# Taxation and Poverty*

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**Abstract**

We explore the implications of four natural axioms in taxation: *continuity* (small changes in the data of a taxation problem should not lead to large changes in the tax allocation), *equal treatment of equals* (agents with the same pre-tax incomes pay equal taxes), *consistency* (the way in which a group allocates a tax burden is immune to secessions of taxpayers) and *composition down* (an increase in the tax burden is handled according to agents' current post-tax incomes). The combination of the four axioms characterizes a large family of rules, which we call *generalized equal-sacrifice* rules, encompassing the so-called *equal-sacrifice* rules (such as the *flat tax*), as well as *constrained equal-sacrifice* rules (such as the *head tax*), and *exogenous poverty-line* rules (such as the *leveling tax*, and some of its possible compromises with the previous ones).

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Do not imagine that mathematics is hard and crabbed, and repulsive to common sense. It is merely the etherealization of common sense.

William Thomson[1]

This quote from William Thomson's namesake nicely summarizes the axiomatic approach to which William has dedicated his research career. There are few who equal him as either scientist or mentor (Thomson, 2015b). This paper is dedicated to him with our sincerest thanks.

# 1 Introduction

The search for the perfect income tax structure has long been of interest to economists and politicians. In a series of influential contributions, Young (1987b, 1988, 1990) studied a long-standing principle of income taxation; namely, the principle of *equal sacrifice*. This principle, which can be traced back to John Stuart Mill, states that tax schemes should be designed so that all taxpayers end up sacrificing equally, according to some cardinal utility function of income. In a stylized model of taxation, Young (1988) provides a characterization of the family of equal-sacrifice rules based on a few compelling principles.[2] However, the equal sacrifice methods (and, by association, some of the principles) can be problematic in a few senses. First, they preclude exogenous concepts such as "poverty lines." A compelling concept in the theory of progressive taxation is that there may be a threshold of income, below which individuals should not be taxed.[3] Equal sacrifice methods rule out such thresholds; indeed, everyone must *sacrifice*, as the name suggests.[4] Second, there are many compelling methods which "almost" satisfy all of Young's principles, but not the strict inequalities posited by them. The two of these principles in question (namely, strict income order preservation and strict endowment monotonicity) exclude some of the focal rules in the literature, such as the so-called head tax and leveling tax, or, in general, *constrained equal-sacrifice* rules, *i.e.*, rules that impose equal sacrifice only among some (typically richer) taxpayers.

[1]Quoted in Thompson (1910), p. 1139.

[2]The mathematical framework was introduced by O'Neill (1982) to analyze the problem of adjudicating conflicting claims.

[3]Except for the case in which all individuals are taxed to lie below this threshold; ending up with equal after-tax incomes.

[4]The recent interest in inequality (see *e.g.,* Piketty, 2014), suggests that a closer look should be taken at other such methods.

The purpose of this paper is to explore the implications of the remaining axioms used by Young (1988), without considering the two strict axioms mentioned above. More precisely, we consider four natural axioms for taxation rules: *Continuity* is the standard technical condition requiring that small changes in the data of a taxation problem should not lead to large changes in the tax allocation. *Equal treatment of equals* states that agents with the same (pre-tax) incomes pay the same taxes. *Consistency* relates the allocation of a given problem to the allocations of the subproblems that appear when we consider a subgroup of agents as a new population, and the amounts gathered in the original problem as the available endowment. The axiom requires that the application of the rule to each subproblem produces the allocation that the subgroup obtained in the original problem.[5] Finally, *composition down*, which pertains to the way in which a rule reacts to tentative changes in the tax burden. More precisely, suppose that after having divided the tax burden among taxpayers, it turns out that the actual value of the revenue to be collected is larger than was initially assumed. Then, two options are open: either the tentative division is canceled altogether and the actual problem is solved, or we add to the initial tax distribution the result of applying the rule to the remaining revenue. The requirement of *composition down* is that both ways of proceeding should result in the same allocations.

Our main result states that the combination of the four axioms described above characterizes a large family of rules, not only encompassing the equal-sacrifice rules characterized by Young (1988), but also *constrained equal-sacrifice* rules (such as the head tax), and *exogenous poverty-line* rules (such as the leveling tax, or the ones mentioned above).

The rest of the paper is organized as follows. We present the model and preliminary definitions (of axioms and rules), as well as some preliminary results, in Section 2. We present the characterization result of our family (the main result of the paper) in Section 3. We provide some further insights in Section 4 and conclude in Section 5. Some technical details have been relegated to the Appendix.

[5]See Thomson (2007a) for an excellent survey of the many applications that have been made on this idea. It is worth mentioning that, although not transparent, solidarity underpinnings have also been provided for this axiom (e.g., Thomson, 2012).

# 2 The model

We rely on the variable-population model of taxation problems used by Young (1988), and earlier by O'Neill (1982) to analyze the problem of adjudicating conflicting claims.[6] The set of potential taxpayers, or **agents**, is indexed by the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element $N$. For each $i \in N$, let $c_i \in \mathbb{R}_+$ be $i$'s taxable **income** (or $i$'s claim) and $c \equiv (c_i)_{i \in N}$ the income (claims) profile. A tax revenue $T \in \mathbb{R}_+$ is to be collected from $N$. Let $E \in \mathbb{R}_+$ denote the resulting overall post-tax income (endowment) of $N$, after collecting such a tax revenue, i.e., $E \equiv \sum_{i \in N} c_i - T$. For ease of exposition, we refer to a taxation problem as the resulting problem of allocating post-tax incomes.[7] Formally, a (taxation) **problem** is a triple consisting of a population $N \in \mathcal{N}$, an income profile $c \in \mathbb{R}^N_+$, and a post-tax total income (endowment) $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $C \equiv \sum_{i \in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the domain of taxation problems with population $N$ and $\mathcal{D} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}$, an **allocation** is a vector $x \in \mathbb{R}^N$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$, and (ii) $\sum_{i \in N} x_i = E$. We refer to (i) as **boundedness**, and (ii) as **balance**.[8]

## 2.1 Tax rules

A (taxation) **rule** on $\mathcal{D}$, $S \colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $S(N, c, E)$ for the problem.

Canonical examples of rules are the **head tax**, which distributes the tax burden equally, provided no agent ends up paying more than her income, the **flat tax**, which equalizes tax rates across agents, and the **leveling tax**, which equalizes post-tax income across agents, provided no agent is subsidized.

Formally, for each $(N, c, E) \in \mathcal{D}$, and each $i \in N$,

$H_i(N, c, E) = \max\{0, c_i - \lambda\}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

$F_i(N, c, E) = \lambda \cdot c_i$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \lambda \cdot c_i = E$.

$L_i(N, c, E) = \min\{c_i, \lambda\}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

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[6] See Thomson (2003, 2014a, 2015a) for surveys of the sizable literature dealing with this model.

[7] In Young (1988), the alternative (dual) interpretation is considered in which the issue is to allocate the tax revenue $T$ among the agents in $N$.

[8] Obviously, for each allocation $x$ of a given problem $(N, c, E) \in \mathcal{D}$, one can always construct its corresponding tax profile $t = c - x$.

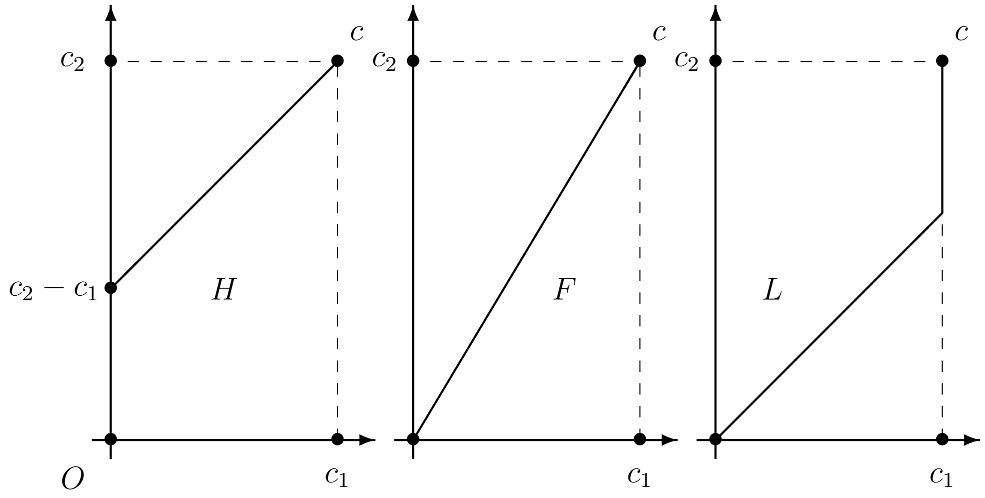


**Figure 1: Focal rules in the two-agent case**. This figure illustrates the "path of allocations" of some rules for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. The path of allocations for $c$ (the locus of the post-tax incomes vector chosen by a rule as the endowment $E$ varies from 0 to $c_1 + c_2$) of $H$ follows the vertical axis until the endowment is large enough, so that the agent with the lowest income is able to get a positive post-tax income, i.e., until $E = c_2 - c_1$. After that, it follows the line of slope 1 until it reaches the vector of incomes $c$. The path of $F$ follows the line joining the origin and the vector of incomes. Finally, the path of $L$ follows the $45^o$ line until the endowment is large enough to allow the agent with the lowest income to be exempted, i.e., until $E = 2c_1$, from where it is vertical until it reaches the vector of incomes.

Young (1988) described a class of taxation rules, known as **equal-sacrifice** rules. Each member of the class is described by a utility function. Formally, let $u : \mathbb{R}_{++} \to \mathbb{R}$ be a continuous and strictly increasing function such that $\lim_{x \to 0_+} u(x) = -\infty$. Then, for each $(N, c, E) \in \mathcal{D}$, with $c > 0$, the **equal-sacrifice rule relative to** $u$**,** $ES^u$, selects the allocation $x$ such that, for some $\lambda \geq 0$, and for each $i \in N$, we have $u(c_i) - u(x_i) = \lambda$. Among the previous examples, only the flat tax belongs to this family.

Young (1988) observed that each equal-sacrifice rule may be described in an alternative way that suggests a potentially useful geometric interpretation. Formally, let $v : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function such that $v(0) = 0$. Then, for each $(N, c, E) \in \mathcal{D}$, with $c > 0$, the **generalized proportional rule relative to** $v$**,** $GP^v$, selects the allocation $x$ such that, for some $\mu \geq 0$, and for each $i \in N$, we have $\frac{v(x_i)}{v(c_i)} = \mu$. Thus, the family is also consistent with the interpretation of proportionality, where proportionality is measured in terms of utility, rather than in terms of actual income.[9]

[9] An alternative proposal for generalized proportional rules, in this context, is introduced by Ju, Miyagawa and Sakai (2007).

It turns out, as noted by Young (1988), that a rule is an equal-sacrifice rule if and only if it is a generalized proportional rule.[10] This suggests the following geometric interpretation. A rule is an equal-sacrifice rule (or, equivalently, a generalized proportional rule), if and only if there is a strictly increasing transformation $v$ for which $v(0) = 0$, and for which, under the map $(y_1, \ldots, y_n) \mapsto (v(y_1), \ldots, v(y_n))$, the paths of allocations for the rule are segments connecting the origin to the point $(v(c_1), \ldots, v(c_n))$. Figure 2 illustrates this interpretation (in the two-agent case) for a focal equal-sacrifice rule; the so-called **Cassel rule**. This rule (as described by Young, 1988) is obtained for the utility function $u : \mathbb{R}_{++} \to \mathbb{R}$ defined by setting $u(x) = -\frac{1}{x}$, for each $x \in \mathbb{R}_{++}$, in the definition of equal-sacrifice rules. More precisely, for each $(N, c, E) \in \mathcal{D}$, the Cassel rule ($C$) selects the allocation $x = C\,(N, c, E)$ such that $-\frac{1}{c_i} + \frac{1}{x_i} = \lambda$, for each $i \in N$, *i.e.*, $x_i = \frac{c_i}{\lambda c_i + 1}$, for each $i \in N$. The corresponding generalized proportional rule would then be associated to the function $v : \mathbb{R}_+ \to \mathbb{R}$ such that $v(x) = \exp\left(-\frac{1}{x}\right)$, for each $x \in \mathbb{R}_{++}$, and $v(0) = 0$.

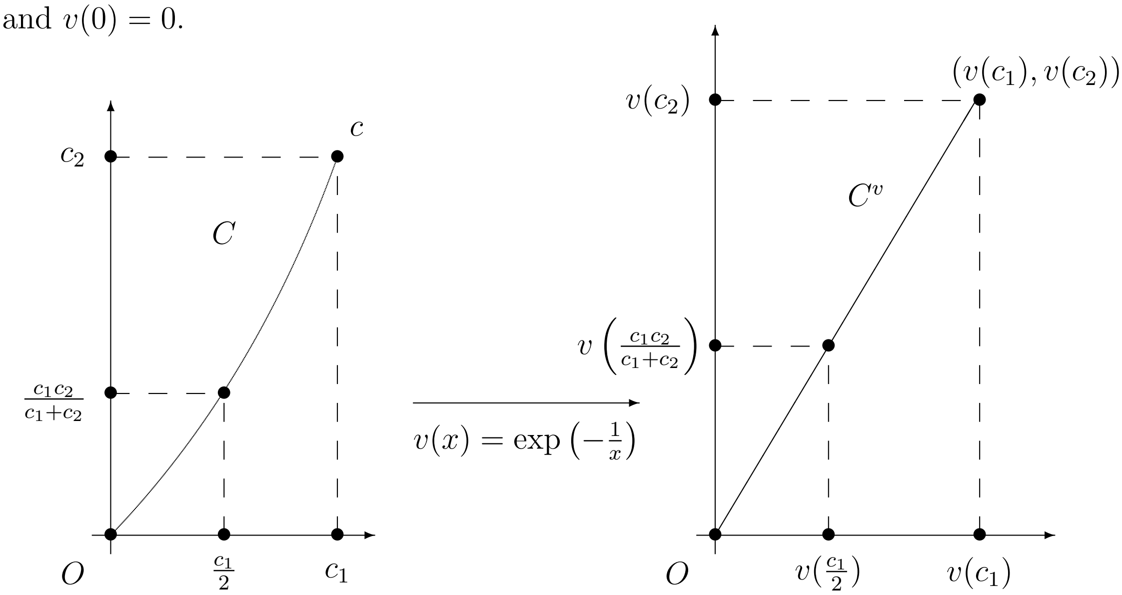


**Figure 2: A generalized proportional rule in the two-agent case**. This figure illustrates the path of allocations of the Cassel rule for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. As the picture on the left shows, the path of allocations of $C$ is a convex curve joining the origin and $c$. A vector $(x_1, x_2)$ belongs to the path if and only if $c_2 x_2 (c_1 - x_1) = c_1 x_1 (c_2 - x_2)$. The picture on the right is obtained from the one on the left via the corresponding function $v(\cdot)$, which expresses the Cassel rule as a generalized proportional rule. In that case, the path of allocations for the rule becomes the segment connecting the origin to the point $(v(c_1), v(c_2))$.

[10]For the equal-sacrifice rule relative to $u$, $ES^u$, let $v(x) = \exp u(x)$. For the generalized proportional rule relative to $v$, $GP^v$, let $u(x) = \log v(x)$. Young (1988) does not use "generalized proportional rule", but rather talks about relative (versus absolute) equal sacrifice.

The family of equal-sacrifice rules is included in the following family introduced (and characterized) by Young (1987a). Each member of this broader family is described by a *parametric representation*. Formally, let $a, b \in \mathbb{R} \cup \{\pm\infty\}$, and let $f : [a, b] \times \mathbb{R}_+ \to \mathbb{R}$ be a jointly continuous (in both variables) and non-decreasing (in the first variable) function, such that, for each $x \in \mathbb{R}_+$, $f(a, x) = 0$ and $f(b, x) = x$; Then, for each $(N, c, E) \in \mathcal{D}$, the **parametric rule relative to $f$,** $S^f$, selects the allocation $S(N, c, E)$ such that, for some $\lambda \in [a, b]$, and for each $i \in N$, $S_i(N, c, E) = f(\lambda, c_i)$.[11]

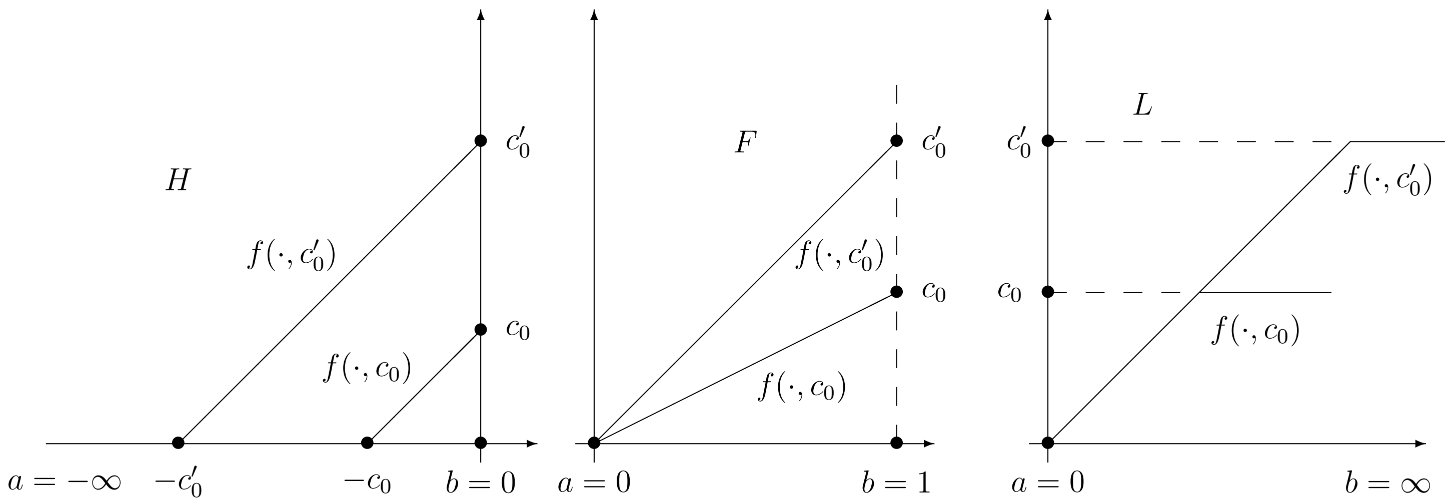


**Figure 3: Parametric representations of the three focal rules**. (a) Head tax: $a \equiv -\infty$ and $b \equiv 0$, and the schedule for each claim $c_0$ follows the horizontal axis until the point of abscissa $-c_0$, then a line of slope 1 up to the point $(0, c_0)$. (b) Flat tax: $[a, b] \equiv [0, 1]$ and the schedule for each claim $c_0$ is a segment through the origin of slope equal to $c_0$. (c) Leveling tax: $a \equiv 0$ and $b \equiv \infty$, and the schedule for each claim $c_0$ follows the $45^o$ line up to the point $(c_0, c_0)$, then continues horizontally.

The following parametric rules, some of which have unquestionable normative appeal for taxation purposes, are not equal-sacrifice rules.

- **Compromises between the leveling tax and the flat tax**;

  Consider the case in which the leveling tax is imposed until some exogenous endowment level, after which the flat tax is imposed (to adjusted down incomes). We interpret such a rule as a *poverty-line* rule, in which the aim is to collect taxes so that agents below

[11] Stovall (2014) characterizes a generalized family, in which the parametric representations are allowed to be agent-dependent.

the poverty line are exempted whenever possible. Once all agents can be guaranteed the minimal subsistence level, flat taxation is imposed. Formally, for each $\alpha \in \mathbb{R}_+ \cup \{0\}$, define the rule $S^\alpha$ such that, for each $N \in \mathcal{N}$, and each $c \in \mathbb{R}^N_+$, the path of allocations of $S^\alpha$ for $c$ is the path of the leveling tax until all taxpayers whose incomes are at least $\alpha$ have received $\alpha$. It concludes with a segment to $c$, unless no agent's income is greater than $\alpha$, in which case this last segment is degenerate (Figure 4a).[12] In other words, consider the parametric rules with a parametric representation of the following form (Figure 4b): they are piece-wise linear in two pieces. For $c_0 \leq \alpha$, the first piece is the segment from the origin to $(c_0, c_0)$, and the second piece is the segment from there to $(b, c_0)$. For $c_0 \geq \alpha$, the first piece is the segment from the origin to $(\alpha, \alpha)$, and the second piece is the segment from there to $(b, c_0)$.

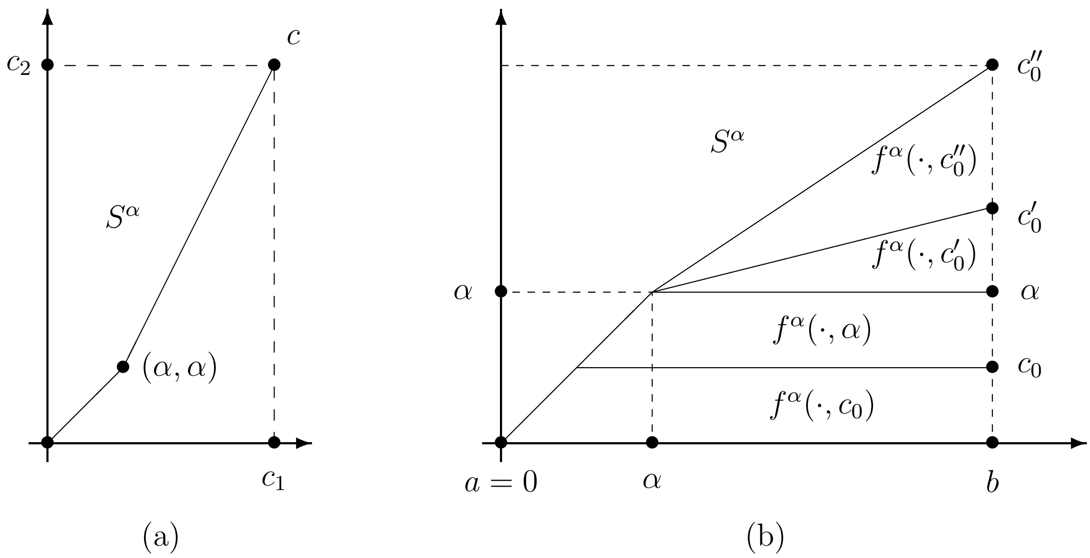


**Figure 4: Compromising between the leveling tax and the flat tax**. **(a)** Path of allocations of $S^\alpha$: for $N = \{1, 2\}$, and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$, it follows the $45^o$ line until $(\alpha, \alpha)$, and then continues with a segment to $c$. **(b)** Parametric representations of $S^\alpha$: for each $c_0 \in \mathbb{R}_+$, the graph of $f^\alpha(\cdot, c_0)$ follows the $45^o$ line until $(\min\{c_0, \alpha\}, \min\{c_0, \alpha\})$ and then continues in a linear way until $(b, c_0)$.

- **Compromises between the flat tax and the leveling tax**;

Consider now the opposite case in which the flat tax is imposed until some exogenous endowment level, after which the leveling tax is imposed (to adjusted down incomes). Formally, for each $A \in \mathbb{R}_+ \cup \{0\}$, consider the parametric rules, $S^A$, with a parametric representation of the following form: they are piece-wise linear in two pieces. For $c_0 \leq A$,

[12] This family was proposed by Thomson (2007b).

the first piece is the segment from the origin to $(A, c_0)$, and the second piece is a horizontal half-line from there. For $c_0 \geq A$, the first piece is the segment from the origin to $(c_0, c_0)$, and the second piece is a horizontal half-line from there (Figure 5a). In particular, for each $N \in \mathcal{N}$ for which $|N| = 2$, and each $c \in \mathbb{R}^N_+$ such that $c_i < c_j$, the path of allocations of $S^A$ for $c$ takes one of the following forms: (i) If $c_i \leq A$, it consists of the broken segment connecting the origin, the point whose $i$-th coordinate is $c_i$ and $j$-th coordinate is $A$, and $c$ (Figure 5b). (ii) If $A < c_i < c_j$, it coincides with the path of the leveling tax. (iii) If $c$ has equal coordinates, it is simply the segment joining the origin and $c$. Note that, for $A = 0$, the previous compromise leads the leveling tax, whereas for $A = \infty$, it leads the flat tax.[13]

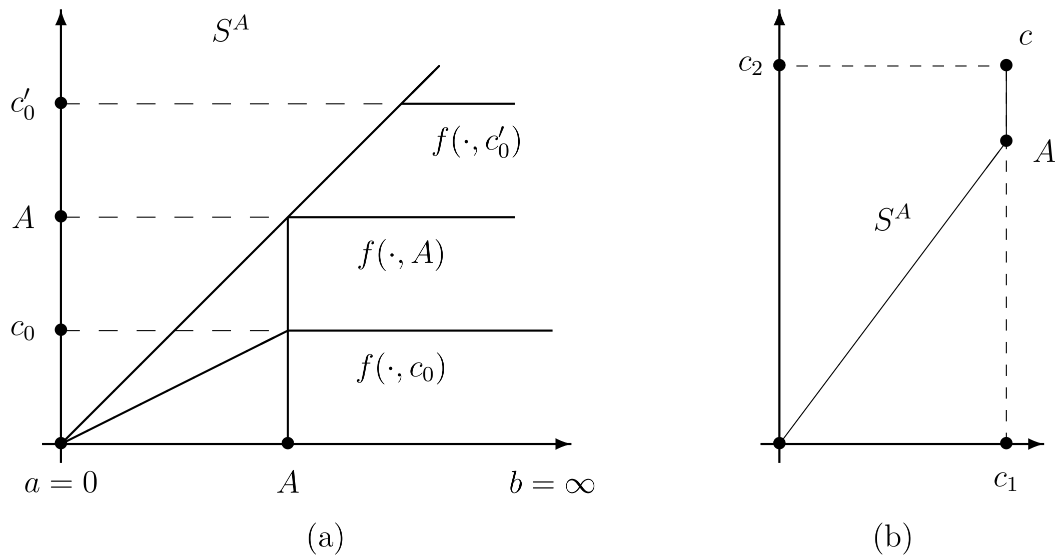


**Figure 5: Compromising between the flat tax and the leveling tax**. **(a)** Parametric representations of $S^A$: for each $c_0 \in \mathbb{R}_+$, the graph of $f^\alpha(\cdot, c_0)$ follows the segment between the origin and $(\max\{c_0, A\}, c_0)$ and then continues with a horizontal half-line. **(b)** Path of allocations of $S^A$: for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$, it follows the segment between the origin and $(c_1, A)$, and then continues vertically until $c$.

- **Constrained equal-sacrifice taxes**;

Consider rules that equalize utility losses (sacrifices) as much as possible, for *general* utility functions. More precisely, let $u : \mathbb{R}_{++} \to \mathbb{R}$ be a continuous and strictly increasing function, without necessarily obeying Young's proviso that $\lim_{x \to 0_+} u(x) = -\infty$, and hence allowing for zero consumption. Let the domain of such utility functions be denoted

[13] This proposal was made by Thomson (2014b).

as $\mathcal{U}$. Then, for each $(N, c, E) \in \mathcal{D}$, with $c > 0$, the **constrained equal-sacrifice rule relative to** $u \in \mathcal{U}$, $CES^u$, selects the tax allocation vector such that, for all $i, j \in N$, if $u(c_i) - u(x_i) < u(c_j) - u(x_j)$ then $x_i = 0$. Note that the head tax is an instance of a constrained equal-sacrifice rule (but not of an equal-sacrifice rule).

### 2.1.1 A new family

The above motivates the following definition of a large family lying between the family of equal-sacrifice rules and the family of parametric rules. A **generalized equal-sacrifice** rule is a rule associated with a partition of $\mathbb{R}_+$ into *left-closed intervals*.[14] Any such partition must be a partition into left-closed right-open intervals, and points. Let us call a generic such partition as $\Lambda$, with typical element $\lambda \in \Lambda$. We refer to elements of $\Lambda$ as **brackets**.[15] Note that $\Lambda$ is naturally ordered by $\geq$. Associated with each left-closed right-open interval $\lambda = [a_\lambda, b_\lambda)$ there is a continuous and strictly increasing function $g_\lambda(\cdot) : [a_\lambda, b_\lambda) \to \mathbb{R} \cup \{-\infty\}$. It is important to note that $b_\lambda = +\infty$ is permitted, but not required. Furthermore, $g_\lambda(a_\lambda) = \lim_{x \to a_\lambda} g_\lambda(x) = -\infty$ is also possible. Let us denote the associated collection of functions to each bracket within the partition as $\Gamma$. With this in mind, the associated rule operates as follows. For any $(N, c, E) \in \mathcal{D}$, the bracket $\lambda \in \Lambda$ and allocation are chosen so that $c_i \leq \lambda$ implies $S_i(N, c, E) = c_i$, and for each pair $i, j \in N$ such that $\neg(c_i \leq \lambda)$ and $\neg(c_j \leq \lambda)$, we have $S_i(N, c, E), S_j(N, c, E) \in \lambda$;[16] and, further, if $\lambda$ is a left-closed right-open interval, we have $g_\lambda(\min\{c_i, b_\lambda\}) - g_\lambda(S_i(N, c, E)) < g_\lambda(\min\{c_j, b_\lambda\}) - g_\lambda(S_j(N, c, E))$ implies that $S_i(N, c, E) = a_\lambda$. We refer to the rule, so defined, as $S^{\Lambda,\Gamma}$.

In words, generalized equal-sacrifice rules impose constrained equal sacrifice with respect to some lower and upper bounds, allowing for a possible set of agents being exempted, to be interpreted as *agents below the poverty line*. Those bounds are exogenously described by the brackets defining the rule, although the ones eventually being used are determined by the characteristics (set of agents, claims and endowment) of the problem at hand.[17]

Alternatively, each rule within the family could be interpreted as a sort of hybrid between

[14] We say that an interval is left-closed if it possesses a minimum.

[15] For ease of exposition, we shall sometimes refer to points as *degenerate brackets*.

[16] Formally, $c_i \leq \lambda$ means that $c_i \leq x$ for all $x \in \lambda$.

[17] Lower and upper bounds have a long tradition of use within normative economics and, in particular, frequent instances occur within the literature on fair allocation (e.g., Thomson, 2013b). Early uses of these notions for the same model considered in this paper are Moreno-Ternero and Villar (2004) and Dominguez and Thomson (2006).

the leveling tax and different constrained equal-sacrifice rules, in which each of the latter are applied at different non-degenerate brackets of the partition, and the leveling tax is used to alternate among them, provided some of those (non-degenerate) brackets are disjoint from the ensuing ones in the partition.

The family is general enough to accommodate all of the rules introduced above. The leveling tax, for instance, is obtained when one considers in the above definition the partition made of all points on the (positive) real line (each interpreted as a degenerate bracket). The flat tax is obtained when the partition is only made of one interval and the corresponding function is the logarithmic one. The head tax is obtained with the same partition and the identity function. More generally, all constrained equal-sacrifice rules (and, in particular, all equal sacrifice-rules) can be obtained with such a partition and a generic utility function with the properties stated at the definition of this family. Furthermore, compromises between the leveling and the flat tax (such as those illustrated in Figures 4 and 5) and more general poverty-line rules can also be described as members of this family. If the leveling tax is part of the definition of the rule, then degenerate brackets should be part of the partition. If only constrained equal-sacrifice rules are considered to describe the rule, then no brackets within the partition can be degenerate.[18]

All the rules within the family have a particularly intuitive and simple description in the two-agent case. More precisely, as illustrated in Figure 6, each rule is associated with a collection of *boxes* along the $45^o$ line. For each bracket within the partition defining the rule, a box with vertices $(a_\lambda, a_\lambda)$ and $(b_\lambda, b_\lambda)$ is constructed.[19] The path of the rule for a given claims vector $c = (c_1, c_2)$ will follow the $45^o$ until it reaches the southeast corner of the box corresponding to the bracket in which $\min\{c_1, c_2\}$ lies. If such a bracket is degenerate, then it continues horizontally or vertically until it reaches $c$. Alternatively, if the bracket is a left-closed right-open interval $[a_\lambda, b_\lambda)$, then it follows a path which equalizes losses (as much as possible) according to $g_\lambda$ for the truncated claims vector $(\min\{c_1, b_\lambda\}, \min\{c_2, b_\lambda\})$, subject to the constraint that the claims vector is at least $a_\lambda$.[20] If $(\min\{c_1, b_\lambda\}, \min\{c_2, b_\lambda\}) = c$, the path concludes there. Otherwise, the path continues vertically, or horizontally (depending on whether $c_2 > c_1$, or vice versa) until it reaches $c$.

[18]The reader is referred to the Appendix for more details about these representations.

[19]The boxes, whose interiors are disjoint, might thus have different length, some of them even being *degenerate*.

[20]Note that the function $g_\lambda$ is uniquely extended to $b_\lambda$.

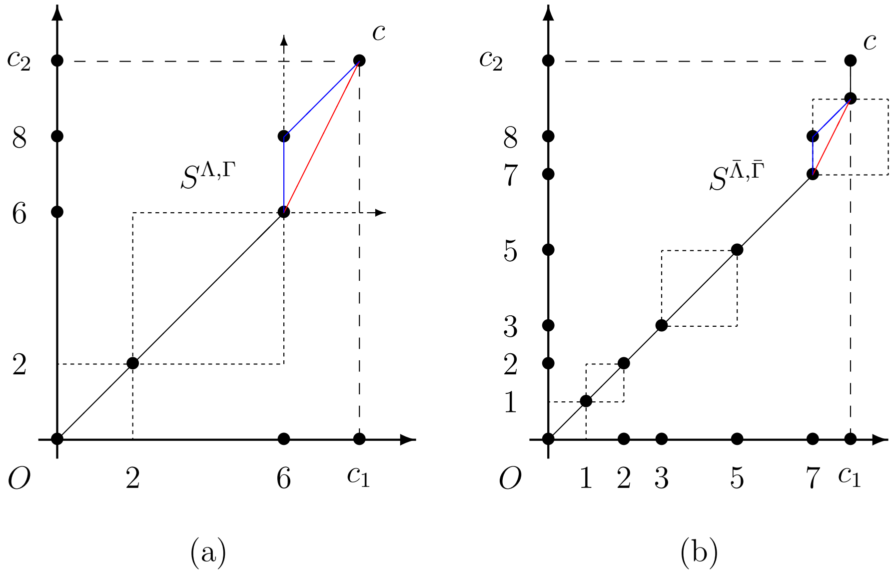


**Figure 6: Generalized equal-sacrifice rules in the two-agent case**. This figure illustrates the path of allocations of several generalized equal-sacrifice rules for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 = 8 < c_2 = 10$. In Figure 6a, we consider the rule for which $\Lambda = \{[0, 2), [2, 6), [6, +\infty)\}$ and $\Gamma = \{g_{[0,2)}(x) = \log(x);\ g_{[2,6)}(x) = x;\ g_{[6,+\infty)}(x) = \log(x - 6)\}$. Thus, the path of allocations of $S^{\Lambda,\Gamma}$ follows the $45^o$ line (i.e., it coincides with the path of $L$) until it reaches the point $(6, 6)$, the vertex of the box determined by the third bracket (the first one containing a claim). After that, it follows the segment joining such a point with the claims vector (i.e., it coincides with the path of $F$, the equal-sacrifice rule corresponding to $g_{[6,+\infty)}$). Note that the path would not be altered for different utility functions in the first two brackets, or for different configurations of brackets before $[6, +\infty)$, the first one in which a claim lies. Now, if instead of using $g(x) = \log(x-6)$ at the bracket $[6, +\infty)$, we would use the identity function, then the last piece of the path (the one included in the third box) would change to endorse the path of $H$ therein (i.e., a vertical line from the point $(6, 6)$ to the point $(6, 8)$ and then the segment from $(6, 8)$ to the claims vector). In Figure 6b, we consider the rule for which $\bar{\Lambda} = \{[0, 1), [1, 2), [3, 5), [7, 9), \{x\} : x \in (2, 3) \cup (5, 7) \cup (9, +\infty)\}$, and $\bar{\Gamma} = \{g_{[0,1)}(x) = \log(x);\ g_{[1,2)}(x) = x;\ g_{[3,5)}(x) = \log(x - 3);\ g_{[7,9)}(x) = x\}$. Thus, the path of allocations of $S^{\bar{\Lambda},\bar{\Gamma}}$ follows the $45^o$ line (i.e., it coincides with the path of $L$) until it reaches the point $(7, 7)$, vertex of the box determined by the fourth non-degenerate bracket (the first one containing a claim). After that, it follows the path of $H$, the equal-sacrifice rule corresponding to the identity function, until the truncated claims vector $(8, 9)$. From there, it follows a vertical line until the original claims vector is reached. If instead of the identity function, the utility function for the last bracket would had been $g(x) = \log(x - 7)$, then the previous to last piece would had been following the path of $F$ (instead of $H$) to such a truncated-claims vector, i.e., the segment between $(7, 7)$ and $(8, 9)$. All other rules within the family coinciding with $S^{\bar{\Lambda},\bar{\Gamma}}$ at the bracket $(7, 9)$ would yield the same path of allocations in this case.

More generally, a generalized equal-sacrifice rule can be associated to a *weakly monotone space-filling tree*. That is, we can describe any generalized equal-sacrifice rule by means of a network of weakly monotone curves emanating from the origin, paths of allocations being subsets of these curves. Figure 7 provides such a "tree representation" for one rule within the family.

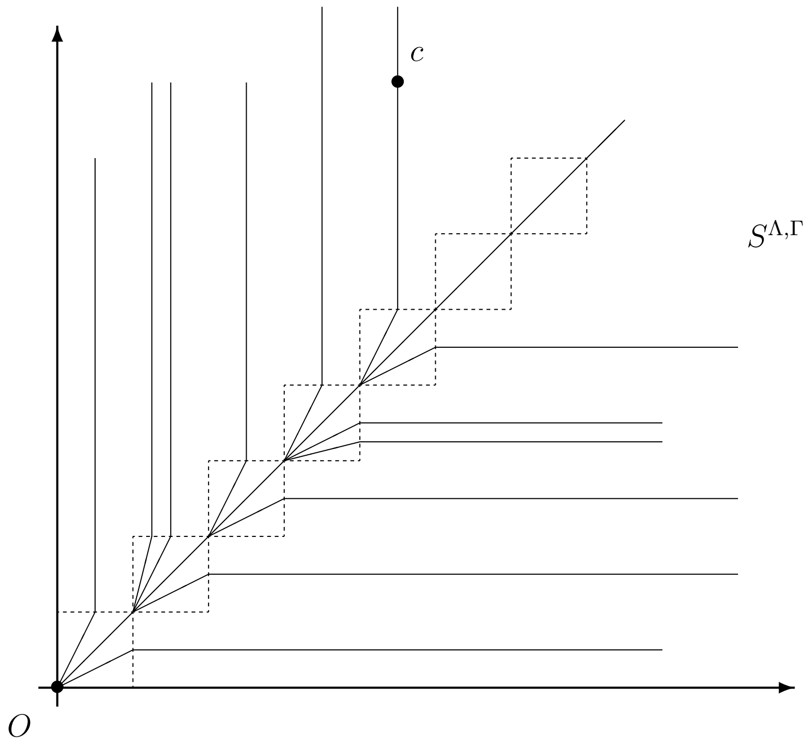


**Figure 7: The "tree description" of a generalized equal-sacrifice rule in the two-agent case**. This figure illustrates the description of a generalized equal-sacrifice rule by means of its possible paths of allocations for $N = \{1, 2\}$. We consider the rule for which $\Lambda = \{[n, n+1) : n \in \mathbb{N}\}$ and $\Gamma = \{g_{[n,n+1)}(x) = \log(x - n)\}$. For such a rule, we consider the tree with a main branch following the $45^o$ line and secondary branches emanating from the set of points $A = \{(n, n) : n \in \mathbb{N}\}$ of such a main branch, involving a segment from such points to either points within the set $B = \{[x, n+1] : n \in \mathbb{N}; x \in [n, n+1]\}$, or points within the set $C = \{[n+1, y] : n \in \mathbb{N}; y \in [n, n+1]\}$, followed by a vertical (respectively, horizontal) line. More precisely, for each $c \in \mathbb{R}^N_+$, the corresponding path of allocations of $S^{\Lambda,\Gamma}$ follows the $45^o$ line from the origin until it reaches the point $(\bar{n}, \bar{n})$, where $\bar{n}$ is the integer part of $\min\{c_1, c_2\}$. Then, we distinguish two cases. If $c_1 < c_2$, the path follows the segment from $(\bar{n}, \bar{n})$ to $(c_1, \bar{n}+1)$, and then the segment from $(c_1, \bar{n}+1)$ to $c$. Alternatively, if $c_1 > c_2$, the path follows the segment from $(\bar{n}, \bar{n})$ to $(\bar{n}+1, c_2)$, and then the segment from $(\bar{n}+1, c_2)$ to $c$. To complete, if $c_1 = c_2$, the path just follows the main branch until $c$.

## 2.2 Axioms

We concentrate on four axioms of rules for our analysis in this paper.

The first one states that small changes in the data of the problem should not lead to large changes in the chosen allocation. Formally,

- **Continuity**: For each sequence $\{(N, c^k, E^k)\}$ of problems in $\mathcal{D}$, and each $(N, c, E) \in \mathcal{D}$, if $(N, c^k, E^k) \to (N, c, E)$, then $S(N, c^k, E^k) \to S(N, c, E)$.

The second one requires that equal pre-tax incomes implies equal post-tax incomes. Formally,

- **Equal treatment of equals**: For each $(N, c, E) \in \mathcal{D}$, and each $i, j \in N$, such that $c_i = c_j$, $S_i(N, c, E) = S_j(N, c, E)$.

The third one relates the allocation of a given problem to the allocations of the subproblems that appear when we consider a subgroup of agents as a new population, and the amounts gathered in the original problem as the available endowment. The axiom requires that the application of the rule to each subproblem produces the allocation that the subgroup obtained in the original problem. Formally,

- **Consistency**: For each $(N, c, E) \in \mathcal{D}$, each $M \subset N$, and each $i \in M$, we have $S_i(N, c, E) = S_i(M, c_M, \sum_{i \in M} S_i(N, c, E))$.

The fourth axiom pertains to the way in which a rule responds to changes in the endowment. Suppose that after having divided the tax burden among taxpayers, it turns out that the actual value of the revenue to be collected is larger than was initially assumed (*i.e.*, the endowment is lower). Then, two options are open: either the tentative division is canceled altogether and the actual problem is solved, or we add to the initial tax distribution the result of applying the rule to the remaining revenue. We require that both ways of proceeding should result in the same allocations.[21] Formally,

[21] The name was coined by Thomson (2013a). The original axiom was first used in this context by Young (1988). This property is reminiscent of the so-called "path independence" axiom for choice functions (*e.g.*, Plott, 1973). It also has a relative in the theory of axiomatic bargaining: the so-called "step-by-step negotiations" axiom introduced by Kalai (1977), which is the basis for the characterization of the egalitarian solution in such a context. The same principle has also been frequently used in other related contexts like rationing, queuing, or resource allocation (e.g., Moulin, 2000; Moulin and Stong, 2002; Chambers, 2006; Moreno-Ternero and Roemer, 2012).

- **Composition down**: For each $(N, c, E) \in \mathcal{D}$ and each $E' < E$, we have $S(N, c, E') = S(N, S(N, c, E), E')$.

It is straightforward to see that composition down implies the following principle of *solidarity*,[22] which says that when the tax burden to be allocated is larger, nobody should benefit from it.[23] Formally,

- **Endowment monotonicity**: For each $(N, c, E) \in \mathcal{D}$ and each $E' < E$, we have $S(N, c, E') \leq S(N, c, E)$.

Thus, in what follows, we will use the following lemma often without explicit mention.

**Lemma 1.** *If a rule satisfies composition down, then it satisfies endowment monotonicity.*

We also use the following lemma, which shows other intermediary implications of the combination of the four axioms we consider. In order to present it, let us introduce first the notion that states that pre-tax and post-tax incomes are equally ordered. Formally,

- **Income order preservation**: For each $(N, c, E) \in \mathcal{D}$ and each pair $i, j \in N$ such that $c_i \geq c_j$, $S_i(N, c, E) \geq S_j(N, c, E)$.

Furthermore, we introduce a notion stating that if a taxpayer strictly benefits from a decrease in the tax burden, so do agents with larger pre-tax incomes.[24] Formally,

- **Exemption monotonicity**: For each $(N, c, E) \in \mathcal{D}$ and each $E' > E$, with $E' \leq \sum c_i$, and each pair $\{i, j\} \in N$, such that $c_i \leq c_j$ and $S_i(N, c, E') > S_i(N, c, E)$, then $S_j(N, c, E') > S_j(N, c, E)$.

[22]Solidarity properties with respect to population changes, and axiomatizations based on it, were actually introduced by Thomson (1983a,b) in related models. Roemer (1986) introduced the solidarity notion referring to the available endowment.

[23]Endowment monotonicity and consistency together are equivalent to another axiom of solidarity, which states that the arrival of new agents, whether or not it is accompanied by changes in the endowment, should affect all the incumbent agents in the same direction (e.g., Chun, 1999; Moreno-Ternero and Roemer, 2006). Thomson (2014a) names such an axiom as *resource-population uniformity*.

[24]This is, to the best of our knowledge, a new axiom in the literature. Nevertheless, it constitutes a weakening of the so-called axiom of *order preservation under endowment variations* (e.g., Thomson, 2014a, page 113), which states that if the endowment increases, given any two taxpayers, the taxpayer with the larger income should face a share of the increment that is at least as large as the share received by the taxpayer with the smaller income. The axiom was named by Dagan, Serrano and Volij (1997), and by Yeh (2006), as "supermodularity", given the connection with the mathematical property of the same name. In that sense, and following Milgrom and Shannon (1994), exemption monotonicity can be interpreted as formalizing "quasi-supermodularity" in this context.

**Lemma 2.** *If a rule satisfies continuity, equal treatment of equals, consistency, and composition down, then it also satisfies income order preservation and exemption monotonicity.*

*Proof.* Let $S$ be a rule satisfying the four axioms in the premise of the statement.

Suppose, by means of contradiction, that $S$ does not satisfy *income order preservation*, *i.e.*, there exists $(N, c, E) \in \mathcal{D}$ for which $c_i \leq c_j$ yet $S_i(N, c, E) > S_j(N, c, E)$. By *consistency*, we may (without loss of generality) assume the economy has only two agents, *i.e.*, $N \equiv \{i, j\}$. By *continuity*, there is some $E' > E$ for which $S_i(N, c, E') = S_j(N, c, E')$. By *composition down*, $S(N, c, E) = S(N, S(N, c, E'), E)$. But then, by *equal treatment of equals*, $S_i(N, c, E) = S_j(N, c, E)$, a contradiction.

As for *exemption monotonicity*, note that, by *composition down* and *consistency*, it suffices to show that $x < y$ implies that if $E < x + y$, then $S_2(\{1, 2\}, (x, y), E) < y$. Note also that, by Theorem 1 in Young (1987a), $S$ is a parametric rule. Let $f$ denote its parametric representation.

Now, suppose $x < y$, and let $\lambda^* = \arg\min\{\lambda : f(\lambda, y) = y\}$, which exists by continuity of $f$ in the first coordinate. By continuity of $f$ in the second coordinate, there exists $z$ for which $f(\lambda^*, z) = x$. Then, $S(\{1, 2\}, (z, y), x + y) = (x, y)$. Furthermore, if $E < x + y$, we have $S_2(\{1, 2\}, (z, y), E) < y$, as $\lambda^*$ was minimal. Now, by *composition down*, for any $E < x + y$, $S(\{1, 2\}, (z, y), E) = S(\{1, 2\}, (x, y), E)$. It then follows that $S_2(\{1, 2\}, (x, y; E) < y)$ whenever $E < x + y$. □

# 3 The main result

Our main result is the following:

**Theorem 1.** *A rule satisfies continuity, equal treatment of equals, consistency, and composition down if and only if it is a generalized equal-sacrifice rule.*

The idea of the proof is anticipated by Young (1988). Indeed, we borrow the main construction of his proof and generalize it according to a result from the mathematics literature (Mostert and Shields (1957), Theorem B, p. 130). This result provides an exhaustive characterization of all parametric rules (Young, 1987a) satisfying *composition down*.

*Proof.* We focus on the non-trivial implication of the theorem. In other words, we assume that $S$ is a rule satisfying the four axioms in the statement (and, hence, by Lemmas 1 and 2, also *endowment monotonicity*, *income order preservation*, and *exemption monotonicity*). Our goal is to show that $S$ is a generalized equal-sacrifice rule, *i.e.*, there exists a partition of brackets $\Lambda$, and the associated collection of functions $\Gamma = \{g_\lambda(\cdot)\}_{\lambda\in\Lambda}$, such that $S \equiv S^{\Lambda,\Gamma}$. We proceed in several steps:[25]

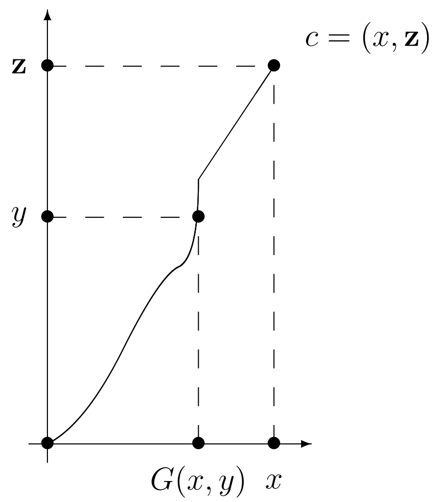


**Figure 8: Definition of $G$**

**Step 1**: Defining a function $G$ to describe the paths of allocations and identifying critical properties of $G$.

First, fix some $\mathbf{z} \in \mathbb{R}$, $\mathbf{z} > 0$. We will define a binary operator on the interval $[0, \mathbf{z}]$. To do so, we need some preliminary results.

Let $N = \{1, 2\}$. Let $G : [0, \mathbf{z}]^2 \to \mathbb{R}_+$ be defined by the requirement that, for each pair $\{x, y\} \subset [0, \mathbf{z}]$, the vector $(G(x, y), y)$ lies on the path of allocations for the claims vector $c = (x, \mathbf{z}) \in \mathbb{R}^2_+$ (see Figure 7).[26]

More formally, $G$ is defined implicitly as the solution to the equation

$$G(x, y) = S_1(\{1, 2\}, (x, \mathbf{z}), G(x, y) + y),$$

where $S_j(\cdot)$ denotes the $j$-th coordinate of the allocation $S(\{1, 2\}, (x, \mathbf{z}), G(x, y)+y)$, for $j = 1, 2$.

- $G$ is well-defined.

  Indeed, by *endowment monotonicity*, $S_2(\{1, 2\}, (x, \mathbf{z}), E)$ is continuous in $E$. Hence, as $E \in [0, x + \mathbf{z}]$, $S_2$ traces out an interval, which is $[0, \mathbf{z}]$. Thus, for each $y \in [0, \mathbf{z}]$, there is $E$ for which $S_2(\{1, 2\}, (x, \mathbf{z}), E) = y$. As $y \leq E$, we can just set $G(x, y) = E - y$ and

[25] The outline of the proof follows the outline of the proof in Thomson (2014a), which slightly rewrites the original proof in Young (1998) for the family of equal-sacrifice rules.

[26] In words, if we draw the path for $(x, \mathbf{z})$, then $G(x, y)$ is the abscissa of the unique point on that path whose ordinate is $y$.

we obtain $(G(x,y),y) = S(\{1,2\},(x,\mathbf{z}),G(x,y)+y)$, as desired. Moreover, by *exemption monotonicity*, $S_2(\{1,2\},(x,\mathbf{z}),E)$ is strictly increasing in $E$, so the previous procedure describes a unique point.

- $G$ is continuous in its second coordinate.[27]

  To show this, let $y_n \to y$. We will show that all subsequences of $G(x,y_n)$ themselves contain a subsequence converging to $G(x,y)$, which is enough to establish that $G(x,y_n)$ converges to $G(x,y)$. Without loss of generality, assume that such a subsequence is $y_n$. As $G(x,y_n) \in [0,x]$, we know that $G(x,y_n)$ possesses a convergent subsequence, say $G(x,y_{n_k}) \to G$. Hence, by *continuity*, we have $(G,y) = S(\{1,2\},(x,\mathbf{z}),y+G)$. But then $G = G(x,y)$, by definition.

- $G$ is independent of agents 1 and 2.

  As $S$ satisfies *equal treatment of equals* and *consistency*, it is also anonymous (e.g., Chambers and Thomson, 2002; Lemma 3).

We now provide some further properties of the binary operator induced by $G$:

- (Associativity) For each $x,y,z \in [0,\mathbf{z}]$, $G(G(x,y),z) = G(x,G(y,z))$.

  The argument is taken directly from Young (1988). Let $x,y,z \in [0,\mathbf{z}]$ and consider $N' = \{1,2,3\}$. Consider the vector of incomes $(G(x,y),y,\mathbf{z})$, and find, by *continuity*, numbers $a$ and $b$ for which $(a,b,z)$ lies on the path for $(G(x,y),y,\mathbf{z})$. In particular, *consistency* applied to the first and third coordinates implies that $(a,z)$ lies on the path for $(G(x,y),\mathbf{z})$, so that $a = G(G(x,y),z)$ by definition of $G$. *Consistency* applied to the second and third coordinates implies that $(b,z)$ lies on the path for $(y,\mathbf{z})$. So, $b = G(y,z)$ by definition of $G$. *Consistency* applied to the first and second coordinates implies that $(a,b)$ lies on the path for $(G(x,y),y)$, which itself lies on the path for $(x,\mathbf{z})$ (by definition of $G$); hence, by *composition down*, $(a,b)$ lies on the path for $(x,\mathbf{z})$. Consequently, $a = G(x,b)$; Equivalently, $G(G(x,y),z) = G(x,G(y,z))$.

- (Continuity) G is continuous.

  Let $(x^*,y^*) \in [0,\mathbf{z}]^2$. Let $\epsilon > 0$. Choose $0 < \eta < \frac{\epsilon}{3}$ small so that $|y-y^*| < \eta$ implies that $G(x^*,y)$ is within $\frac{\epsilon}{3}$ of $G(x^*,y^*)$. Let $\overline{E} = y^* + \eta + G(x^*,y^*+\eta)$; similarly, $\underline{E} = y^* - \eta +$

[27] We will later show that it is indeed jointly continuous.

$G(x^*, y^* - \eta)$. Choose $\delta > 0$ small so that $|x - x^*| < \delta$ ensures that $S_2(\{1,2\}, (x, \mathbf{z}), \underline{E}) < y^* < S_2(\{1,2\}, (x, \mathbf{z}), \overline{E})$. Now let $0 < \beta < \frac{\epsilon}{3}$ be small so that $|y - y^*| < \beta$ ensures $S_2(x, \mathbf{z}, \underline{E}) < y < S_2(x, \mathbf{z}, \overline{E})$. Note, therefore, that, for such $x$ within $\delta$ of $x^*$, and $y$ within $\beta$ of $y^*$, $y^* - \eta + G(x^*, y^* - \eta) = \underline{E} < G(x, y) + y < \overline{E} = y^* + \eta + G(x^*, y^* + \eta)$. Hence, $G(x^*, y^*) - \epsilon < G(x, y) < G(x^*, y^*) + \epsilon$.

- (Zero) $G(0, x) = G(x, 0) = 0$, for each $x \in [0, \mathbf{z}]$.

  $G(0, x) = 0$ follows by boundedness, and $G(x, 0) = 0$ by *income order preservation*.

- (One) $G(\mathbf{z}, x) = G(x, \mathbf{z}) = x$, for each $x \in [0, \mathbf{z}]$.

  $G(\mathbf{z}, x) = x$ follows from *equal treatment of equals*, and $G(x, \mathbf{z}) = x$ from *exemption monotonicity*.

**Step 2**: Constructing $\Lambda$ and $\Gamma$ from $G$.

We now apply the aforementioned characterization, due to Mostert and Shields (1957). We use a representation which is due to Marichal (2000, Theorem 4.2).

We have verified that $G$ satisfies the properties of Theorem $B$ of Mostert and Shields. As a consequence, there exist a countable index set $K$, a family of disjoint open subintervals $\{(\alpha_k, \beta_k) : k \in K\}$ of $[0, \mathbf{z}]$ and a family $\{f_k : k \in K\}$ of continuous strictly decreasing functions $f_k : [\alpha_k, \beta_k] \to [0, +\infty]$, with $f_k(\beta_k) = 0$, such that, for all $x, y \in [0, \mathbf{z}]$,

$$G(x,y) = \begin{cases} f_k^{-1}\left(\min\{f_k(x) + f_k(y), f_k(\alpha_k)\}\right), & \text{if there exists } k \in K \text{ such that } x, y \in [\alpha_k, \beta_k] \\ \min\{x, y\}, & \text{otherwise .} \end{cases}$$

By considering the function $g_k = -f_k$ on $[\alpha_k, \beta_k]$, it is quite easy to see that this can equivalently be written as:

$$G(x,y) = \begin{cases} g_k^{-1}\left(\max\{g_k(x) + g_k(y), g_k(\alpha_k)\}\right), & \text{if there exists } k \in K \text{ such that } x, y \in [\alpha_k, \beta_k] \\ \min\{x, y\}, & \text{otherwise .} \end{cases}$$

Here, $g_k$ is obviously strictly increasing and satisfies $g_k(\beta_k) = 0$. In particular, the argument of Young (1988) focuses on the case in which there is only one interval (namely, $(0, \infty)$), and $g_k(0) = -\infty$. Let

$$\Lambda = \{[\alpha_k, \beta_k) : k \in K\} \cup \left\{ \{x\} : x \in \mathbb{R}_+ \setminus \bigcup_{k \in K} [\alpha_k, \beta_k) \right\},$$

and consider the associated collection of functions $\Gamma = \{g_k(\cdot)\}_{k \in K}$.

**Step 3**: Restricted to problems with incomes bounded by $\mathbf{z}$, $S \equiv S^{\Lambda,\Gamma}$.

It is easy to see that, for a vector of incomes $c \in [0, \mathbf{z}]^N$, by considering the profile $(c, \mathbf{z})$ obtained by adding an individual with claim $\mathbf{z}$, and applying *consistency*, the path generated for the income vector $c$ is given by $(G(c_1, \lambda), \ldots, G(c_n, \lambda))$, as $\lambda$ ranges between 0 and $\mathbf{z}$.

We now demonstrate that the rule coincides with our generalized equal-sacrifice solution, *i.e.*, $S \equiv S^{\Lambda,\Gamma}$. To do so, we distinguish two cases:

- $\lambda \notin \bigcup_{k \in K} [\alpha_k, \beta_k)$

  In this case, it is straightforward to see, by the definition of $G$, that the rule assigns everybody either their claim, or an equal amount, *i.e.*, the allocation proposed by the leveling tax.

- $\lambda \in [\alpha_k, \beta_k)$ for some $k \in K$.

  In this case, for each $i \in N$,

$$S_i(N, c, E) = \begin{cases} c_i, & \text{if } c_i < \alpha_k \\ g_k^{-1}\left(\max\{g_k(c_i) + g_k(\lambda), g_k(\alpha_k)\}\right), & \text{if } \alpha_k \leq c_i \leq \beta_k \\ \lambda, & \text{if } c_i > \beta_k. \end{cases}$$

  Note that for agents of the third type (*i.e.*, for $i$ such that $c_i > \beta_k$), the operation of $G$ produces $G(c_i, \lambda) = \lambda$, so that $S_i(N, c, E) = \lambda$; inspection of the definition of $G$ verifies that this is the same as $G(\beta_k, \lambda)$, as $g_k(\beta_k) = 0$. Thus,

$$S_i(N, c, E) = \begin{cases} c_i, & \text{if } c_i < \alpha_k \\ G(c_i, \lambda), & \text{if } \alpha_k \leq c_i \leq \beta_k \\ G(\beta_k, \lambda), & \text{if } c_i > \beta_k. \end{cases}$$

  We now claim that the formula for $G$ implies that these are constrained equal-sacrifice taxes above $\alpha_k$, for the function $g_k$. In order to show that we distinguish two cases.

  1. $G(c_i, \lambda) > \alpha_k$.

     In this case, by definition of $G$, $g_k(\alpha_k) \leq g_k(c_i) + g_k(\lambda)$, and thus $S_i(N, c, E) = g_k^{-1}(g_k(c_i) + g_k(\lambda))$. As $g_k$ operates on $[\alpha_k, \beta_k)$, and as $S_i(N, c, E) \in [\alpha_k, \beta_k]$, this means that $g_k(S_i(N, c, E)) = g_k(c_i) + g_k(\lambda)$, or $g_k(c_i) - g_k(S_i(N, c, E)) = -g_k(\lambda)$.

In particular, this tells us that, for any two agents with allocations greater than $\alpha_k$, sacrifices are equalized.

2. $G(c_i, \lambda) = \alpha_k$

   In this case, it follows that $g_k(\alpha_k) \geq g_k(c_i) + g_k(\lambda)$. Hence $g_k(c_i) - g_k(S_i(N, c, E)) = g_k(c_i) - g_k(\alpha_k) \leq -g_k(\lambda)$. In other words, such an agent's sacrifice (measured by $g_k$) is only less than the remaining agents' sacrifices if they consume $\alpha_k$.

**Step 4**: Establishing the general result.

We start by demonstrating that any two representations of the same (generalized equal-sacrifice) rule, restricted to an interval $[0, \mathbf{z}]$, are unique up to affine transformations. More precisely, $(\Lambda, \Gamma) = \left(\{[\alpha_k, \beta_k) : k \in K\} \cup \left\{\{x\} : x \in \mathbb{R}_+ \setminus \bigcup_{k \in K}[\alpha_k, \beta_k)\right\}, \{g_k(\cdot)\}_{k \in K}\right)$ and $(\overline{\Lambda}, \overline{\Gamma}) = \left(\{[\gamma_l, \delta_l) : l \in L\} \cup \left\{\{x\} : x \in \mathbb{R}_+ \setminus \bigcup_{l \in L}[\gamma_l, \delta_l)\right\}, \{h_l(\cdot)\}_{l \in L}\right)$ represent $S$ if and only if there is a bijective function $M : K \to L$, and constants $p_k > 0$ and $r_k \in \mathbb{R}$, for each $k \in K$, such that $[\gamma_{M(k)}, \delta_{M(k)}) = [\alpha_k, \beta_k)$ and $h_{M(k)} = p_k g_k + r_k$.

One direction is straightforward. For the other direction, let us suppose that $(\Lambda, \Gamma)$ and $(\overline{\Lambda}, \overline{\Gamma})$ each represent $S$ on the interval $[0, \mathbf{z}]$. Fix $x, y \in [0, \mathbf{z}]$ where $x \neq y$. Consider a representation of $S$. Associated with each bracket $[\alpha_k, \beta_k)$ in the representation is an upper-closed, lower-open interval $(\alpha_k, \beta_k]$. It is easy to see, by definition of the rule, that $x$ and $y$ are in the same upper-closed lower-open interval if and only if there is $E > 0$ for which

- $S_1(\{1, 2\}, (x, y), E) < x$,
- $S_2(\{1, 2\}, (x, y), E) < y$,
- $S_1(\{1, 2\}, (x, y), E) \neq S_2(\{1, 2\}, (x, y), E)$,
- if $w \leq x$ and $S_1(\{1, 2\}, (w, y), E) = S_1(\{1, 2\}, (x, y), E)$, then $w = x$,
- if $z \leq y$ and $S_2(\{1, 2\}, (x, z), E) = S_2(\{1, 2\}, (x, y), E)$, then $z = y$.

This immediately tells us that for any pair of representations of the same rule, the corresponding brackets $[\alpha_k, \beta_k)$ are the same.

Now, fix such a bracket $[\alpha_k, \beta_k)$. Pick any two interior points $x < y$ in the bracket. We can normalize the functions $h_{M(k)}$ and $g_k$ so that $g_k(x) = h_{M(k)}(x) = 0$ and $g_k(y) = h_{M(k)}(y) = 1$. We now claim that, after such a normalization, the two functions are equal, which establishes the result. Now, consider the point $x_{1/2}$ for which $g_k(x_{1/2}) = 1/2$. Note that $S(\{1, 2\}, (y, x_{1/2}), x +$

$x_{1/2}) = (x_{1/2}, x)$. In turn, this implies that $h_{M(k)}(x_{1/2}) = 1/2$. Inductively, we can determine coincidence of $g_k$ and $h_{M(k)}$ on any dyadic rational combination of $x$ and $y$. Thus coincidence is extended to all points in between $x$ and $y$ by *continuity*. The construction can be carried on outside of this interval in the obvious manner.

Hence, any two representations of the same rule restricted to an interval $[0, \mathbf{z}]$ are unique in the preceding manner. This therefore allows us to complete the construction as follows. For any pair $x, y \in [0, \infty)$, $x, y$ are in the same bracket if there is $\mathbf{z} > \max\{x, y\}$ for which $x$ and $y$ are in the same bracket of the $\mathbf{z}$-representation. Once we obtain the class of brackets, we can choose $g_k$ functions by picking, for each bracket, two points $x_k < y_k$ in the interior of the bracket and defining $g_k(x_k) = 0$ and $g_k(y_k) = 1$ (this does not rely on the axiom of choice and can be done algorithmically; say, if the interval is bounded, picking $x_k = \alpha_k + \frac{\beta_k - \alpha_k}{3}$ and $y_k = \alpha_k + \frac{2(\beta_k - \alpha_k)}{3}$; or if it is unbounded, picking $x_k = \alpha_k + 1$ and $y_k = \alpha_k + 2$). The functions $g_k$ can then be uniquely constructed everywhere by the preceding observations. □

# 4 Further insights

Theorem 3 of Young (1985) reports, without proof, an axiomatization of all continuous rules satisfying equal treatment of equals, consistency, composition down and **homogeneity**, which states that if claims and endowment are multiplied by the same positive number, then all post-tax incomes should be multiplied by the same number. Formally, for each $(N, c, E) \in \mathcal{D}$ and $\lambda > 0$, $S(N, \lambda c, \lambda E) = \lambda S(N, c, E)$. The following is a restatement of Young's theorem, described in our notation.

**Theorem 2.** *A rule satisfies continuity, equal treatment of equals, consistency, composition down, and homogeneity if and only if it is either the leveling tax, or $CES^u$ with one of the following utility indices:*

- *For $p > 0$, $u(x) = x^p$,*
- *$u(x) = \ln(x)$,*
- *For $p < 0$, $u(x) = -x^p$.*

As shown in its proof below, the theorem is easily derived from our Theorem 1 and the following claim, which states a property that characterizes the left endpoints of the (non-degenerate) brackets associated to a generalized equal-sacrifice rule.

**Claim 1.** *Let $S^{\Lambda,\Gamma}$ be a generalized equal-sacrifice rule, and $\lambda = [a_\lambda, b_\lambda) \in \Lambda$. Then, for each $x \geq a_\lambda$ and each $E \in [2a_\lambda, x + a_\lambda]$, $a_\lambda = S_1(\{1, 2\}, (a_\lambda, x), E)$. Conversely, if $y \in \mathbb{R}_+$ is such that, for each $x \geq y$ and each $E \in [2y, x + y]$, $S_1(\{1, 2\}, (y, x), E) = y$, then there exists $\lambda = [a_\lambda, b_\lambda) \in \Lambda$ such that $y = a_\lambda$.*

*Proof.* Let $S^{\Lambda,\Gamma}$ be a generalized equal-sacrifice rule, $\lambda = [a_\lambda, b_\lambda) \in \Lambda$, and $x \geq a_\lambda$. Then, the path of allocations of $S^{\Lambda,\Gamma}$ for $(a_\lambda, x)$ follows the $45^o$ line until it reaches $(a_\lambda, a_\lambda)$, from where it continues vertically until it reaches $(a_\lambda, x)$. Thus, for each $E \in [2a_\lambda, x + a_\lambda]$, it follows that $a_\lambda = S_1(\{1, 2\}, (a_\lambda, x), E)$, as desired. Conversely, let $y \in \mathbb{R}_+$ be such that, for each $x \geq y$ and each $E \in [2y, x + y]$, $S_1(\{1, 2\}, (y, x), E) = y$. Then, for each $x \geq y$, the path of allocations of $S^{\Lambda,\Gamma}$ for $(y, x)$ follows the $45^o$ line until it reaches $(y, y)$, from where it continues vertically until it reaches $(y, x)$, i.e., the path of the leveling tax. This implies that $y$ is a left endpoint of a (non-degenerate) bracket in $\Lambda$ as no constrained equal-sacrifice rule shares its path with the leveling tax. □

In words, the claim states that the left endpoints of the (non-degenerate) brackets, associated to a generalized equal-sacrifice rule, are precisely those claims that are fully honored, when it is feasible to do so. We are now ready to prove Theorem 2.

*Proof.* We focus on the non-trivial implication of the theorem. In other words, we assume that $S$ is a rule satisfying the five axioms in the statement. By Theorem 1, $S$ is a generalized equal-sacrifice rule, *i.e.*, there exists a partition of brackets $\Lambda$, and the associated collection of functions $\Gamma = \{g_\lambda(\cdot)\}_{\lambda\in\Lambda}$, such that $S \equiv S^{\Lambda,\Gamma}$. By homogeneity and Claim 1, either no positive real numbers are left endpoints of (non-degenerate) brackets in $\Lambda$ or all of them are. In other words, $\Lambda$ either contains the bracket $[0, \infty)$; or only degenerate brackets. In the latter case, $S$ would coincide with the leveling tax. In the other cases, $S$ would be a constrained equal sacrifice rule. Furthermore, this constrained equal sacrifice rule would have the property that for any $0 \leq y \leq x$, $0 \leq w \leq z$, if $u(x) - u(y) = u(z) - u(w)$, then for any $\lambda > 0$, we have $u(\lambda x) - u(\lambda y) = u(\lambda z) - u(\lambda w)$. By Theorem 1 in (Section 2, page 28) Azcel (1987), it follows that the general continuous nonconstant solutions of that equation are given by $u(x) = ax^p$ for some $a > 0$ and $p > 0$, $u(x) = a \ln(x)$ for some $a > 0$, or $u(x) = -ax^p$ for some $a > 0$ and $p < 0$.[28] □

[28]This is basically the argument that characterizes constant relative risk aversion (CRRA) utility indices; early appearances of such a result are in Burk (1936) and Hardy, Littlewood and Pólya (1952, p. 68). See also Lemma 1 in Young (1987b) for a similar treatment to the one considered in this paper.

The statement of Theorem 2 gives a parametric family including the head tax ($p = 1$), the leveling tax (which can be understood as the pointwise limit as $p \to -\infty$), and the flat tax $u(x) = \ln(x)$, (which can also be understood as $p = 0$). Interestingly, if we consider the pointwise limit as $p \to \infty$, the rule tends to approach a kind of "fully regressive" taxation rule, which is discontinuous in incomes. We discuss this rule more in the Appendix (item 4).

# 5 Conclusion

In this paper, we have investigated the implications of four natural axioms for taxation rules, which lead to a new family of rules. The family is a hybrid of the leveling tax, and constrained equal sacrifice methods. It also allows flexibility to incorporate poverty lines into the theory of taxation and bankruptcy. The family can also be understood as an exhaustive characterization of all parametric rules satisfying composition down. As such, the family lies directly in between the two families investigated by Peyton Young, and characterized as follows:

**Theorem 3.** *(Young, 1987a) A rule satisfies continuity, equal treatment of equals, and consistency, if and only if it is a parametric rule.*

**Theorem 4.** *(Young, 1988) A rule satisfies continuity, equal treatment of equals, consistency, composition down, strict income order preservation, and strict endowment monotonicity if and only if it is an equal-sacrifice rule.*

To conclude, it is worth mentioning that additional (counterpart) characterization results to the ones presented in this paper can be obtained upon exploiting the notion of duality (see Thomson (2014a, Chapter 7) for further details about the notion and its applications in this setting).

# 6 Appendix

## 6.1 The tightness of the characterization result

The rules presented next are all chosen to be homogeneous, which hence also shows the tightness of Theorem 2.

1. A rule that satisfies all properties except for *equal treatment of equals* is a lexicographic dictatorship.

2. A rule that satisfies all properties except for *composition down* is the so-called Talmud rule (introduced by Aumann and Maschler, 1985).

3. A rule that satisfies all properties except for *consistency* is a rule which coincides with the head tax for pairs of agents, and the leveling tax otherwise.

4. A rule that satisfies all properties except for *continuity* is the fully regressive tax, which orders agents according to the size of their incomes, and taxes the poorest agents first. An agent with a high income only pays a tax if all poorer agents have fully exhausted their incomes. Agents with equal incomes are taxed equally.

5. A rule that satisfies all properties except *balance* is the rule that always taxes everybody their entire income.

## 6.2 Some generalized equal-sacrifice rules and their representations

Assume, for ease of notation, that, for each $(N, c, E) \in \mathcal{D}$, the claims vector is such that $c_1 \leq c_2 \leq \cdots \leq c_n$, where $n$ denotes the cardinality of $N$.

- **The leveling tax**: Let $\Lambda = \{\{x\} : x \in \mathbb{R}_+\}$.[29] Then, for each $(N, c, E) \in \mathcal{D}$, let $k$ be the smallest non-negative integer in $\{0, ..., n\}$ such that $E \leq ((\sum_{i=1}^{k} c_i) + (n-k)c_{k+1})$. Let $\lambda = \frac{E - (\sum_{i=1}^{k} c_i)}{n-k}$. It follows that, for each $i \in N$ such that $c_i \leq \lambda$, $L_i(N, c, E) = c_i$. Furthermore, for each $c_i > \lambda$, $L_i(N, c, E) = \lambda$.

- **The flat tax**: Let $\Lambda = \{[0, +\infty)\}$ and $\Gamma = \{g_\lambda\}$, where $g_\lambda : [0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ is such that, for each $x \in [0, +\infty)$, $g_\lambda(x) = \log(x)$. Then, for each $(N, c, E) \in \mathcal{D}$, and each pair $i, j \in N$, $g_\lambda(c_i) - g_\lambda(F_i(N, c, E)) = g_\lambda(c_j) - g_\lambda(F_j(N, c, E))$.

- **The head tax**: Let $\Lambda = \{[0, +\infty)\}$ and $\Gamma = \{g_\lambda\}$, where $g_\lambda : [0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ such that, for each $x \in [0, +\infty)$, $g_\lambda(x) = x$. Then, for each $(N, c, E) \in \mathcal{D}$, and each pair $i, j \in N$, $g_\lambda(c_i) - g_\lambda(H_i(N, c, E)) < g_\lambda(c_j) - g_\lambda(H_j(N, c, E))$ implies that $H_i(N, c, E) = 0$.

- **Equal-sacrifice rules**: We proceed as with the flat tax, but considering a generic utility function with the properties stated at the definition of this family (not necessarily the logarithmic function, as we considered for the flat tax). More precisely, let $\Lambda = \{[0, +\infty)\}$ and $\Gamma = \{g_\lambda\}$, where $g_\lambda : [0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ such that, for each $x \in [0, +\infty)$, $g_\lambda(x) =$

[29] As all brackets are degenerate, there is no need to define the set of associated functions $\Gamma$.

$u(x)$. Then, for each $(N, c, E) \in \mathcal{D}$, and each pair $i, j \in N$, $g_\lambda(c_i) - g_\lambda(ES_i^u(N, c, E)) = g_\lambda(c_j) - g_\lambda(ES_j^u(N, c, E))$.

- **Constrained equal-sacrifice rules**: We proceed as with the head tax, but considering a generic utility function with the properties stated at the definition of this family (not necessarily the identity function, as we considered for the head tax). More precisely, let $\Lambda = \{[0, +\infty)\}$ and $\Gamma = \{g_\lambda\}$, where $g_\lambda : [0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ such that, for each $x \in [0, +\infty)$, $g_\lambda(x) = u(x)$. Then, for each $(N, c, E) \in \mathcal{D}$, and each pair $i, j \in N$, $g_\lambda(c_i) - g_\lambda(CES_i^u(N, c, E)) < g_\lambda(c_j) - g_\lambda(CES_j^u(N, c, E))$ implies that $CES_i^u(N, c, E) = 0$.

- **Compromises between the flat tax and the leveling tax**: We concentrate on the rule illustrated at Figure 4, i.e., leveling tax until all agents are guaranteed a certain income ($\alpha$) and then flat tax. Let $\Lambda = \{\{x\}, [\alpha, +\infty) : 0 \leq x \leq \alpha\}$, i.e., degenerate brackets from 0 to $\alpha$ and a unique non-degenerate bracket $[\alpha, +\infty)\}$.[30] Now, let $\Gamma = \{g_\alpha\}$, where $g_\alpha : [\alpha, +\infty) \to \mathbb{R} \cup \{-\infty\}$ such that, for each $x \in [\alpha, +\infty)$, $g_\alpha(x) = \log(x - \alpha)$. Then, for each $(N, c, E) \in \mathcal{D}$, let $k_\alpha$ denote the smallest integer number for which $c_{k_\alpha} < \alpha$.[31] Then, we distinguish two cases:

  **Case 1**: $E \leq ((\sum_{i=1}^{k_\alpha} c_i) + (n - k_\alpha)c_{k+1})$. In this case, let $k$ be the smallest non-negative integer in $\{0, ..., k_\alpha\}$ such that $E \leq ((\sum_{i=1}^{k} c_i) + (n - k)c_{k+1})$. Let $\lambda = \frac{E - (\sum_{i=1}^{k} c_i)}{n-k}$. It follows that, for each $i \in N$ such that $c_i \leq \lambda$, $S_i(N, c, E) = c_i$. Furthermore, for each $c_i > \lambda$, $S_i(N, c, E) = \lambda = L_i(N, c, E)$.

  **Case 2**: $E > ((\sum_{i=1}^{k_\alpha} c_i) + (n - k_\alpha)c_{k+1})$. In this case, let $\lambda = [\alpha, +\infty)$. It follows that, for each $i \in N$ such that $c_i \leq \lambda$, $S_i(N, c, E) = c_i$. Furthermore, for each pair $i, j \in N$, such that $\min\{c_i, c_j\} > \alpha$, $g_\alpha(c_i) - g_\alpha(S_i(N, c, E)) = g_\alpha(c_j) - g_\alpha(S_j(N, c, E))$.

- **Poverty-line rules**: Let $\Lambda = \{\{x\}, [\alpha_k, \beta_k) : k \in K; x \in \mathbb{R}_+ \setminus \cup_{k \in K}[\alpha_k, \beta_k)\}$, i.e., besides the intervals $[\alpha_k, \beta_k)$, all points lying outside those intervals are also considered in this partition as degenerate brackets. Let $\Gamma = \{g_k : k \in K\}$, where, for each $k \in K$, $g_k : [\alpha_k, \beta_k) \to \mathbb{R} \cup \{-\infty\}$. Then, for each $(N, c, E) \in \mathcal{D}$, let $k_\alpha$ denote the smallest integer number for which $c_{k_\alpha} < \alpha_1$.

  **Case 1**: $E \leq ((\sum_{i=1}^{k_\alpha} c_i) + (n - k_\alpha)c_{k+1})$. In this case, let $k$ be the smallest non-negative

[30] Note that $\alpha$ is exogenously given.

[31] Recall we are assuming that claims are increasingly ordered.

integer in $\{0,...,k_\alpha\}$ such that $E \leq ((\sum_{i=1}^{k} c_i) + (n-k)c_{k+1})$. Let $\lambda = \frac{E - \left(\sum_{i=1}^{k} c_i\right)}{n-k}$. It follows that, for each $i \in N$ such that $c_i \leq \lambda$, $S_i(N,c,E) = c_i$. Furthermore, for each $c_i > \lambda$, $S_i(N,c,E) = \lambda = L_i(N,c,E)$.

**Case 2**: $E > ((\sum_{i=1}^{k_\alpha} c_i) + (n-k_\alpha)c_{k+1})$. In this case, several subcases can be considered. Before presenting them, we need to introduce some notation. For each $k \in K$, let $N_k^c = \{i \in N : \alpha_k < c_i \leq \beta_k\}$ and $n_k^c$ denote its cardinality. Furthermore, let $N_\infty^c \equiv \{i \in N : \beta_{\sup K} < c_i\}$ and $n_\infty^c$ denote its cardinality.

**Case 2.1**: $E \leq \sum_{i\in\{1,...,k_\alpha\}\cup N_1^c} c_i + (n-k_\alpha-n_1^c)\beta_1$. In this subcase, let $\lambda = [\alpha_1, \beta_1)$. It follows that, for each $i \in N$ such that $c_i \leq \lambda$, $S_i(N,c,E) = c_i$. Furthermore, for each pair $i,j \in N$, such that $\min\{c_i,c_j\} > \alpha_1$, $g_1(\min\{c_i,\beta_1\}) - g_1(S_i(N,c,E)) < g_1(\min\{c_j,b_1\}) - g_1(S_j(N,c,E))$ implies that $S_i(N,c,E) = \alpha_1$.

**Case 2.2**: $\sum_{i\in\{1,...,k_\alpha\}\cup N_1^c} c_i + (n-k_\alpha-n_1^c)\beta_1 < E \leq \sum_{i\in\{1,...,k_\alpha\}\cup N_1^c} c_i + (n-k_\alpha-n_1^c)\alpha_2$. In this subcase, let $k_1$ be the smallest non-negative integer in $\{k_\alpha,\ldots,n\}$ such that $\hat{E}_1 = E - \sum_{i\in\{1,...,k_\alpha\}\cup N_1^c} c_i + (n-k_\alpha-n_1^c)\beta_1 \leq (\sum_{i=1}^{k_1} c_i) + (n-k_1)c_{k_1+1}$. Let $\lambda = \beta_1 + \frac{\hat{E}_1 - \left(\sum_{i=1}^{k_1} c_i\right)}{n-k_1} \in (\beta_1, \alpha_2)$. It follows that, for each $i \in N$ such that $c_i \leq \lambda$, $S_i(N,c,E) = c_i$. Furthermore, for each $c_i > \lambda$, $S_i(N,c,E) = \lambda = L_i(N,c,E)$.

**Case 2.3**: $\sum_{i\in\{1,...,k_\alpha\}\cup N_1^c} c_i + (n-k_\alpha-n_1^c)\alpha_2 < E \leq \sum_{i\in\{1,...,k_\alpha\}\cup N_1^c} c_i + (n-k_\alpha-n_1^c)\beta_2$. In this subcase, let $\lambda = [\alpha_2, \beta_2)$. It follows that, for each $i \in N$ such that $c_i \leq \lambda$, $S_i(N,c,E) = c_i$. Furthermore, for each pair $i,j \in N$, such that $\min\{c_i,c_j\} > \alpha_2$, $g_2(\min\{c_i,\beta_2\}) - g_2(S_i(N,c,E)) < g_2(\min\{c_j,b_2\}) - g_2(S_j(N,c,E))$ implies that $S_i(N,c,E) = \alpha_2$.

...

The remaining cases are similarly obtained

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