An ordinal characterization of the linear opinion pool

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Abstract

Given a collection of individual ordinal probabilities on a finite state space, we discuss an ordinal condition that is necessary and sufficient for an ordinal probability to be represented as a weighted average of probability representations of the individual probabilities. We also give necessary and sufficient conditions for when such an ordinal probability can be represented as an unweighted average of probability representations. Keywords: Opinion pool, linear opinion pool, ordinal probability, probability aggregation, utilitarian. JEL classification: D71, D81.

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1 Introduction

The purpose of this paper is to shed light on a commonly used method of aggregating the likelihood assessments of a collection of experts. Working within a binary decision model, and a finite set of states of the world, a "social planner" seeks to come up with some social assessment of the likelihoods of events. The social assessment is based on the assessments of certain agents (one may wish to interpret these agents as "experts"). The agents in society do not attach probabilities to events, but can only rank them in an ordinal sense. Nevertheless, we imagine that these ordinal rankings can be represented by probability measures. Such an ordinal likelihood assessment is termed an *ordinal probability*. Many probability measures may represent the same ordinal probability.

To motivate, in a binary decision model, the only statements that have empirical content are statements about the relative likelihood of certain events. Such a model is not rich enough to deliver unique, well-defined probabilities (in contrast to the models of Savage (1972) or Anscombe-Aumann (1963)).

Nevertheless, probabilistic likelihood assessments are common in the real-world. In this environment, perhaps the most common method of coming to a social assessment is the linear opinion pool (Genest and Zidek (1986), Stone (1961)). The linear opinion pool ranks events according to some convex combination of individual probability representations. A convex combination of individual probability representations appears to be an entirely natural social assessment.

However, the implications of taking such convex combinations are not at all obvious. As we mentioned above, it is possible that many probability measures may represent the same ordinal probability. Moreover; even if we know which probability measures to use, there is a question of how heavily each individual's probability measure should be weighted. In this sense, there are at least two degrees of freedom in the choice of such a social assessment.

We would like to understand exactly which social ordinal probabilities can be arrived at by an aggregation procedure of the type described above. To this end, our main result is an *ordinal* statement of the complete implications of using a linear opinion pool as a social likelihood assessment. That is, we provide a condition which is both necessary and sufficient for a social likelihood to be written as a convex combination of individual probability representations. Working in a traditional multi-profile social choice framework, Lainé, LeBreton, and Trannoy (1986) and Weymark (1997) consider several normative conditions that a "good" aggregation procedure should satisfy, none of which coincide with the linear opinion pool (except for an extreme example in which all agents but one are given zero weight).

To get some feeling for the condition characterizing such assessments, we first consider a related condition of Kraft, Pratt, and Seidenberg (1959) (hereafter KPS) . They establish a necessary and sufficient condition for an ordinal likelihood assessment to be representable by a probability measure in a finite-states environment. Given that a convex combination of probability measures is itself a probability measure, our condition must imply that the social likelihood assessment should satisfy the KPS property. The KPS condition is best viewed in the following way. Any probability measure can be used to perform integration, and in this sense, extends to a unique linear functional. Hence, the ordinal probability can also be extended to an ordinal relation on the class of integrable functions. This ordinal relation has an integral representation–so that it is linear. Conditions for representation by a linear functional are known. In particular, such an extension must satisfy something analogous to the independence axiom of decision theory. The KPS condition tells us exactly when such an extension will not lead to contradictory rankings of functions.

Now, our condition builds off of the KPS condition in the following way. Much is known about aggregation of ordinal rankings satisfying independence axioms. A classical result due to Harsanyi (1955) tells us that if a group ranking of such ordinal rankings satisfies a simple Pareto condition with respect to the individual rankings, then it can be represented as some convex combination of individual representations. The Pareto condition simply says that if all individuals rank one alternative higher than another, then the social ranking should be the same as the individual rankings. Our condition is best understood as stating this Pareto condition on the collection of linear extensions of ordinal probabilities. Of course, our condition implies that if all individuals rank an event as more likely than another event, then so does society.

We use a version of the Theorem of the Alternative in order to establish the condition. The role of this theorem in formal social choice models has been known at least since Fishburn (1973, 1973). In fact, our condition can be viewed as a type of combination of conditions first proposed by Fishburn and KPS. That is, the KPS condition tells us when an ordinal probability can be represented by a probability measure. Fishburn's work addresses the question as to when a social choice rule can be represented as the maximal elements of some utilitarian social welfare function. We combine these two ideas: we require our social welfare function to be utilitarian, but we require it to be utilitarian with respect to probabilistic representations.

Suppose that instead of working in a finite-state model, we were to work in a model in which individual likelihood assessments were representable as *non-atomic* measures. In this environment, as long as the social assessment is also representable by a non-atomic measure, the Pareto condition is necessary and sufficient for it to be represented as a convex combination of individual measures (see Mongin (1995)). This occurs because in this environment, each probability measure is *uniquely* extended to the space of functions. Thus, at least in this environment, there is a strong motivation for the linear opinion pool.

We recognize that allowing for any possible linear opinion pool admits a very large number of possibilities. To this end, we also provide a necessary and sufficient condition for a social likelihood assessment to be represented as an unweighted average of individual probability representations. We know of no analogous condition in the literature on nonatomic probability aggregation. The interpretation of this condition is as of a Pareto condition "with transfers."

Section 2 introduces the basic model and main result. Section 3 discusses the stronger condition needed to characterize symmetric linear opinion pools. Finally, section 4 concludes.

2 The model and basic result

Let Ω be a finite set of **states of the world**. Let $\{1, ..., n\}$ be a set of **agents**. Each agent is endowed with a likelihood relation $\preceq_i$ over 2^Ω–the set of **events**. We assume that likelihoods are **ordinal probabilities** in the sense of Kraft, Pratt, and Seidenberg (KPS) (so that there exists a probability measure $p_i : 2^\Omega \to \mathbb{R}$ that represents $\preceq_i$).[1]

For a set of probability measures $\{p_i\}_{i=1}^n$, say that a probability measure p_0 is a **linear opinion pool** with respect to $\{p_i\}_{i=1}^n$ if there exists some $\lambda \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n \lambda_i = 1$, where

$$p_0 = \sum_{i=1}^n \lambda_i p_i.$$

For a set of ordinal probabilities $\{\preceq_i\}_{i=1}^n$, say that an ordinal probability $\preceq_0$ is a **linear opinion pool** with respect to $\{\preceq_i\}_{i=1}^n$ if for all $i \in \{0, ..., n\}$, there exists a probability measure p_i that represents $\preceq_i$ so that p_0 is a linear opinion pool with respect to $\{p_i\}_{i=1}^n$.

The linear opinion pool is the most common way of combining the probabilities of different agents to produce a social probability. Here we ask the following question: Given a collection of ordinal probabilities (but not specific representations for them), what are the set of necessary and sufficient conditions that an ordinal probability must satisfy in order to be a linear opinion pool with respect to the given probabilities?

We provide a condition telling us exactly when a social likelihood relation is a linear opinion pool with respect to individual ordinal probabilities. For two finite sequences of elements of 2^Ω of equal length, say, $(A_1, ..., A_k)$ and $(B_1, ..., B_k)$, we write (as in Weymark (1997)) $(A_1, ..., A_k) \risingdotseq (B_1, ..., B_k)$ to mean that for all $\omega \in \Omega$, $|\{i : \omega \in A_i\}| = |\{i : \omega \in B_i\}|$. Thus, the number of sets in which ω is a member in each of the sequences is the same.

We can now formally state the definition of an ordinal probability. An **ordinal probability** $\preceq$ on 2^Ω is a likelihood relation satisfying the following properties:

i) $\preceq$ is complete

ii) For all $A \subseteq \Omega$, $A \succeq \varnothing$

iii) $\Omega \succ \varnothing$

iv) For all finite sequences $(A_1, ..., A_k)$,$(B_1, ..., B_k)$ of equal length for which $(A_1, ..., A_k) \risingdotseq (B_1, ..., B_k)$ and $A_j \succeq B_j$ for all $j \in \{1, ..., k\}$, it must be that $A_j \sim B_j$ for all j.

[1] As usual, $\prec$ denotes the asymmetric part of $\preceq$, whereas $\sim$ denotes the symmetric part.

It is of note that the axioms $i)$ and $iv)$ imply transitivity.[2]

Clearly, a linear opinion pool satisfies completeness. What other conditions are necessary? Obviously, some notion of dominance is necessary–i.e. if for all $i \in \{1, ..., n\}$, $A \succeq_i B$, then $A \succeq_0 B$, with social likelihood strict if at least one individual likelihood is strict. In the case in which social likelihoods are nonatomic and countably additive, this dominance condition is both necessary and sufficient (with the axioms of probability) to ensure that a social likelihood is a linear opinion pool. Here, however, something stronger is needed.

We introduce the condition now and then discuss it. The condition is a joint condition on $\preceq_0$ and $\{\preceq_i\}_{i=1}^n$.

The linear opinion condition (LOC) For all $i \in \{0, ..., n\}$, let $k^i \in \mathbb{Z}_+$, and let $\left(A_1^i, ..., A_{k^i}^i\right)$, $\left(B_1^i, ..., B_{k^i}^i\right)$ be two sequences such that for all $i \in \{0, ..., n\}$ and for all $j \in \left\{1, ..., k^i\right\}$, $A_j^i \succeq_i B_j^i$. Suppose that for all $i \in \{1, ..., n\}$, $\left(A_1^0, ..., A_{k^0}^0, A_1^i, ..., A_{k^i}^i\right) \doteqdot \left(B_1^0, ..., B_{k^0}^0, B_1^i, ..., B_{k^i}^i\right)$. Then for all $i \in \{0, ..., n\}$ and for all $j \in \left\{1, ..., k^i\right\}$, $A_j^i \sim_i B_j^i$.

If for all $i \in \{1, ..., n\}$, $k^i = 0$, then the LOC is the statement of the KPS axiom $iv)$ for $\preceq_0$.

Before we proceed with the characterization, we first discuss the intuition behind the LOC. Our claim is that it is essentially a Pareto criterion.

Recall the KPS property $iv)$ for a given ordinal probability: it states that for all $(A_1, ..., A_k), (B_1, ..., B_k)$ such that $(A_1, ..., A_k) \doteqdot (B_1, ..., B_k)$, if for all $j \in \{1, ..., k\}$, $A_j \succeq B_j$, then for all $j \, \{1, ..., k\}$, $A_j \sim B_j$.

Let us define a piece of notation. The indicator function of $A \subset \Omega$ is a function $1_A : \Omega \to \mathbb{R}$ defined by

$$1_A(\omega) \equiv \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases}.$$

A necessary condition for an ordinal probability to be represented as a probability measure is that it is possible to define a binary relation $\preceq^*$ on the set of functions $f : \Omega \to \mathbb{Z}$ such that $1_A \succeq^* 1_B$ if and only if $A \succeq B$ (we will call $\preceq^*$ an **extension** of $\preceq$) and which is "additive." By additivity, we mean that $f \succeq^* g$, $f' \succeq^* g'$ imply $f - g' \succeq^* g - f'$, with $f - g' \succ^* g - f'$ if either $f \succ^* g$ or $f' \succ^* g'$.[3] Given a probability measure p representing $\preceq$, it is easy to define $\preceq^*$ as

$$f \succeq^* g \Leftrightarrow \int_\Omega f(\omega)\, dp(\omega) \geq \int_\Omega g(\omega)\, dp(\omega).$$

We now show how an additive extension of $\preceq$ can be derived from $\preceq$, without reference to a particular probability measure representing $\preceq$. Let $\mathcal{F}$ be the

[2] Suppose that $A \succeq B$ and $B \succeq C$. By means of contradiction, suppose that it is not the case that $A \succeq C$. By $i)$, $C \succ A$. Now, label $(A_1, A_2, A_3) = (A, B, C)$ and $(B_1, B_2, B_3) = (B, C, A)$ and note that $(A_1, A_2, A_3) \doteqdot (B_1, B_2, B_3)$. Moreover, $A_j \succeq B_j$ for $j = 1, 2, 3$, yet $A_3 \sim B_3$ is false. This is a contradiction to $iv)$.

[3] Note that this condition then implies that $f + f' \succeq^* g + g'$, with strict ranking if either individual ranking is strict.

class of $\mathbb{Z}$-valued functions defined on Ω. Let $\preceq^*$ be defined on $\mathcal{F}$ by $f \succeq^* g$ if and only if there exists $(A_1, ..., A_k), (B_1, ..., B_k), (C_1, ..., C_{k'}), (D_1, ..., D_{k'})$ for which $f = \sum_{j=1}^{k} 1_{A_j} - \sum_{j=1}^{k'} 1_{C_j}$, $g = \sum_{j=1}^{k} 1_{B_j} - \sum_{j=1}^{k'} 1_{D_j}$, and $A_j \succeq B_j$ for all $j \in \{1, ..., k\}$ and $D_j \succeq C_j$ for all $j \in \{1, ..., k'\}$, with $f \succ^* g$ if any of the individual likelihood rankings is strict. Note that $\preceq^*$ is typically incomplete. If $\preceq^*$ is well-defined, it satisfies the two conditions discussed in the previous paragraph (it is an extension of $\preceq$ and it is additive). This is exactly the purpose of the KPS condition: it simply guarantees that $\preceq^*$ is well-defined, so that if $f \succeq^* g$, it is not the case that $g \succ^* f$. To see, this suppose, by means of contradiction that the KPS condition is satisfied, yet there exist f and g for which $f \succeq^* g$ and $g \succ^* f$. Then in particular, as $f \succeq^* g$, there exist $(A_1, ..., A_k), (B_1, ..., B_k), (C_1, ..., C_{k'}), (D_1, ..., D_{k'})$ for which $f = \sum_{j=1}^{k} 1_{A_j} - \sum_{j=1}^{k'} 1_{C_j}$, $g = \sum_{j=1}^{k} 1_{B_j} - \sum_{j=1}^{k'} 1_{D_j}$, and $A_j \succeq B_j$ for all $j \in \{1, ..., k\}$ and $D_j \succeq C_j$ for all $j \in \{1, ..., k'\}$. As $g \succ^* f$, there exist $(A'_1, ..., A'_l), (B'_1, ..., B'_l), (C'_1, ..., C'_{l'}), (D'_1, ..., D'_{l'})$ for which $g = \sum_{j=1}^{l} 1_{A'_j} - \sum_{j=1}^{l'} 1_{C'_j}$, $f = \sum_{j=1}^{l} 1_{B'_j} - \sum_{j=1}^{l'} 1_{D'_j}$, and $A'_j \succeq B'_j$ for all $j \in \{1, ..., l\}$ and $D'_j \succeq C'_j$ for all $j \in \{1, ..., l'\}$, with some preference strict. Clearly,

$$\sum_{j=1}^{k} 1_{A_j} + \sum_{j=1}^{k'} 1_{D_j} + \sum_{j=1}^{l} 1_{A'_j} + \sum_{j=1}^{l'} 1_{D'_j} = \sum_{j=1}^{k} 1_{B_j} + \sum_{j=1}^{k'} 1_{C_j} + \sum_{j=1}^{l} 1_{B'_j} + \sum_{j=1}^{l'} 1_{C'_j}.$$

But this is a contradiction to the LOC, as $A_j \succeq B_j$ for all $j \in \{1, ..., k\}$, $D_j \succeq C_j$ for all $j \in \{1, ..., k'\}$, $A'_j \succeq B'_j$ for all $j \in \{1, ..., l\}$, and $D'_j \succeq C'_j$ for all $j \in \{1, ..., l'\}$. Hence, if the LOC were satisfied, there would have to be indifference among all the preceding relations. However, we know at least one of them is strict. This is a contradiction.

Within this framework, it is easy to understand the LOC as a Pareto-like condition on these incomplete binary relations. Recall the classical result of Harsanyi (1995), and the later results obtained in a series of papers involving Mongin (Mongin (1995), DeMeyer and Mongin (1995)). These results state that if a social preference over some convex set is additive and if each individual preference is additive, if the social preference satisfies a simple Pareto monotonicity condition with respect to the individual preferences, social preference can be represented as some weighted sum of the individual preferences. Our statement of the Pareto monotonicity condition will be on the extensions $\preceq_i^*$ of $\preceq_i$ for $i \in \{0, ..., n\}$. It states that if $f \succeq_i^* g$ for all $i \in \{1, ..., n\}$, then $g \succ_0^* f$ is impossible, with $g \succeq_0^* f$ impossible if for some $i \in \{1, ..., n\}$, $f \succ_i^* g$. Thus, the Pareto condition we refer to merely states that a unanimous ranking across all agents in society cannot be refuted by the social ranking (with a statement holding as well for strict ranking). It is a natural generalization of Harsanyi's strong Pareto concept to potentially incomplete binary relations.

Our claim is that the LOC for the ordinal probabilities $\{\preceq_i\}_{i=0}^n$ is equivalent to Pareto monotonicity of the extensions $\{\preceq_i^*\}_{i=0}^n$. To see this, suppose that $\{\preceq_i^*\}_{i=0}^n$ satisfies the Pareto monotonicity condition. We will show that $\{\preceq_i\}_{i=0}^n$

satisfies the LOC. Consider families of sequences as discussed in the LOC, such that for all $i \in \{1,...,n\}$, $\left(A_1^0,...,A_{k^0}^0,A_1^i,...,A_{k^i}^i\right) \risingdotseq \left(B_1^0,...,B_{k^0}^0,B_1^i,...,B_{k^i}^i\right)$ and $A_j^i \succeq_i B_j^i$ for all i,j. Suppose, by means of contradiction, that there exists some $i \in \{0,...,n\}$ for which for some $j \in \left\{1,...,k^i\right\}$, $A_j^i \succ_i B_j^i$. In particular, we can conclude that for all $i \in \{0,1,...,n\}$, $\sum_{j=1}^{k^i} 1_{A_j^i} \succeq_i^* \sum_{j=1}^{k^i} 1_{B_j^i}$, with strict ranking for some $i \in \{0,...,n\}$. By additivity of each $\preceq_i^*$, we conclude that $\sum_{j=1}^{k^i} 1_{A_j^i} - \sum_{j=1}^{k^i} 1_{B_j^i} \succeq_i^* 0$ for all $i \in \{0,...,n\}$, with strict ranking for some $i \in \{0,...,n\}$. But for all $i,l \in \{1,...,n\}$, $\sum_{j=1}^{k^i} 1_{A_j^i} - \sum_{j=1}^{k^i} 1_{B_j^i} = \sum_{j=1}^{k^l} 1_{A_j^l} - \sum_{j=1}^{k^l} 1_{B_j^l}$. Moreover, for all $i \in \{1,...,n\}$, $\sum_{j=1}^{k^i} 1_{A_j^i} - \sum_{j=1}^{k^i} 1_{B_j^i} = \sum_{j=1}^{k^0} 1_{B_j^0} - \sum_{j=1}^{k^0} 1_{A_j^0}$. Consequently, by the Pareto monotonicity condition, it is not the case that $0 \succ_0^* \sum_{j=1}^{k^0} 1_{B_j^0} - \sum_{j=1}^{k^0} 1_{A_j^0}$. By the hypothesis of the LOC, $A_j^0 \succeq_0 B_j^0$ for all $j \in \left\{1,...,k^0\right\}$. Thus, $\sum_{j=1}^{k^0} 1_{B_j^0} - \sum_{j=1}^{k^0} 1_{A_j^0} \preceq_0^* 0$. Therefore, $\sum_{j=1}^{k^0} 1_{B_j^0} - \sum_{j=1}^{k^0} 1_{A_j^0} \sim_0^* 0$, which implies that $A_j^0 \sim_0 B_j^0$ for all $j \in \left\{1,...,k^0\right\}$. Hence, in order for our supposition to hold, there must exist some $i \in \{1,...,n\}$ and some $j \in \left\{1,...,k^i\right\}$ for which $A_j^i \succ_i B_j^i$. But then the Pareto monotonicity condition implies that it is not the case that $0 \succeq_0^* \sum_{j=1}^{k^0} 1_{B_j^0} - \sum_{j=1}^{k^0} 1_{A_j^0}$, a contradiction.

Conversely, suppose that the LOC is satisfied, let $f,g \in \mathcal{F}$, and suppose that for all $i \in \{1,...,n\}$, $f \succeq_i^* g$. Without loss of generality, we may suppose that for all $i \in \{1,...,n\}$, there exist $\left(A_1^i,...,A_{k^i}^i\right)$ and $\left(B_1^i,...,B_{k^i}^i\right)$ for which $A_j^i \succeq_i B_j^i$ for all $j \in \left\{1,...,k^i\right\}$ for which $f = \sum_{j=1}^{k^i} 1_{A_j^i}$ and $g = \sum_{j=1}^{k^i} 1_{B_j^i}$. Suppose, by means of contradiction, that $g \succ_0^* f$. Then there exist $\left(A_1^0,...,A_{k^0}^0\right)$ and $\left(B_1^0,...,B_{k^0}^0\right)$ for which $g = \sum_{j=1}^{k^0} 1_{A_j^0}$ and $f = \sum_{j=1}^{k^0} 1_{B_j^0}$ for which $A_j^0 \succeq_0 B_j^0$ for all $j \in \left\{1,...,k^0\right\}$, with strict ranking for some j. But note that for all $i \in \{1,...,n\}$, $\left(A_1^0,...,A_{k^0}^0,A_1^i,...,A_{k^i}^i\right) \risingdotseq \left(B_1^0,...,B_{k^0}^0,B_1^i,...,B_{k^i}^i\right)$. This directly contradicts the LOC. The case of an individual with strict ranking follows similarly.

We now give formal proofs that the LOC is both necessary and sufficient for the linear opinion pool.

Proposition 1: If $\preceq_0$ is a linear opinion pool with respect to $\{\preceq_i\}_{i=1}^n$, then it satisfies the LOC.

Proof. For all $i \in \{0,...,n\}$, let p_i be a probability measure representing $\preceq_i$, so that $p_0 = \sum_{i=1}^n \lambda_i p_i$ for some collection of positive weights λ_i. Let $\left(A_1^i,...,A_{k^i}^i\right)$, $\left(B_1^i,...,B_{k^i}^i\right)$ be two sequences satisfying the hypotheses of the LOC. In particular, as

$$\left(A_1^0,...,A_{k^0}^0,A_1^i,...,A_{k^i}^i\right) \risingdotseq \left(B_1^0,...,B_{k^0}^0,B_1^i,...,B_{k^i}^i\right),$$

we conclude that for all $i \in \{1, ..., n\}$,

$$\sum_{j=1}^{k^0} 1_{A_j^0} + \sum_{j=1}^{k^i} 1_{A_j^i} = \sum_{j=1}^{k^0} 1_{B_j^0} + \sum_{j=1}^{k^i} 1_{B_j^i}.$$

We may rewrite the preceding so that

$$\sum_{j=1}^{k^0} \left(1_{A_j^0} - 1_{B_j^0}\right) = \sum_{j=1}^{k^i} \left(1_{B_j^i} - 1_{A_j^i}\right).$$

For all i,

$$\int_\Omega \sum_{j=1}^{k^0} \left(1_{A_j^0}(\omega) - 1_{B_j^0}(\omega)\right) dp_i(\omega) = \int_\Omega \sum_{j=1}^{k^i} \left(1_{B_j^i}(\omega) - 1_{A_j^i}(\omega)\right) dp_i(\omega).$$

Rewrite this expression as

$$\sum_{j=1}^{k^0} \left(p_i\left(A_j^0\right) - p_i\left(B_j^0\right)\right) = \sum_{j=1}^{k^i} \left(p_i\left(B_j^i\right) - p_i\left(A_j^i\right)\right).$$

But, recall that $A_j^i \succeq_i B_j^i$ for all i; hence,

$$\sum_{j=1}^{k^i} \left(p_i\left(B_j^i\right) - p_i\left(A_j^i\right)\right) \leq 0$$

(with strict inequality if any individual likelihood is strict). Consequently,

$$\sum_{j=1}^{k^0} \left(p_i\left(A_j^0\right) - p_i\left(B_j^0\right)\right) \leq 0$$

(again with strict inequality if any individual likelihood is strict). Hence,

$$\sum_{i=1}^{n} \lambda_i \sum_{j=1}^{k^0} \left(p_i\left(A_j^0\right) - p_i\left(B_j^0\right)\right) \leq 0$$

(with strict inequality if any individual likelihood is strict as $\lambda_i > 0$ for all $i \in \{1, ..., n\}$), so

$$\sum_{j=1}^{k^0} \left(\sum_{i=1}^{n} \lambda_i p_i\left(A_j^0\right) - \sum_{i=1}^{n} \lambda_i p_i\left(B_j^0\right)\right) \leq 0.$$

Recall that $p_0 = \sum_{i=1}^{n} \lambda_i p_i$ represents social likelihood, so that the preceding is

$$\sum_{j=1}^{k^0} \left(p_0\left(A_j^0\right) - p_0\left(B_j^0\right)\right) \leq 0.$$

But $A_j^0 \succeq_0 B_j^0$ for all j. Therefore

$$\sum_{j=1}^{k^0} \left(p_0\left(A_j^0\right) - p_0\left(B_j^0\right)\right) \geq 0,$$

with strict inequality if social likelihood is strict. Consequently,

$$\sum_{j=1}^{k^0} \left(p_0\left(A_j^0\right) - p_0\left(B_j^0\right)\right) = 0,$$

so that neither social nor individual likelihood can be strict. Hence, for all $i \in \{0, ..., n\}$, for all $j \in \left\{1, ..., k^j\right\}$, $A_j^i \sim_i B_j^i$. ■

The main purpose of this paper is that the converse is also true.

Proposition 2: Suppose that the ordinal probability $\preceq_0$ satisfies the LOC with respect to the ordinal probabilities $\{\preceq_i\}_{i=1}^n$. Then $\preceq_0$ is a linear opinion pool with respect to $\{\preceq_i\}_{i=1}^n$.[4]

Proof. For this direction, we employ the theorem of the alternative, as found in Fishburn (1973), Theorem 3.3 (see also Aumann (1964)). To this end, note that $\preceq_0$ is a linear opinion pool with respect to the ordinal probabilities $\{\preceq_i\}_{i=1}^n$ if and only if for all $i \in \{1, ..., n\}$, and for all $A \in 2^\Omega \backslash \varnothing$, there exists a number $p_i(A)$ such that

1. $A \succ_0 B \Longrightarrow \sum_{i=1}^n p_i(A) > \sum_{i=1}^n p_i(B)$

2. $A \sim_0 B \Longrightarrow \sum_{i=1}^n p_i(A) = \sum_{i=1}^n p_i(B)$

3. for all $i \in \{1, ..., n\}$, $A \succ_i B \Longrightarrow p_i(A) > p_i(B)$

4. for all $i \in \{1, ..., n\}$, $A \sim_i B \Longrightarrow p_i(A) = p_i(B)$

5. for all $i \in \{1, ..., n\}$ and for all $A, B \in 2^\Omega \backslash \varnothing$ for which $A \cap B = \varnothing$, $p_i(A \cup B) = p_i(A) + p_i(B)$.

Note that if there exists such $\{p_i(A)\}_{i \in n, A \in 2^\Omega \backslash \varnothing}$, then in fact for all $i \in \{1, ..., n\}$ and for all ω^*, $p_i(\{\omega^*\}) \geq 0$. This follows as each $\preceq_i$ is an ordinal probability, and hence $\Omega \succeq_i \Omega \backslash \{\omega^*\}$, from which we conclude that

$$\sum_{\omega \in \Omega} p_i(\{\omega\}) \geq \sum_{\omega \in \Omega \backslash \{\omega^*\}} p_i(\{\omega\}),$$

[4] Technically, if $\preceq_0$ is an arbitrary binary relation on 2^Ω satisfying the LOC with respect to the ordinal probabilities $\{\preceq_i\}_{i=1}^n$, it is easy to show directly that if it is complete, it is itself an ordinal probability. So there is no need to assume that $\preceq_0$ is itself an ordinal probability (moreover, the proof searches for a specific representation of $\preceq_0$; if it has a representation of this type, then it is representable by a probability measure and is also clearly an ordinal probability).

or $p_i(\{\omega^*\}) \geq 0$. This in turn implies that $p_i(A) \geq 0$ for all i, A. Lastly, it can be similarly shown that $p_i(\Omega) > 0$ for all Ω. Therefore, the existence of such p_i functions indeed implies that $\preceq_0$ is a linear opinion pool with respect to $\{\preceq_i\}_{i=1}^n$. Note that the terms $p_i(A)$ do not necessarily represent probabilities; rather they represent probabilities weighted by λ_i as in the representation.

Moreover, note that there exist constraints of type 1 and 3 since $\preceq_0$ and $\{\preceq_i\}_{i=1}^n$ are all ordinal probabilities.

We will formulate the existence of such a p as an integer linear programming problem. We adopt the convention that a row vector multiplied by a column vector of the same dimension results in a real number. We construct a matrix X which has $n\left(\left|2^\Omega \backslash \varnothing\right|\right)$ columns and which has as many rows as there are constraints in $1-5$. Formally, if there are two symmetric constraints in $1-5$, they will be treated as distinct (for example, $A \sim_i B$ is treated as distinct from $B \sim_i A$). Each row of X will correspond to a constraint of type $1-5$. Columns will be indexed by (i, A), where $i \in \{1, ..., n\}$ and $A \in 2^\Omega \backslash \varnothing$.

For a row vector of dimension $n\left(\left|2^\Omega \backslash \varnothing\right|\right)$, the vector $1_{(i,A)}$ indicates a vector which places a 1 in the (i, A) column, and a zero everywhere else.

We will construct the rows corresponding to constraints of all types, 1-5. For rows corresponding to constraints of type $A \succ_0 B$, label this row as $X_{A \succ_0 B}$. Then

$$X_{A \succ_0 B} = \sum_{i=1}^n 1_{(i,A)} - \sum_{i=1}^n 1_{(i,B)}.$$

For rows corresponding to constraints of type 2, label the row as $X_{A \sim_0 B}$. Then

$$X_{A \sim_0 B} = \sum_{i=1}^n 1_{(i,A)} - \sum_{i=1}^n 1_{(i,B)}.$$

For rows corresponding to constraints of type 3, label the row as $X_{A \succ_i B}$. Then

$$X_{A \succ_i B} = 1_{(i,A)} - 1_{(i,B)}.$$

For rows corresponding to constraints of type 4, label the row as $X_{A \sim_i B}$. Then

$$X_{A \sim_i B} = 1_{(i,A)} - 1_{(i,B)}.$$

For rows corresponding to constraints of type 5, label the row as $X_{(i,A,B)}$. Then

$$X_{(i,A,B)} = 1_{(i,A \cup B)} - 1_{(i,A)} - 1_{(i,B)}.$$

The existence of a p satisfying the inequalities $1-5$ is therefore equivalent to the existence of a column vector $p \in \mathbb{R}^{n\left(\left|2^\Omega \backslash \varnothing\right|\right)}$ for which

1'. $X_{A \succ_0 B} p > 0$

2'. $X_{A \sim_0 B} p = 0$

3'. $X_{A \succ_i B} p > 0$

4'. $X_{A\sim_i B}p = 0$

5'. $X_{(i,A,B)}p = 0.$

We will refer to rows of X by a subscript and columns of X by a superscript. Note that X is a matrix of zeroes and ones (integers). It follows by the theorem of the alternative that if such a p does not exist, then there exists a column vector of integers z whose dimension is the number of constraints for which

$$z^T X = 0.$$

Here, 0 is a row vector of dimension equal to the number of columns of X. Thus, corresponding to each constraint of type $1-5$ is an integer. The theorem of the alternative further implies that for at least one strict inequality constraint, the corresponding value of z is strictly positive. All elements of z corresponding to strict inequality constraints must have nonnegative z component; and all other constraints (equality constraints) may have an arbitrary integer component. We may, of course, without loss of generality assume that all constraints of type $1-4$ have a z value which is nonnegative, as $z_{A\sim_i B}X_{A\sim_i B} = (-z_{A\sim_i B})\, X_{B\sim_i A}$.

Let $i \in \{0,1,...,n\}$ be arbitrary. Let $k^i = \sum_{\{(A,B):A\succ_i B\}} z_{A\succ_i B} + \sum_{\{(A,B):A\sim_i B\}} z_{A\sim_i B}$. First, define the sequences $(A,...,A)\,(B,...,B)$ (each A,B replicated $z_{A\succ_i B}$ times or $z_{A\sim_i B}$ times), and then define the sequences $\left(A_1^i,...,A_{k^i}^i\right), \left(B_1^i,...,B_{k^i}^i\right)$ by concatenating the previous sequences. Therefore, for all $j \in \left\{1,...,k^i\right\}$, $A_j^i \succeq_i B_j^i$. Furthermore, as some component of z corresponding to a strict inequality is positive, it follows that for some $i \in \{0,...,n\}$, some element of the sequence $\left(A_1^i,...,A_{k^i}^i\right), \left(B_1^i,...,B_{k^i}^i\right)$ corresponds to $A_j^i \succ_i B_j^i$.

We will now establish that for all $i \in \{1,...,n\}$, $\left(A_1^0,...,A_{k^0}^0, A_1^i,...,A_{k^i}^i\right) \doteqdot \left(B_1^0,...,B_{k^0}^0, B_1^i,...,B_{k^i}^i\right)$. This will contradict the LOC as for all $i \in \{0,...,n\}$, $A_j^i \succeq_i B_j^i$, yet it is not the case that for all $i \in \{0,....,n\}$, $A_j^i \sim_i B_j^i$.

So, let $i \in \{1,...,n\}$ be arbitrary. For arbitrary $A \in 2^\Omega \backslash \varnothing$, $z^T X^{(i,A)} = 0$. Fix $\omega \in \Omega$. Thus

$$\sum_{\{A:\omega\in A\}} z^T X^{(i,A)} = 0.$$

Writing out this expression formally, we obtain

$$\begin{aligned}
& \sum_{\{(A,B):A\succ_0 B,\omega\in A\}} z_{A\succ_0 B} + \sum_{\{(A,B):A\sim_0 B,\omega\in A\}} z_{A\sim_0 B} + \sum_{\{(A,B):A\succ_i B,\omega\in A\}} z_{A\succ_i B} \\
& + \sum_{\{(A,B):A\sim_i B,\omega\in A\}} z_{A\sim_i B} - \sum_{\{(A,B):A\succ_0 B,\omega\in B\}} z_{A\succ_0 B} - \sum_{\{(A,B):A\sim_0 B,\omega\in B\}} z_{A\sim_0 B} \\
& - \sum_{\{(A,B):A\succ_i B,\omega\in B\}} z_{A\succ_i B} - \sum_{\{(A,B):A\sim_i B,\omega\in B\}} z_{A\sim_i B} \\
= & \sum_{\{(A,B):A\cap B=\varnothing,\omega\in A\}} z_{(i,A,B)} + \sum_{\{(A,B):A\cap B=\varnothing,\omega\in B\}} z_{(i,A,B)} - \sum_{\{(A,B):A\cap B=\varnothing,\omega\in A\cup B\}} z_{(i,A,B)}.
\end{aligned}$$

Note that the right hand side of this equality is equal to zero (as for a fixed (A,B) for which $A \cap B = \varnothing$),

$$1_{\omega \in A} + 1_{\omega \in B} = 1_{\omega \in A \cup B},$$

so that

$$z_{(i,A,B)} 1_{\omega \in A} + z_{(i,A,B)} 1_{\omega \in B} = z_{(i,A,B)} 1_{\omega \in A \cup B},$$

and summing over all (A,B) for which $A \cap B = \varnothing$, we obtain

$$\sum_{\{(A,B):A\cap B=\varnothing,\omega\in A\}} z_{(i,A,B)} + \sum_{\{(A,B):A\cap B=\varnothing,\omega\in B\}} z_{(i,A,B)} = \sum_{\{(A,B):A\cap B=\varnothing,\omega\in A\cup B\}} z_{(i,A,B)}.$$

We conclude that

$$\begin{aligned} & \sum_{\{(A,B):A\succ_0 B,\omega\in A\}} z_{A\succ_0 B} + \sum_{\{(A,B):A\sim_0 B,\omega\in A\}} z_{A\sim_0 B} \\ & + \sum_{\{(A,B):A\succ_i B,\omega\in A\}} z_{A\succ_i B} + \sum_{\{(A,B):A\sim_i B,\omega\in A\}} z_{A\sim_i B} \\ = & \sum_{\{(A,B):A\succ_0 B,\omega\in B\}} z_{A\succ_0 B} + \sum_{\{(A,B):A\sim_0 B,\omega\in B\}} z_{A\sim_0 B} \\ & + \sum_{\{(A,B):A\succ_i B,\omega\in B\}} z_{A\succ_i B} + \sum_{\{(A,B):A\sim_i B,\omega\in B\}} z_{A\sim_i B}. \end{aligned}$$

The left hand side of this equality is the number of times that ω appears in $\left(A_1^0, ..., A_{k^0}^0, A_1^i, ..., A_{k^i}^i\right)$, whereas the right hand side is the number of times it appears in $\left(B_1^0, ..., B_{k^0}^0, B_1^i, ..., B_{k^i}^i\right)$. Hence, as ω was arbitrary,

$$\left(A_1^0, ..., A_{k^0}^0, A_1^i, ..., A_{k^i}^i\right) \doteqdot \left(B_1^0, ..., B_{k^0}^0, B_1^i, ..., B_{k^i}^i\right).$$

■

3 Symmetric linear opinion pools

The preceding section gave necessary and sufficient conditions for a group likelihood assessment to be written as *some* convex combination of individual probability assessments. This would allow the weights to vary; perhaps giving more weight to experts whose opinion we valued more highly. But there was no *a priori* constraint required of the weights, except that they be positive and sum to one.

Perhaps the most commonly used linear opinion pool is the one in which each expert's probability assessment is weighted equally. Motivating this on representational form, we would claim that there is no reason to favor one expert over another, so that their opinions should be treated equally. Now, the question comes about as to what are the complete set of *ordinal* conditions implied by the use of such belief aggregation methods. If a belief aggregation method is to be a symmetric linear opinion pool, it must of course be a linear opinion pool. Therefore, the conditions we derive here imply the LOC, but are stronger.

Formally, for a set of probability measures $\{p_i\}_{i=1}^n$, say that a probability measure p_0 is a **symmetric linear opinion pool** with respect to $\{p_i\}_{i=1}^n$ if

$$p_0 = \sum_{i=1}^{n} \frac{1}{n} p_i.$$

For a set of ordinal probabilities $\{\preceq_i\}_{i=1}^n$, say that an ordinal probability $\preceq_0$ is a **symmetric linear opinion pool** with respect to $\{\preceq_i\}_{i=1}^n$ if for all $i \in \{0, ..., n\}$, there exists a probability measure p_i that represents $\preceq_i$ so that p_0 is a symmetric linear opinion pool with respect to $\{p_i\}_{i=1}^n$.

To this end, we introduce a new piece of notation. For two sequences $(A_1, ..., A_k), (B_1, ..., B_k)$ of elements of 2^Ω, and for all $m \in \mathbb{Z}$, we write $(A_1, ..., A_k) \doteq_m (B_1, ..., B_k)$ to mean that for all $\omega \in \Omega$, $|\{i : \omega \in A_i\}| = |\{i : \omega \in B_i\}| + m$. If $m < 0$, this means that for each $\omega \in \Omega$, there are m more sets in the B sequence to which ω belongs than in the A sequence. If $m > 0$, this means that for each $\omega \in \Omega$, there are m more sets in the A sequence to which ω belongs than in the B sequence. It is obvious that $(A_1, ..., A_k) \doteq (B_1, ..., B_k)$ if $(A_1, ..., A_k) \doteq_0 (B_1, ..., B_k)$.

Now we are ready to define the stronger condition (implying the LOC) which is both necessary and sufficient for a group probability assessment to be represented as a symmetric linear opinion pool.

The strong linear opinion condition (SLOC) For all $i \in \{0, ..., n\}$, let $k^i \in \mathbb{Z}_+$ and for all $i \in \{1, ..., n\}$, let $m^i \in \mathbb{Z}$ such that $\sum_{i=1}^n m^i = 0$, and let $\left(A_1^i, ..., A_{k^i}^i\right)$, $\left(B_1^i, ..., B_{k^i}^i\right)$ be two sequences such that for all $i \in \{1, ..., n\}$ and for all $j \in \left\{0, ..., k^i\right\}$, $A_j^i \succeq_i B_j^i$. Suppose that for all $i \in \{1, ..., n\}$, $\left(A_1^0, ..., A_{k^0}^0, A_1^i, ..., A_{k^i}^i\right) \doteq_{m^i} \left(B_1^0, ..., B_{k^0}^0, B_1^i, ..., B_{k^i}^i\right)$. Then for all $i \in \{0, ..., n\}$ and for all $j \in \left\{1, ...k^i\right\}$, $A_j^i \sim_i B_j^i$.

The SLOC is both necessary and sufficient for $\preceq_0$ to be a symmetric linear opinion pool with respect to $\{\preceq_i\}_{i=1}^n$.

In terms of the additive extensions $\{\preceq_i^*\}_{i=0}^n$ of the ordinal probabilities discussed in the previous section, the strong linear opinion pool requires that if $m^i \in \mathbb{Z}$ for all $i \in \{1, ..., n\}$ and $\sum_{i=1}^n m^i = 0$, then if $f \succeq_i^* g + m^i 1_\Omega$ for all $i \{1, ..., n\}$, then $g \succ_0^* f$ is false (with weak social ranking ruled out if some individual ranking is strict). This is again a type of Pareto monotonicity condition, but it is of a very strong type. One might think of it as a "Pareto monotonicity with state-independent transfers" property. The function g cannot strictly beat f if there is a way for the individuals in society to "trade" state-independent units amongst themselves which result in f being ranked at least as high as g by all individuals. It is therefore reasonable to expect, as it is reminiscent of a transferable utility model, that the "utilitarian" weights for each agent can be chosen to be identical.

Proposition 3: If $\preceq_0$ is a symmetric linear opinion pool with respect to $\{\preceq_i\}_{i=1}^n$, then it satisfies the SLOC.

Proof. For all $i \in \{0, ..., n\}$, let p_i be a probability measure representing $\preceq_i$, so that $p_0 = \sum_{i=1}^n \frac{1}{n} p_i$. For all $i \in \{0, ..., n\}$, let $\left(A_1^i, ..., A_{k^i}^i\right)$ and $\left(B_1^i, ..., B_{k^i}^i\right)$ be such that for all i, j, $A_j^i \succeq_i B_j^i$. For all $i \in \{1, ..., n\}$, let $m^i \in \mathbb{Z}$ be such that $\sum_{i=1}^n m^i = 0$, and suppose that for all $i \in \{1, ..., n\}$,

$$\left(A_1^0, ..., A_{k^0}^0, A_1^i, ..., A_{k^i}^i\right) \doteqdot_{m^i} \left(B_1^0, ..., B_{k^0}^0, B_1^i, ..., B_{k^i}^i\right).$$

By definition, we conclude that for all $i \in \{1, ..., n\}$,

$$\sum_{j=1}^{k^0} 1_{A_j^0} + \sum_{j=1}^{k^i} 1_{A_j^i} = \sum_{j=1}^{k^0} 1_{B_j^0} + \sum_{j=1}^{k^i} 1_{B_j^i} + m^i 1_\Omega.$$

We may rewrite the preceding so that

$$\sum_{j=1}^{k^0} \left(1_{A_j^0} - 1_{B_j^0}\right) = \sum_{j=1}^{k^i} \left(1_{B_j^i} - 1_{A_j^i}\right) + m^i 1_\Omega.$$

In particular,

$$\int_\Omega \sum_{j=1}^{k^0} \left(1_{A_j^0}(\omega) - 1_{B_j^0}(\omega)\right) dp_i(\omega) = \int_\Omega \sum_{j=1}^{k^i} \left(1_{B_j^i}(\omega) - 1_{A_j^i}(\omega)\right) dp_i(\omega) + m^i p_i(\Omega).$$

Rewrite this expression as

$$\sum_{j=1}^{k^0} \left(p_i\left(A_j^0\right) - p_i\left(B_j^0\right)\right) = \sum_{j=1}^{k^i} \left(p_i\left(B_j^i\right) - p_i\left(A_j^i\right)\right) + m^i.$$

But, recall that $A_j^i \succeq_i B_j^i$ for all i; hence,

$$\sum_{j=1}^{k^i} \left(p_i\left(B_j^i\right) - p_i\left(A_j^i\right)\right) + m^i \leq m^i.$$

(with strict inequality if any individual likelihood is strict). Consequently,

$$\sum_{j=1}^{k^0} \left(p_i\left(A_j^0\right) - p_i\left(B_j^0\right)\right) \leq m^i$$

(again with strict inequality if any individual likelihood is strict). Hence,

$$\sum_{i=1}^{n} \sum_{j=1}^{k^0} \left(p_i\left(A_j^0\right) - p_i\left(B_j^0\right)\right) \leq \sum_{i=1}^{n} m^i$$

(with strict inequality if any individual likelihood is strict), so

$$\sum_{j=1}^{k^0}\left(\sum_{i=1}^{n}p_i\left(A_j^0\right)-\sum_{i=1}^{n}p_i\left(B_j^0\right)\right)\leq\sum_{i=1}^{n}m^i.$$

Recall that $\sum_{i=1}^{n}m^i=0$, so that the preceding is just

$$\sum_{j=1}^{k^0}\left(\sum_{i=1}^{n}p_i\left(A_j^0\right)-\sum_{i=1}^{n}p_i\left(B_j^0\right)\right)\leq 0.$$

Dividing each side by n, we obtain

$$\sum_{j=1}^{k^0}\left(\sum_{i=1}^{n}\frac{1}{n}p_i\left(A_j^0\right)-\sum_{i=1}^{n}\frac{1}{n}p_i\left(B_j^0\right)\right)\leq 0.$$

Recall that $p_0=\sum_{i=1}^{n}\frac{1}{n}p_i$, so that the preceding is

$$\sum_{j=1}^{k^0}\left(p_0\left(A_j^0\right)-p_0\left(B_j^0\right)\right)\leq 0.$$

The preceding inequality is strict if any individual likelihood ranking is strict.

The measure p_0 represents social likelihood, and $A_j^0\succeq_0 B_j^0$ for all j. Therefore

$$\sum_{j=1}^{k^0}\left(p_0\left(A_j^0\right)-p_0\left(B_j^0\right)\right)\geq 0,$$

with strict inequality if social likelihood is strict. Consequently,

$$\sum_{j=1}^{k^0}\left(p_0\left(A_j^0\right)-p_0\left(B_j^0\right)\right)=0,$$

so that neither social nor individual likelihood can be strict. Hence, for all $i\in\{0,...,n\}$, for all $j\in\left\{1,...,k^j\right\}$, $A_j^i\sim_i B_j^i$. ■

The next proposition shows that the SLOC is also sufficient for a symmetric linear opinion pool.

Proposition 4: Suppose that the ordinal probability $\preceq_0$ satisfies the SLOC with respect to the ordinal probabilities $\{\preceq_i\}_{i=1}^n$. Then $\preceq_0$ is a symmetric linear opinion pool with respect to $\{\preceq_i\}_{i=1}^n$.

Proof. The proof of this result mimics that of Proposition 2. The only difference is that we add an additional linear constraint. To this end, note that $\preceq_0$ is a linear opinion pool with respect to the ordinal probabilities $\{\preceq_i\}_{i=1}^n$ if and only if for all $i\in\{1,...,n\}$, and for all $A\in 2^{\Omega}\backslash\varnothing$, there exists a number $p_i(A)$ such that constraints $1-5$ as used in the proof of Proposition 2 are satisfied, as well as the additional constraint

6. for all $i, j \in \{1, ..., n\}$, $p_i(\Omega) = p_j(\Omega)$.

We will formulate this problem identically to the matrix formulation constructed in the proof of Proposition 2, with the addition of new rows for constraints of type 6. All rows of X corresponding to constraints of type $1-5$ are the same as in Proposition 2 and are labelled the same.

For rows corresponding to constraints of type 6, label the row as $X_{(i,j)}$. Then

$$X_{(i,j)} = 1_{(i,\Omega)} - 1_{(j,\Omega)}.$$

The existence of a p satisfying the inequalities $1-6$ is therefore equivalent to the existence of a column vector $p \in \mathbb{R}^{n(|2^\Omega \backslash \varnothing|)}$ for which all of the constraints from Proposition 2 are satisfied $(1' - 5')$, in addition to

6'. $X_{(i,j)} p = 0$.

It follows by the theorem of the alternative that if such a p does not exist, then there exists a column vector of integers z whose dimension is the number of constraints for which

$$z^T X = 0.$$

Thus, corresponding to each constraint of type $1-6$ is an integer. The theorem of the alternative further implies that for at least one strict inequality constraint, the corresponding value of z is strictly positive. All elements of z corresponding to strict inequality constraints must have nonnegative z component; and all other constraints (equality constraints) may have an arbitrary integer component. We may again without loss of generality assume that all constraints of type $1-4$ have a z value which is nonnegative.

The numbers k^i for $i \in \{0, 1, ..., n\}$ and the sequences $(A_1^i, ..., A_{k^i}^i), (B_1^i, ..., B_{k^i}^i)$ are defined exactly as in the proof of Proposition 2. As was the case there, for some $i \in \{0, ..., n\}$, some element of the sequence $(A_1^i, ..., A_{k^i}^i), (B_1^i, ..., B_{k^i}^i)$ corresponds to $A_j^i \succ_i B_j^i$.

For all $i \in \{1, ..., n\}$, let $m^i = \sum_{j \neq i} z_{(j,i)} - \sum_{j \neq i} z_{(i,j)}$. Clearly, $\sum_{i=1}^n m^i = 0$.

We will now establish that for all $i \in \{1, ..., n\}$, $(A_1^0, ..., A_{k^0}^0, A_1^i, ..., A_{k^i}^i) \doteq_{m^i} (B_1^0, ..., B_{k^0}^0, B_1^i, ..., B_{k^i}^i)$. This will contradict the SLOC as for all $i \in \{0, ..., n\}$, $A_j^i \succeq_i B_j^i$, yet it is not the case that for all $i \in \{0, ..., n\}$, $A_j^i \sim_i B_j^i$.

So, let $i \in \{1, ..., n\}$ be arbitrary. For arbitrary $A \in 2^\Omega \backslash \varnothing$, $z^T X^{(i,A)} = 0$. Fix $\omega \in \Omega$. Thus

$$\sum_{\{A : \omega \in A\}} z^T X^{(i,A)} = 0.$$

Writing out this expression formally, we obtain

$$
\begin{aligned}
& \sum_{\{(A,B):A\succ_0 B,\omega\in A\}} z_{A\succ_0 B} + \sum_{\{(A,B):A\sim_0 B,\omega\in A\}} z_{A\sim_0 B} \\
& + \sum_{\{(A,B):A\succ_i B,\omega\in A\}} z_{A\succ_i B} + \sum_{\{(A,B):A\sim_i B,\omega\in A\}} z_{A\sim_i B} \\
& - \sum_{\{(A,B):A\succ_0 B,\omega\in B\}} z_{A\succ_0 B} - \sum_{\{(A,B):A\sim_0 B,\omega\in B\}} z_{A\sim_0 B} \\
& - \sum_{\{(A,B):A\succ_i B,\omega\in B\}} z_{A\succ_i B} - \sum_{\{(A,B):A\sim_i B,\omega\in B\}} z_{A\sim_i B} \\
= & \sum_{\{(A,B):A\cap B=\varnothing,\omega\in A\}} z_{(i,A,B)} \sum_{\{(A,B):A\cap B=\varnothing,\omega\in B\}} z_{(i,A,B)} \\
& - \sum_{\{(A,B):A\cap B=\varnothing,\omega\in A\cup B\}} z_{(i,A,B)} - \sum_{j\neq i} z_{(i,j)} + \sum_{j\neq i} z_{(j,i)}.
\end{aligned}
$$

As was noted in the end of Proposition 2, the first three terms of the right hand side sum to zero. Hence, the right hand side of this equality is equal to m^i. But the left hand side of this equality is the difference between the number of times that ω appears in $\left(A_1^0, ..., A_{k^0}^0, A_1^i, ..., A_{k^i}^i\right)$ with the number of times that ω appears in $\left(B_1^0, ..., B_{k^0}^0, B_1^i, ..., B_{k^i}^i\right)$. Hence,

$$\left(A_1^0, ..., A_{k^0}^0, A_1^i, ..., A_{k^i}^i\right) \doteq_{m^i} \left(B_1^0, ..., B_{k^0}^0, B_1^i, ..., B_{k^i}^i\right).$$

■

4 Conclusion

We have characterized the complete ordinal implications of both the linear opinion pool and the symmetric linear opinion pool in finite-state environments. The complete ordinal implications of the linear opinion pool have already been established in non-atomic environments, assuming that individual likelihood assessments and society's likelihood assessment can be represented by non-atomic and countably additive probability measures (see Mongin (1995)). Ordinal linear opinion pools are there characterized by a standard Pareto criterion on the likelihood relations. The ordinal linear opinion pool in a finite-state world is characterized by the LOC, which is equivalent to a Pareto-like condition on particular additive "extensions" of the original likelihood relations. Moreover, we have also provided a related ordinal characterization of the symmetric linear opinion pool. The additional requirement here is a much stronger Pareto criterion on the additive "extensions" of the likelihood relations.

A related open question is the ordinal characterization of linear opinion pools in general environments (where the event space is modelled as a Boolean algebra). Conditions for representability of a likelihood relation by a probability measure in this environment are known (see Chateauneuf (1985)). However, not surprisingly, the ordinal linear opinion pool seems much more difficult to characterize in this general environment.

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