
\pi(\omega^i)u_\alpha(x^i) &= \pi(\omega^i)u_\alpha\left(\frac{1}{p^i(\omega^i)}\right) \\
&\geq \frac{\pi(\omega^i)}{p^i(\omega^i)} \\
&> \frac{\pi(\omega)(1+\epsilon)}{p^i(\omega^i)} \\
&\geq \pi(\omega)u_\alpha\left(\frac{1}{p^i(\omega^i)}\right).
\end{aligned}$$

Since a (continuous) convex function is always maximized at an extreme point, by Bauer's Maximum Principle (Theorem 7.69 in Aliprantis and Border, 2006), the result follows. □

We illustrate the economic relevance of these results with the following Corollary, which lists several classes of scalable families of utility functions, including the quadratic family, (iii.), which is strictly increasing only in a neighborhood of the origin.

Corollary. *If the dataset $\mathcal{D} := \{(x^i, p^i)\}_{i=1}^k$ is SEU-rationalizable by a strictly concave and strictly increasing utility function under a full-support probability measure π and each x^i is non-diversified, then $\mathcal{D}$ is also SEU-rationalizable under π by a:*

(i) u_1^α, for any $\alpha \in (0,1)$, such that for some $c_\alpha > 0$, $u_1^\alpha(x) := (x + c_\alpha)^\alpha$ (DARA[6]-risk-averse with positive fixed initial wealth);[7]

(ii) u_2 such that, for some $\beta > 0$, $u_2(x) := 1 - e^{-\beta x}$ (CARA-risk-averse);

(iii) u_3 such that, for some $\lambda > 0$, $u_4(x) := x - \lambda x^2$ (IARA-risk-averse/increasing in a neighborhood of 0);

(iv) u_4 such that, for some $\gamma > 0$, $u_3(x) := \dfrac{x}{1+\gamma x}$ (IARA-risk-averse/increasing in $\mathbb{R}_+$);

(v) Linear u;

(vi) Strictly convex v.

Proof. We apply Proposition 1 separately to the first four cases. It is immediate that the last two satisfy the conditions of Proposition 2 too, and that many suitable classes of functions can be constructed for the last case.
(i) Fix $\alpha \in (0,1)$. Let $\mathcal{U}$ denote the set of all utility indices u for which there exists $\kappa > 0$ such that $u(x) = (1 + \kappa x)^\alpha$. Then it is obvious that $\mathcal{U}$ is scalable. Further, fixing $\kappa = 1$, say, $u(x) = (1 + x)^\alpha$ is concave and continuous, there is a supergradient at 0, and all supergradients are strictly positive. By Proposition 1, there is $\kappa > 0$ for which $u(x) = (1+\kappa x)^\alpha$ rationalizes the data. The result concludes by observing that this utility index is cardinally equivalent to $v(x) = (\kappa^{-1} + x)^\alpha$, where we then set $c_\alpha = \kappa^{-1}$.
(ii) Let $\mathcal{U}$ denote the set of functions u for which there exists $\beta > 0$ so that $u(x) = 1-e^{-\beta x}$ and observe that this family is scalable.
(iii) Observe that the family $\mathcal{U}$ defined by $u \in \mathcal{U}$ if there exist $\theta > 0$ and $\mu > 0$ for which $u(x) = \theta x - \mu x^2$ is scalable. Finally, each such $u \in \mathcal{U}$ is cardinally equivalent to $v(x) = x - \frac{\mu}{\theta} x^2$, from which the result follows.
(iv) Observe that the family $\mathcal{U}$ defined by $u \in \mathcal{U}$ if there exists $\beta, \gamma > 0$ for which $u(x) = \dfrac{\beta x}{1+\gamma x}$ is scalable. Finally, each such $u \in \mathcal{U}$ is cardinally equivalent to $v(x) = \dfrac{x}{1+\gamma x}$. □

References

Aliprantis, Charalambos D. and Kim C. Border, *Infinite Dimensional Analysis*, 3rd edition, Berlin Heidelberg: Springer, 2006.

[6] D(C)(I)ARA refers to decreasing (constant) (increasing) absolute risk aversion.

[7] When c_α is negative, it is also known as the *subsistence parameter* (Ogaki and Zhang, 2001).

Chen, Jie, Xunhua Su, Xuan Tian, and Bin Xu, "Does Customer-Base Structure Influence Managerial Risk-Taking Incentives?," *Journal of Financial Economics*, 2022, *143*, 462–483.

Choi, Syngjoo, Raymond Fisman, Douglas Gale, and Shachar Kariv, "Consistency and Heterogeneity of Individual Behavior under Uncertainty," *American Economic Review*, 2007, *97*, 1921–1938.

Echenique, Federico and Kota Saito, "Savage in the Market," *Econometrica*, 2015, *83*, 1467–1495.

Galanis, Spyros, "Financial Complexity and Trade," *Games and Economic Behavior*, 2018, *112*, 219–230.

Halevy, Yoram, Dotan Persitz, and Lanny Zrill, "Parametric Recoverability of Preferences," *Journal of Political Economy*, 2018, *126* (4), 1558–1593.

Inada, Ken-Ichi, "On a Two-Sector Model of Economic Growth: Comments and a Generalization," *Review of Economic Studies*, 1963, *30*, 119–127.

Ogaki, Masao and Qiang Zhang, "Decreasing Relative Risk Aversion and Tests of Risk Sharing," *Econometrica*, 2001, *69*, 515–526.

Polisson, Matthew, John K.-H. Quah, and Ludovic Renou, "Revealed Preferences over Risk and Uncertainty," *American Economic Review*, 2020, *110*, 1782–1820.

Rockafellar, R. Tyrrell, *Convex Analysis*, Princeton: Princeton University Press, 1970.

Savage, Leonard J., *The Foundations of Statistics*, New York: Wiley, 1954.